リスクとリターンが語る円・正方形・直角三角形と標準正規分布の対称的な幾何学的特性

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日本証券アナリスト協会関西地区交流会の皆様，大阪大学と大阪工業大学の教職員の皆様に厚く御礼を述べます．
また，若い頃の恩師 中易秀敏 先生，栗山仙之助 先生と，リスクと確率論について手解きしてくれた
若い頃の上司 亀島鉱二 先生，志垣一郎 先生に感謝いたします．

掲示図の出典：　日本オペレーションズ・リサーチ学会論文と研究発表会資料 (ORSJ),
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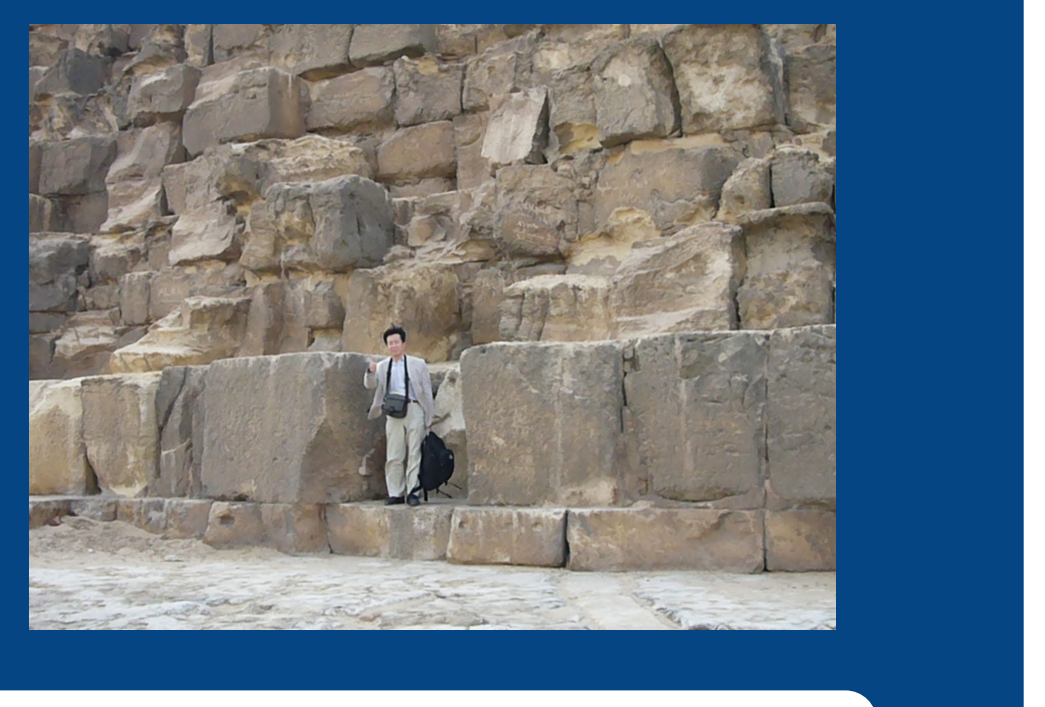
URL:　http://www.oit.ac.jp/center/~nakanishi/
2019 年 9 月 13 日に
大阪工業大学梅田キャンパスで開催の
イノベーションデイズにおいて
展示した作品をベースに作成
Version 2

