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Geometric Characterizations of Standard Normal Distribution

- Two Types of Differential Equations, Relationships with Square and Circle, and Their Similar Characterizations -

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1. Introduction

We confirm various kinds of geometric characterizations of a standard normal distribution with Pearson's finding probabilistic point 0.612003 and its cumulative probability 0.2702678 on a standard normal distribution [1]. After Pearson proposed that, their numbers were studied by Kelley [2, 3], Mosteller [4], Cox [5] and many researchers [6-12]. Our studies [13-17] are also some of them. Sclove and Johari [6, 7] explained Cox's proposal to us as the clustering of normal distribution. Nakamori *et. al.* [8] also mentioned that Cox's proposal was one of the original papers of K-means Algorisms. Kelley's formulation, $\phi(\lambda) = 2\lambda\Phi(-\lambda)$, is also known as the 27 percent rule [9-12]. We found the other characterizations about 0.612003 from the parabolic curves of the cost of the repetitions game of coin tossing [13] and the square on a standard normal distribution with the probabilistic point 0.612003 [14-17].

In this paper, we confirm several characterizations based on this paper title. First, we show the geometric characterizations both Bernoulli differential equations [18] of inverse Mills ratios [19, 20] as the hazard functions and survival functions on standard normal distributions. At the same time, we find that Pearson's 0.612003 should be also an important role in their relations. Second, we clarify that variable coefficient second-order linear homogeneous differential equations [21] are related to a standard normal distribution. Especially, Self-adjoint differential equation [22, 40] with a probability density function of a standard normal distribution is shown concisely. Moreover, we illustrate that the value 0.612003 shows the curve which is the integral form of the cumulative distribution has the tangent lines on essential key points.

Furthermore, we also find the similar tendencies from European and Oriental historical cultures about the relations close to theology between circles and squares. Coincidentally, we get a practical approximated value 0.612004 from an equilateral triangle on the geometric characterization between circles, squares and a standard normal distribution instead of Pearson's finding true value 0.612003.

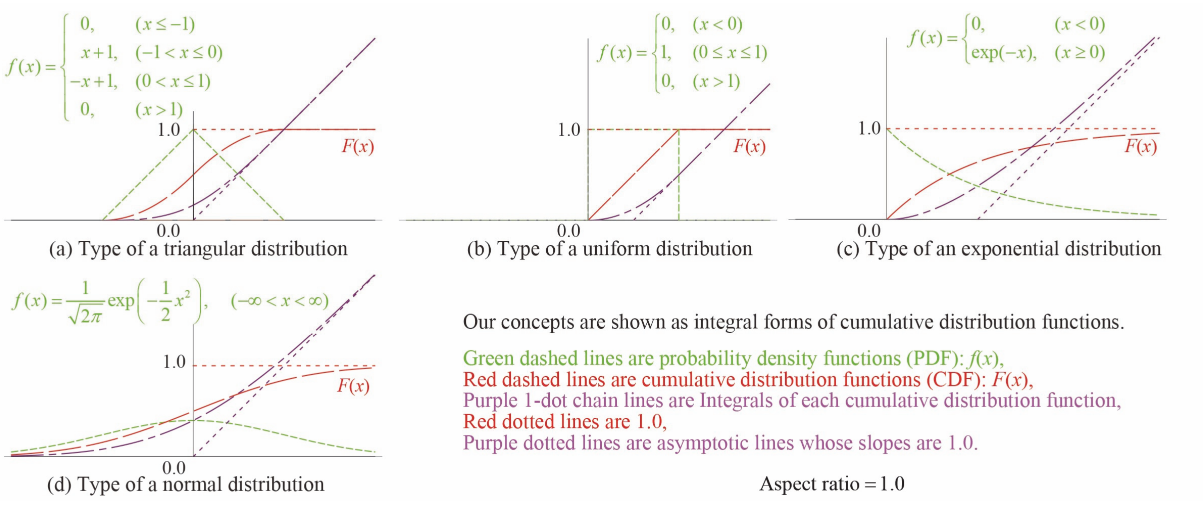

Figure 1 Concepts of integral forms of various kinds of cumulative distribution functions and their probability density functions. (a) Triangular distribution, (b) Uniform distribution, (c) Exponential distribution and (d) Standard normal distribution.