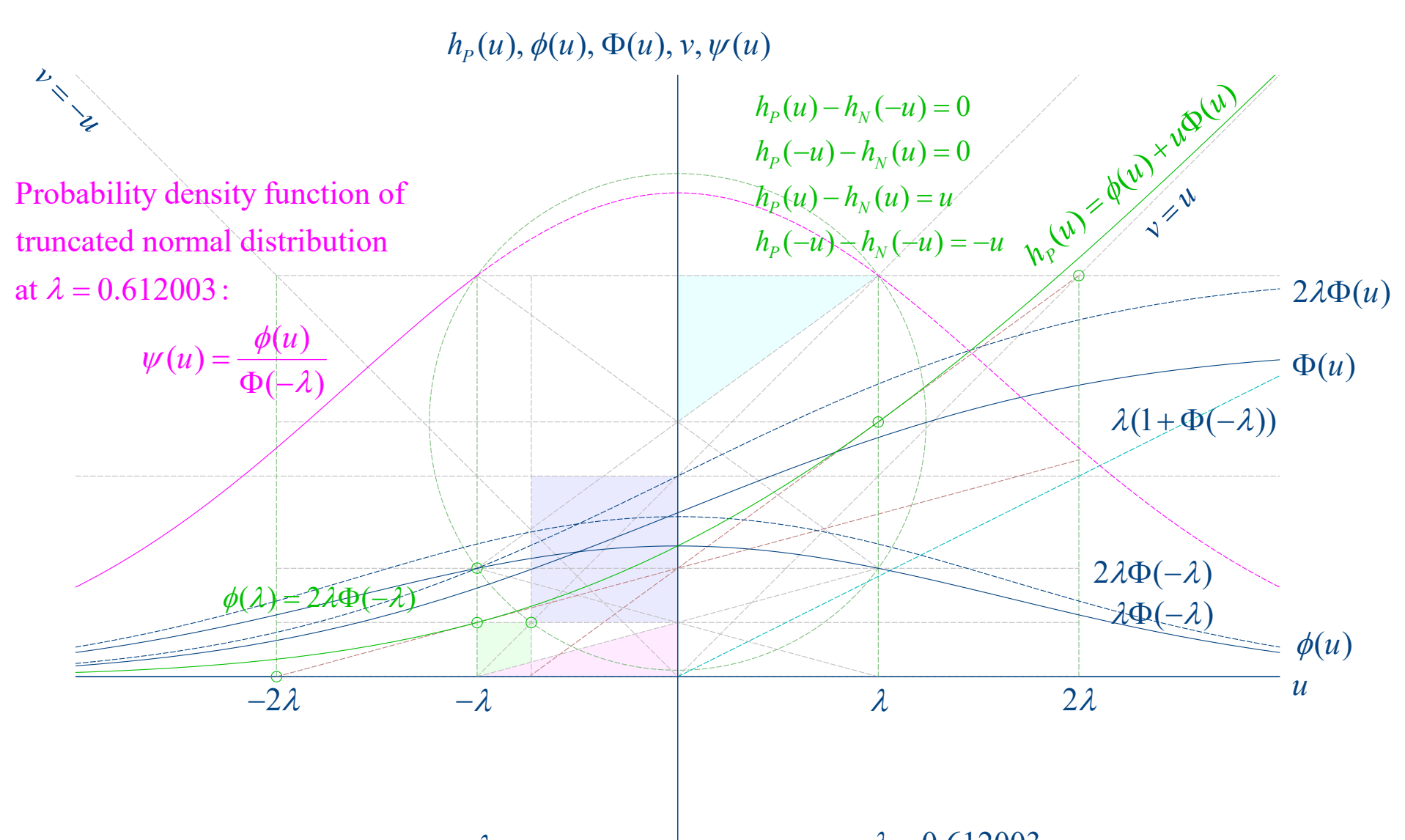


勝者, 敗者, 胴元の均衡点:

$$\phi(\lambda) = 2\lambda\Phi(-\lambda)$$

$$\lambda = 0.612003$$

標準正規分布に二階線形微分方程式とベルヌーイ型微分方程式を用いた．古代エジプトの絵画図の技法のように奥行きを取り除いて回転させて重ね合わせてみると，ピタゴラスの定理とパラメトリック方程式による円，正方形，直角三角形の美しい幾何学模様が描ける．



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＆ 数理・データ科学教育研究センター）

直角三角形による接線の傾きは，累積分布関数の確率で，標準正規分布と逆ミルズ比の切片系の方程式である．

勝者の切片系の方程式：

$$-\frac{1}{\frac{\phi(k)}{\Phi(-k)}}+\frac{1}{\phi(k)}=1$$

敗者の切片系の方程式：

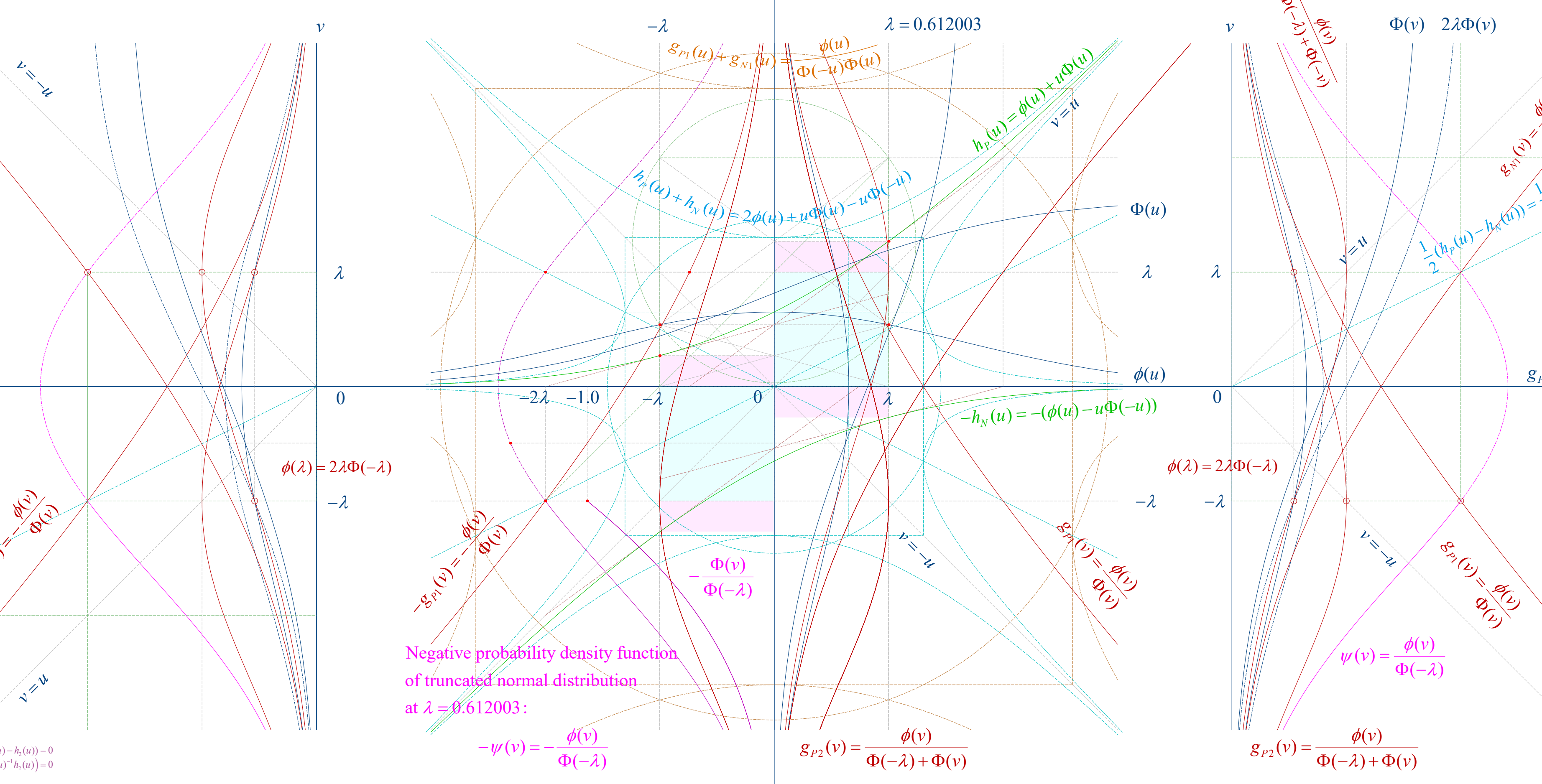
$$-\frac{1}{\frac{\phi(k)}{\Phi(k)}}+\frac{1}{\phi(k)}=1$$