2. Our Concepts of This Study

Prof. Sandoh who was one of two referees of the first author's doctoral thesis [14] advised us to reconfirm the calculations practically about our previous research [13, 14] such as a general integral form [23]. We reconsider that as the following equation

$$E(U|U<-\lambda) = \frac{\int_{-\infty}^{-\lambda} u\phi(u)du}{\int_{-\infty}^{-\lambda} \phi(u)du} = -\frac{\phi(\lambda)}{\Phi(-\lambda)} = -2\lambda \tag{1}$$

as well as using the regression analysis [13, 14]. $\phi(u)$ is a probability density function of a standard normal distribution on its probabilistic point u. $\Phi(-\lambda)$ is a calculated value of a cumulative distribution function from $-\infty$ to $-\lambda$ on u. λ that is about 0.612003 should be an important role of this study on u. $\phi(u)/\Phi(-u)$ is an inverse Mills ratio [19, 20]. Prof. Oya who was the other referee recommended us to investigate the tendencies between squares and normal distributions about our previous research [14]. We think of two precious advices as the concepts such as the integral forms of cumulative distribution functions like an idea by Nagasaka, Tsukamasa and Hashimoto [24] in Figure 1. That is, several purple 1-dot lines are shown in Figure 1. In this study, we clarify that the curve of an integral form of a cumulative distribution function of standard normal distribution has a principal role such as two types of differential equations and passes through several crucial points on their equations in this paper. By the way, Kelley also proposed the 27 percent rule formulation about $\lambda = 0.612003$ as follows

$$\frac{\phi(\lambda)}{\Phi(-\lambda)} = 2\lambda \text{ from } \frac{d\phi(u)}{du} = -u\phi(u) \text{ and } \frac{d\Phi(-u)}{du} = -\phi(u). \tag{2}$$

At this time, we show that the fact, $\phi(\lambda)$ is equal to $2\lambda\Phi(-\lambda)$ from Equations (1) and (2), brings us the equilibrium points in Figure 2. That is to say, a positive return is equal to λ times of its standard deviation as maximal loss [16, 17] and a negative return is equal to that as maximal profit [13-15, 17] under the condition that has the expected value as λ and its standard deviation 1.0 in Figure 2. From Figure 2, we confirm that the square whose length is 2λ is an absolute value of expected value of a truncated normal distribution [25].