胴元の切片系の方程式：

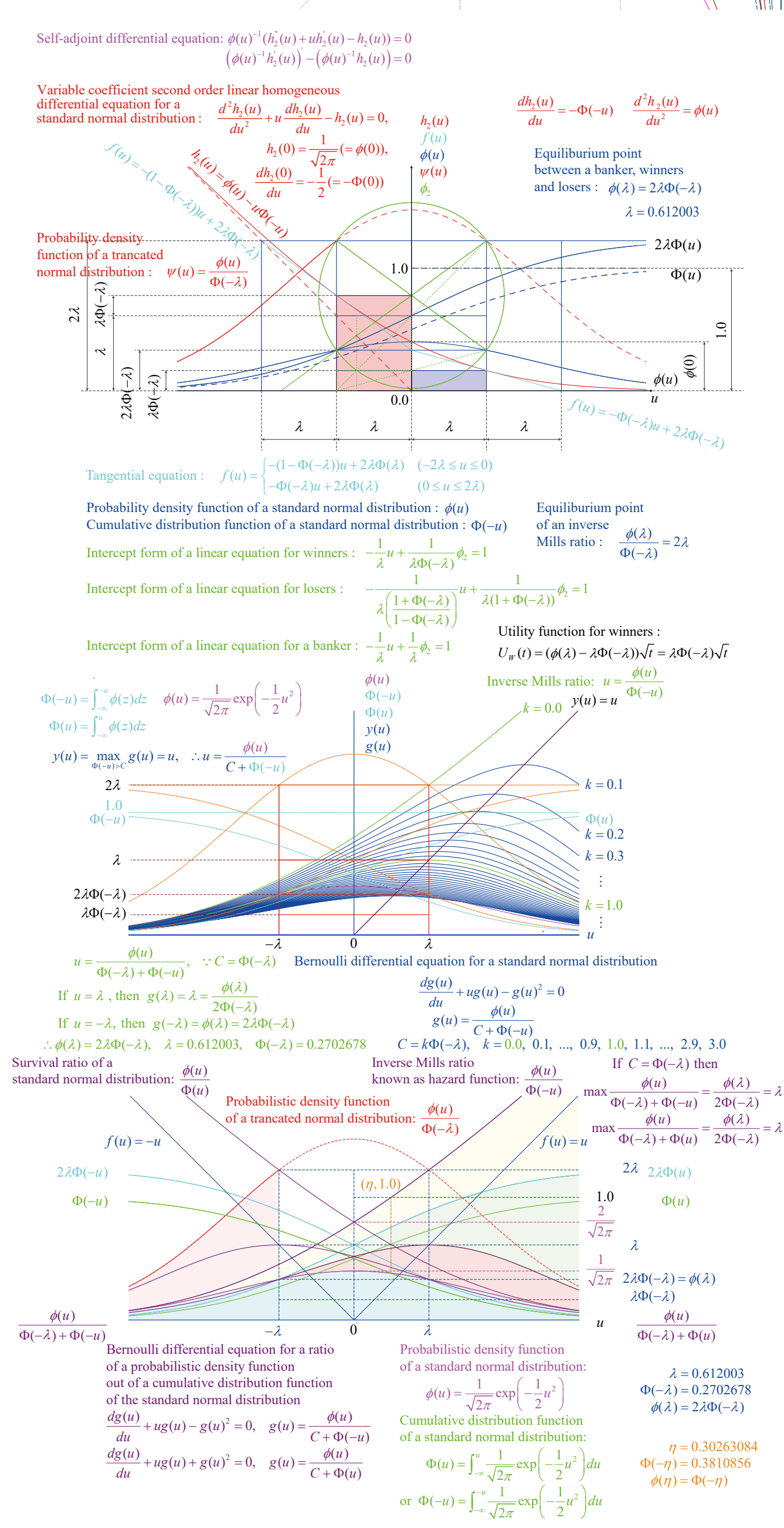
$$-\frac{1}{\frac{\phi(k)}{\Phi(-k)+\Phi(k)}}+\frac{1}{\phi(k)}=1$$