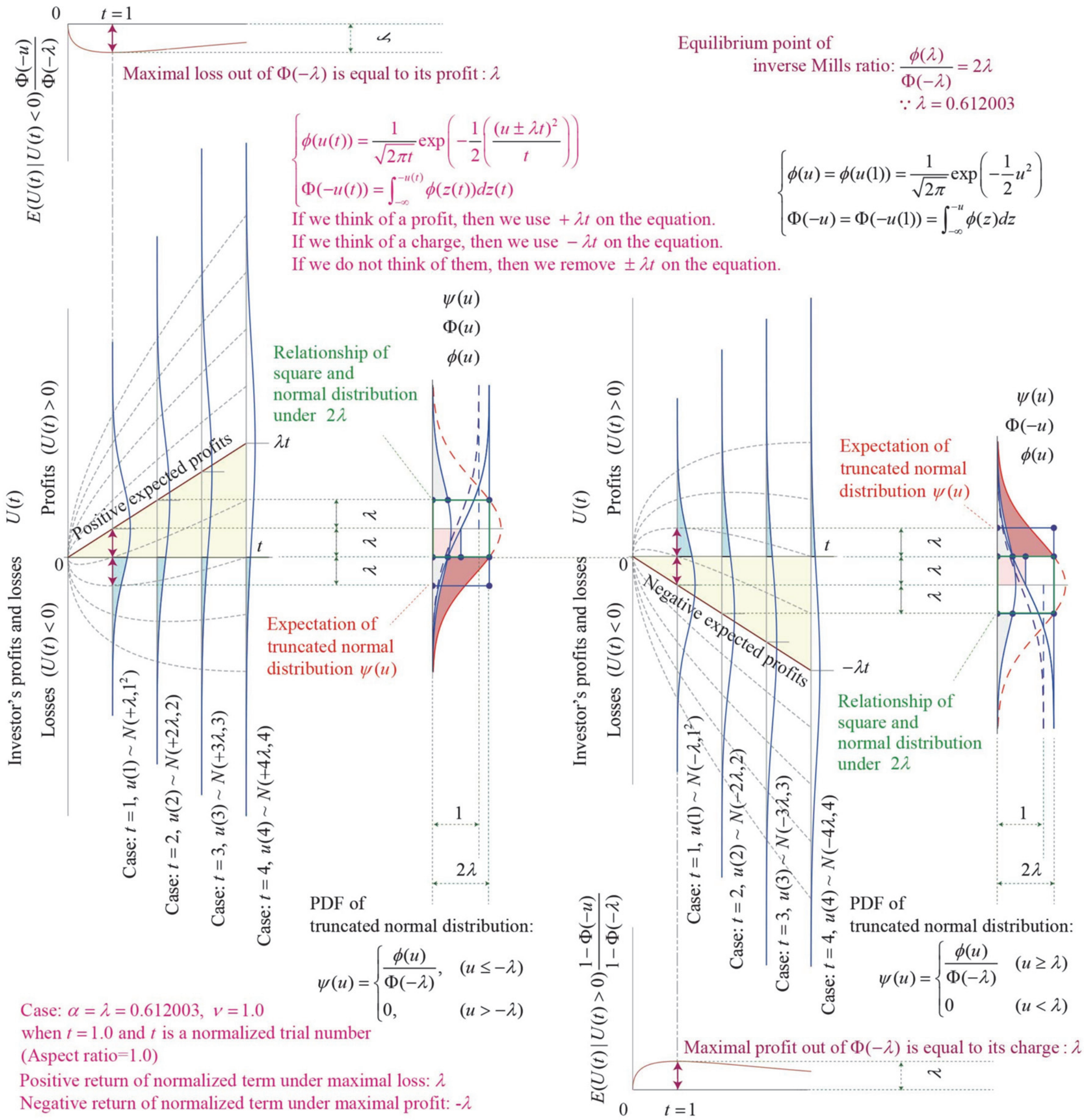

Figure 2 Equilibrium points both positive or negative returns and their risks under the condition on a standard normal distribution ($\lambda = 0.612003$, $t = 1.0$, Original reference [17]).

Especially, our concept about a negative return is similar to the meaningful insists of both Prof. Watanabe's website [26] and Prof. Tanioka's textbook [27] as the fee of gambling. On the other hand, the other about a positive return is similar to the general thinking about that a capital return is greater than its growth return historically such as Piketty's proposal [28].

3. Bernoulli Differential Equations for Inverse Mills Ratio

In section 2, we confirm that the equilibrium points on a standard normal distribution are shown based on $\lambda = 0.612003$. According to Isa's research [20], we rethink of the following formulations as Bernoulli differential equation with a standard normal distribution when we consider the tendencies of an inverse Mills ratio and a hazard function as the special cases on λ. The equation such as Bernoulli differential equation with a standard normal distribution is also mentioned in Prof. Kijima's special lecture [18]. However, since we are interested in the geometric characterizations of that, we propose two types of homogeneous Bernoulli differential equations. That is

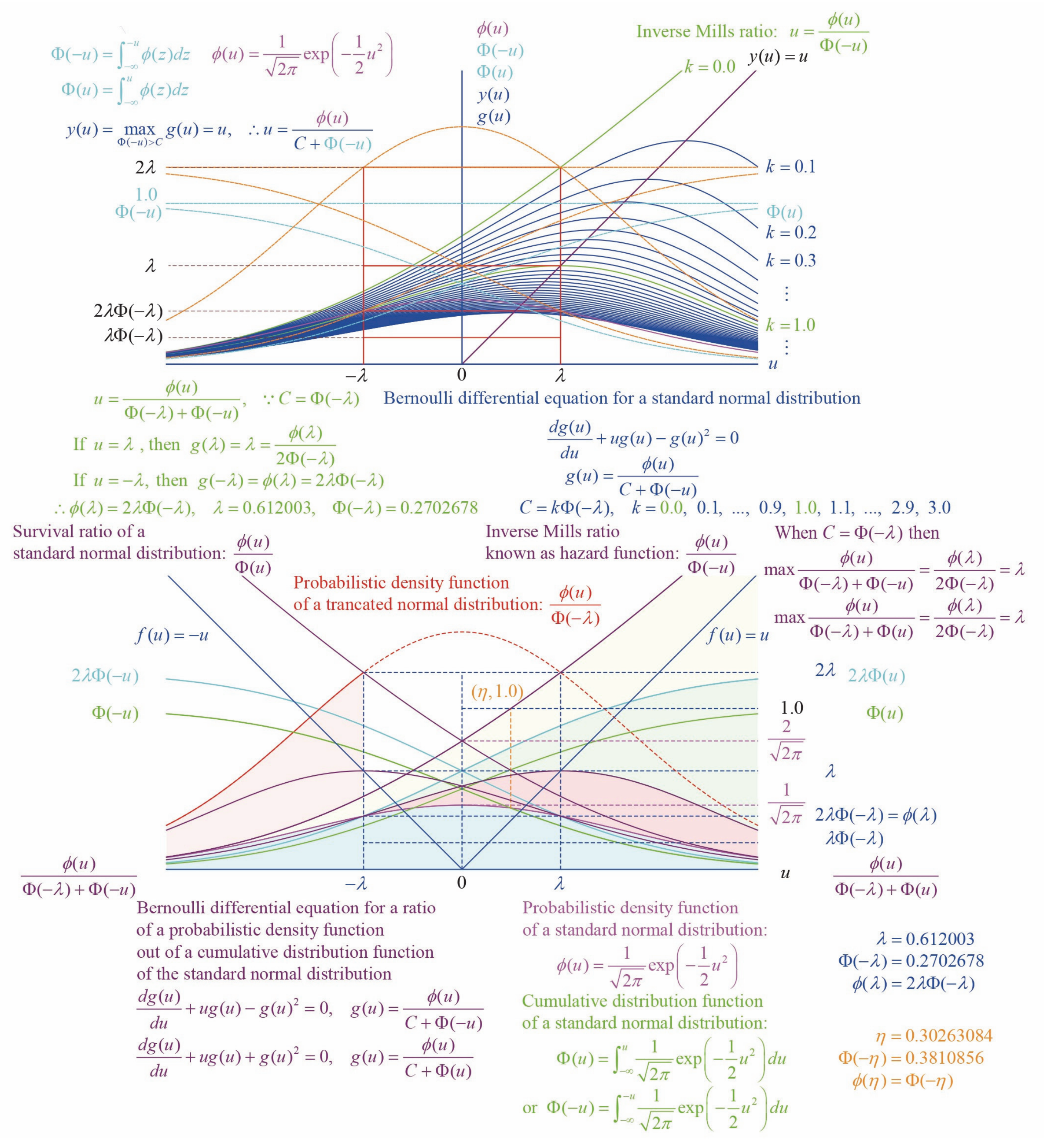

Figure 3 Bernoulli differential equation of the ratios of probability density function of standard normal distribution out of its cumulative distribution function with thinking of its truncated normal distribution on the probabilistic point $\lambda = 0.612003$.

$$\frac{dg(u)}{du} + ug(u) - g(u)^2 = 0, \quad g(u) = \frac{\phi(u)}{C + \Phi(-u)}, \tag{3}$$

$$\frac{dg(u)}{du} + ug(u) + g(u)^2 = 0, \quad g(u) = \frac{\phi(u)}{C + \Phi(u)}. \tag{4}$$

Based on the ideas of asymptotic lines by Gordon [19] and Isa [20], we illustrate the Bernoulli differential equations as the geometric curves in the upper chart of Figure 3 with $\lambda = 0.612003$. At this time, a constant C is changed by k from 0.0 to 3.0 by 0.1 to illustrate those curves clearly. Especially, we confirm that their tendencies are shown as the special types with $\lambda = 0.612003$ when $k = 0.0$ or 1.0. Moreover, 0.30263084 that is also a probabilistic point with the same values of both its cumulative distribution function and probability density function is another important point because the inverse Mills ratio is equal to 1.0. Under the condition of $u = 0$, we find the inverse Mills ratio is $2\phi(0)(= 2/\sqrt{2\pi})$ as 2 times of $\phi(0)$. The lower chart also illustrates the geometric tendencies about Equations (3) and (4) divided by $u = 0$ symmetrically.