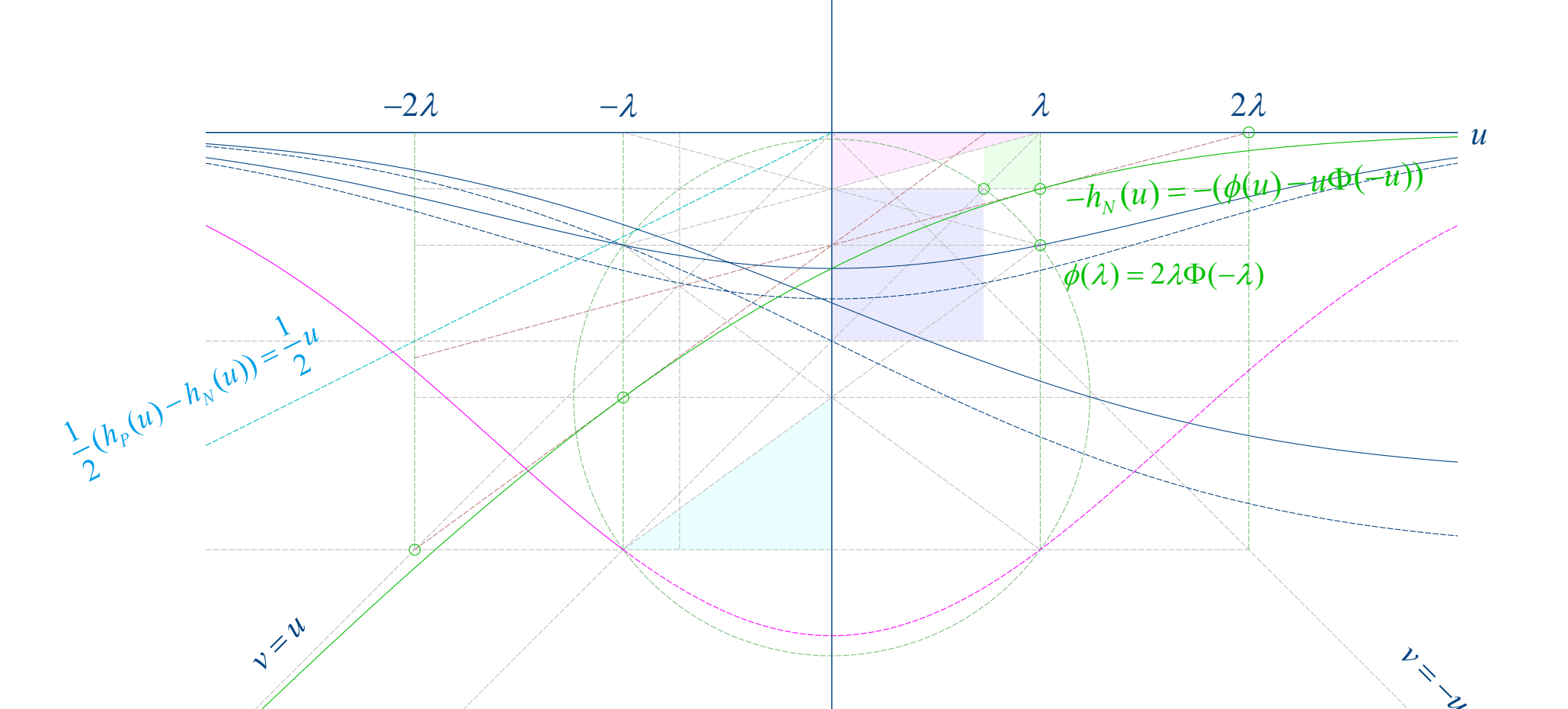
緑色は
標準正規分布
による二階線
形微分方程式
に関連する曲線