4. Self-Adjoint Differential Equations for a Standard Normal Distribution

In section 2, we mentioned our research concepts as the integral forms of a cumulative distribution of a standard normal distribution. We searched for their several characterizations [29, 30] and their differential equations about normal distribution [30, 31]. From our investigations, we propose the variable coefficient second-order linear homogeneous differential equation in Figure 4.

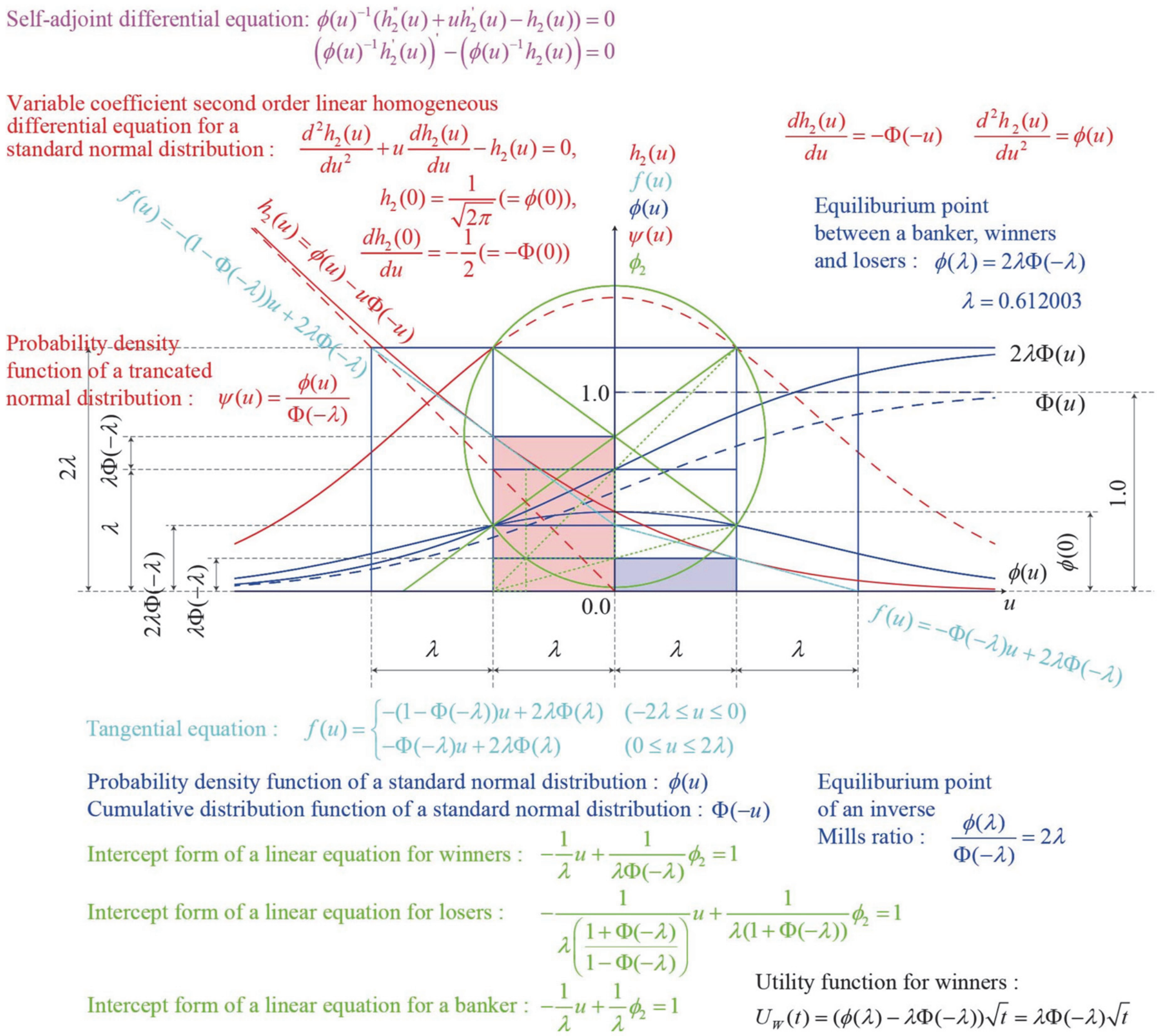

Figure 4 Geometric relationships between Self-adjoint differential equation and standard normal distribution with a circle, squares, meaningful rectangles, their diagonals, and special tangent lines under $\lambda = 0.612003$.

That is

$$\frac{d^2h_2(u)}{du^2} + u\frac{dh_2(u)}{du} - h_2(u) = 0. \tag{5}$$

The way of an expression of $h_2(u)$ in Equation (5) is based on the descriptions of our previous research [14]. The formulations of $h_2(u)$ are shown as the following equation

$$h_2(u) = A\big(\phi(u) - u\Phi(-u)\big) + B\big(\phi(u) + u\Phi(u)\big). \tag{6}$$

If we consider the conditions $dh_2(u)/du = 1/\sqrt{2\pi}$ and $h_2(u) = 1/2$ in Equation (5), we get the answer with $A = 1.0$ and $B = 0.0$ in Equation (6). Therefore, we also confirm that the special curve in Figure 4 passes through several fundamental points with $\lambda\Phi(-\lambda)$ and $\lambda\big(1+\Phi(-\lambda)\big)$ [17]. At the same time, we find their points also have the crucial tangent lines shown in Figure 4. Moreover, we illustrate a true circle with a square and a standard normal distribution in Figure 4 [17]. The center of this circle is an intersection of meaningful rectangle diagonals. Without passing through the intersection, we find the other important line to distinguish into the separated squares by the proportion $\Phi(-\lambda): 1 - \Phi(-\lambda)$.

We reconfirm the geometric characterizations about $\lambda = 0.612003$ with considering a square, a circle and a normal distribution. And their tendencies have a special differential equation as the following Self-adjoint differential equation [22]

$$\big(\phi(u)^{-1}h_2'(u)\big)' - \big(\phi(u)^{-1}h_2(u)\big) = 0. \tag{7}$$

At this time, to investigate the tendencies of the combinations between A and B in Equation (6), A and B are changed by the values from 0.0 to 1.0 by 0.25 respectively in Figure 5.