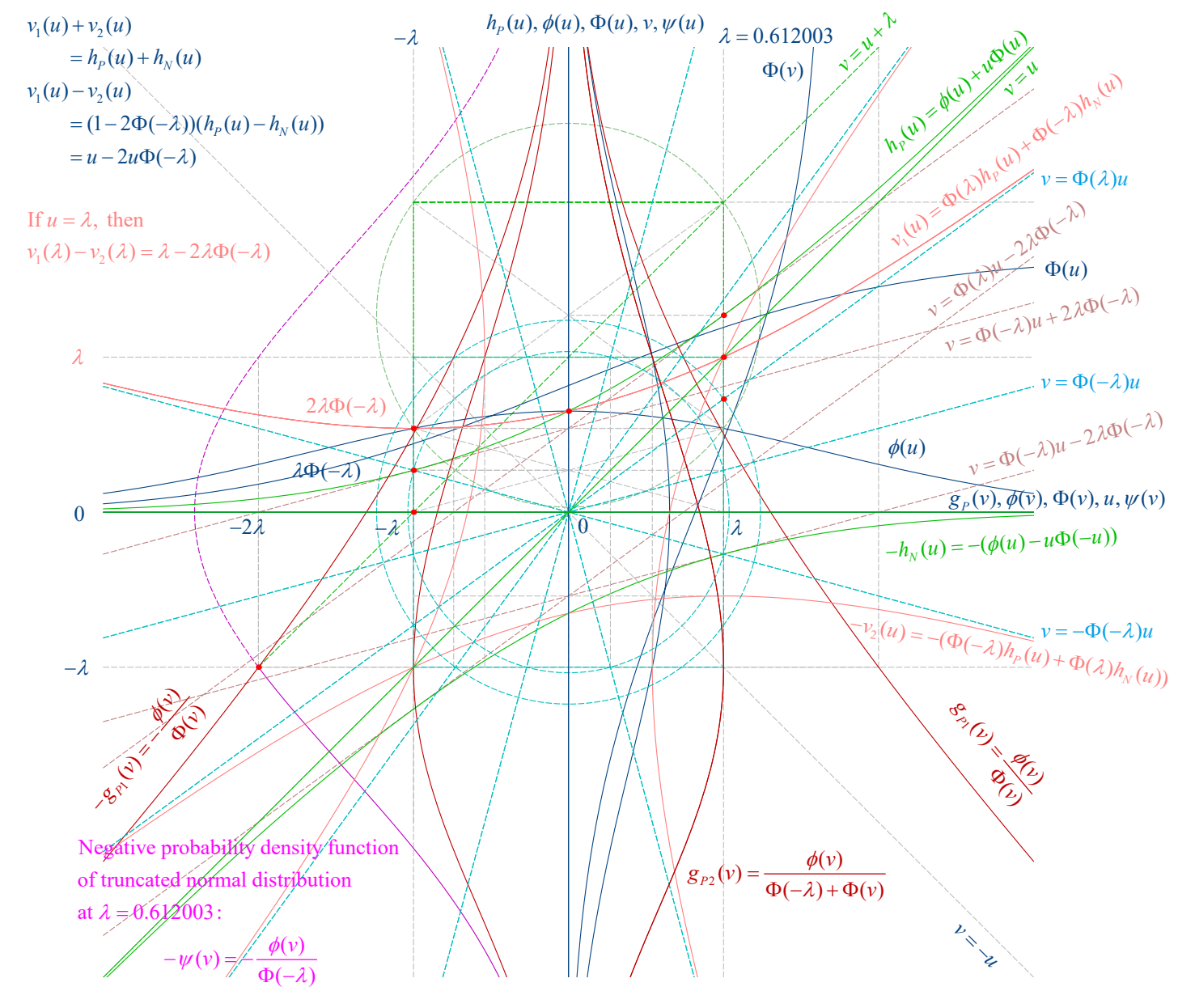
赤色は
標準正規分布
（逆ミルズ比）に
よるベルヌーイ
型微分方程式
に関連する曲
線