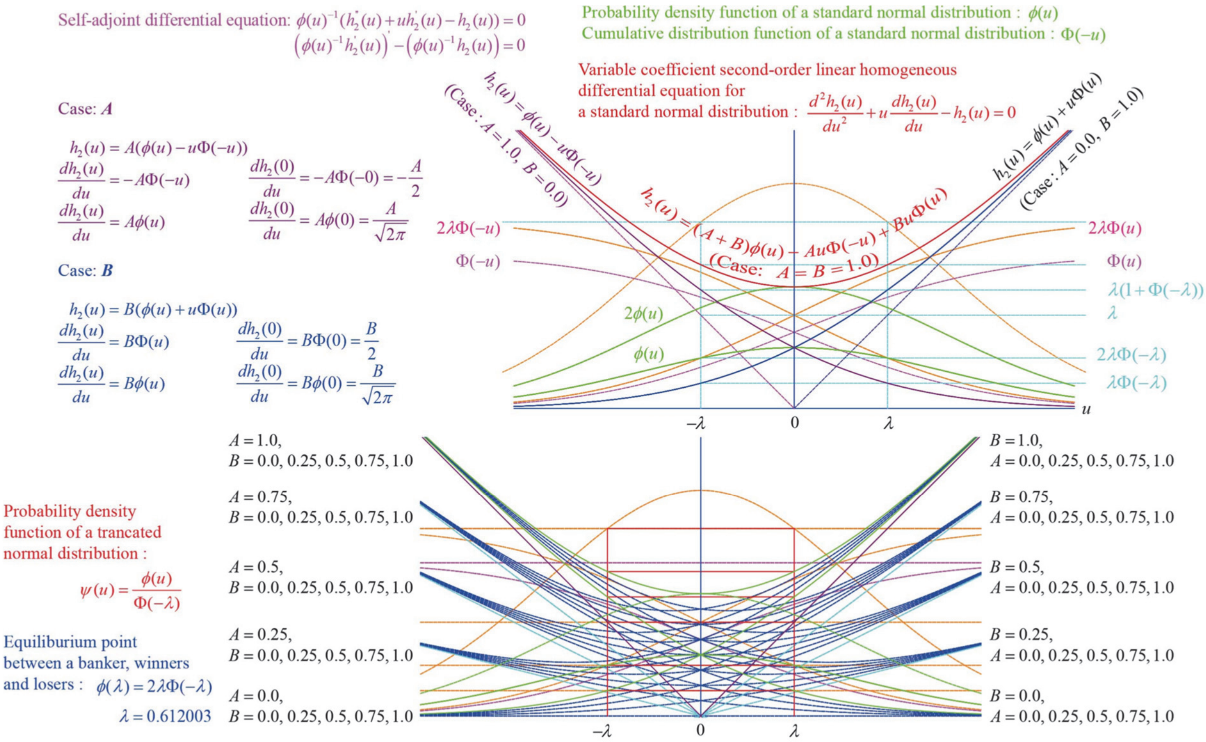

Figure 5 Various geometric characterizations of Strum-Liouville differential equations with standard normal distribution.

5. Square and Circle on Standard Normal Distribution and Approximated Value about Polygons

Although we confirm that there is a relationship between a circle, a square and a normal distribution, we cannot get a correct value λ as well as drawing an equilateral triangle and a regular hexagon from their characterizations which are similar to the problem of squaring the circle [32, 33]. Instead of that, we can get the practical approximated Pearson's finding value. That is

$$\lambda \cong \frac{1+\sqrt{3}}{1+2\sqrt{3}} = 0.612004. \tag{8}$$

This approximated value is estimated by drawing equilateral triangles and regular hexagons shown in Figure 6.

By the way, there are a lot of thinking about circles and squares for a long time all over the world religiously and historically [32-39]. We also show one of their tendencies with a standard normal distribution based on $\lambda = 0.612003$. Coincidentally, since its value is close to an inverse of golden ratio, we illustrate an imitated figure like one of da Vinci's works "Vitruvian Man" [34.35] and based on "Mandalas" [36, 37]. According to Prof. Ida's opinion, it is said that the proportion of that by da Vinci should be less than the inverse of golden ratio ($\cong 0.618$) slightly.

From oriental areas to western areas, there are many theological cultures about circles and squares. And from Egyptian civilization to Greek era and Roman era, the normal distribution had not been known before De Moivre (1733), Laplace (1812), Legendre (1805), and Gauss (1809) studied it [38,39]. At that time of Renaissance era, it is said that da Vinci was interested in the golden ratio, the Fibonacci number, and theology about squares and circles.

Originally, mathematics and statistics have been developed in search of scientific evidences for the development of business, politics, sociology, life science, technology and any other fields for happiness of human beings. In order to pursue that, we propose their geometric characterizations of the standard normal distribution in this paper.

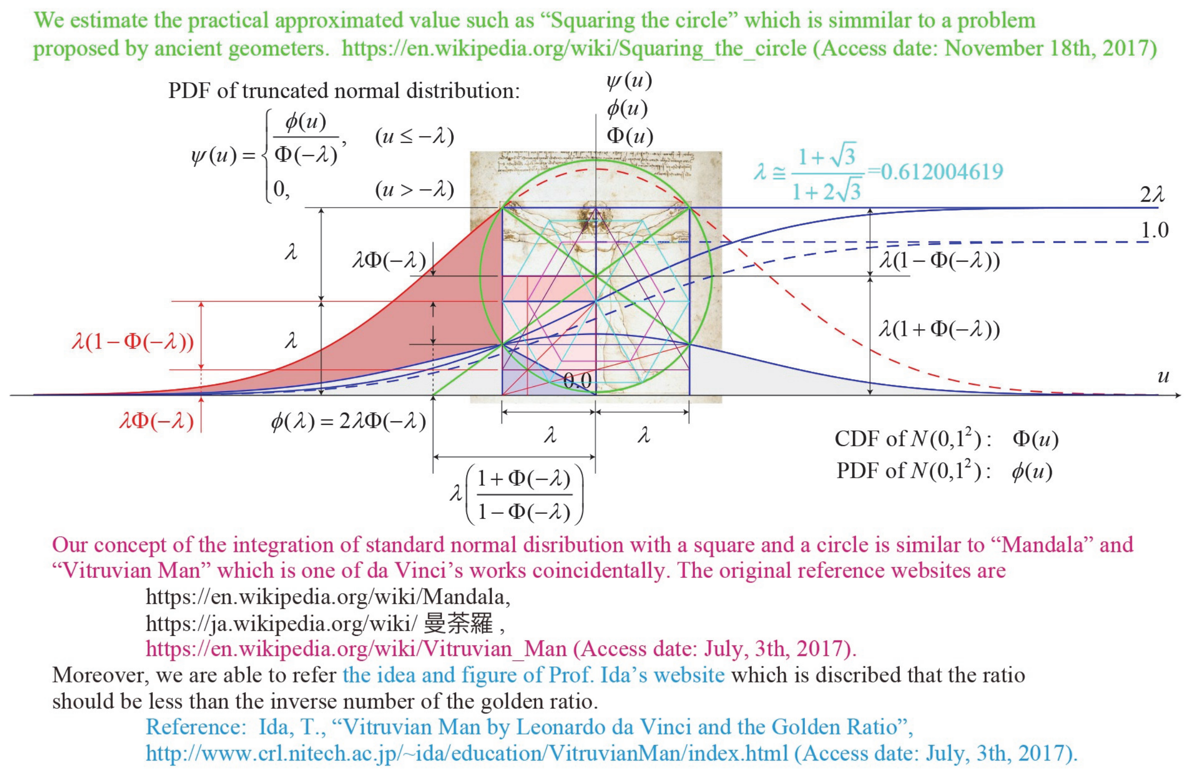

Figure 6 Approximated value of $\lambda = 0.612003$ on the standard normal distribution which associates with the equilateral triangles and hexagons based on the squaring the circle with da Vinci's work.

6. Conclusions

In this paper, we deal with the geometric characterizations about a standard normal distribution.

First, we clarify that the inverse Mills ratio shown by Pearson's finding value 0.612003 is formulated as Bernoulli differential equations. Second, we confirm that the integral form of a cumulative distribution function of standard normal distribution is expressed as Self-adjoint differential equation. Third, although we admit we cannot draw a equilateral triangle correctly, we estimate the approximated value 0.612004 and illustrate the similar chart which is our mathematical concept about da Vinci's art "Vitruvian Man" and "Mandalas." Finally, we realize that the true height of densities of a standard normal distribution is much smaller than we thought of that shown in many textbooks under the aspect ratio is 1.0 throughout this study.

Acknowledgments

本研究の発表に至るまでに， [13]の覆面論文査読者 2 名の先生，三道弘明 先生，大屋幸輔 先生，渡辺隆裕 先生，田畑吉雄 先生，寺岡義伸 先生，仲川勇二 先生，木島正明 先生，穴太克則 先生，黒沢健 先生，長塚豪己 先生，堀口正之 先生，西原理 先生，蓮池隆 先生をはじめ，OR 学会の先生方から多くの助言や知見を頂きました．ここに謝意を表します．また，第一著者の中西真悟は若い頃の恩師 中易秀敏 先生，栗山仙之助 先生と，リスクと確率論について手解きしてくれた若い頃の上司 亀島紘二 先生，志垣一郎 先生に感謝します．加えて特に，第一著者が[13]に取組んだ時から本論文作成までに，何度も繰り返しじゃんけんの勝敗の行方を楽しみ，一緒に寺社を巡り，傍で社会科に興味を持ちながら，自力で算数の問題を解くために一生懸命に補助線を描いていた第一著者の愛娘に感謝します．最後に，本研究には大阪工業大学研究奨励金による経済的支援を受けたことを記します．

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