水色は
標準正規分布による
二階線形微分方程式を
用いたパラメトリック
方程式に関連する曲線





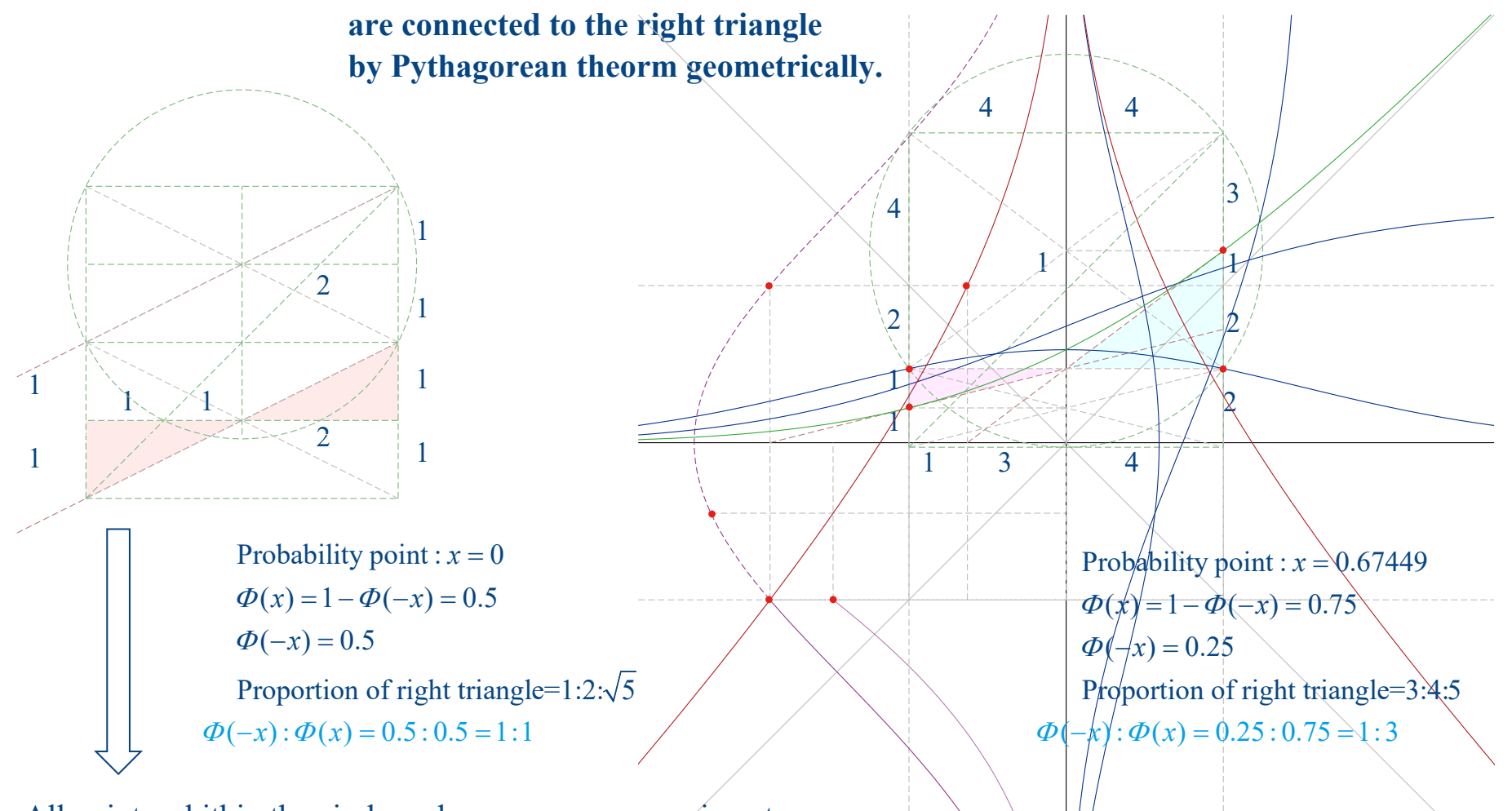


大阪の街から見える生駒山と六甲山の山の高さは
どことなく縦横比が同じときの標準正規分布の高さに似ている．
同時に，歴史を通じた文化を感じながら曼荼羅のような幾何学模様を描ける．

PowerPoint Visual Animation

MP4 Visual Animation

Probabilities of standard normal distribution are connected to the right triangle by Pythagorean theorm geometrically.



Probability point : $x=0$
$\Phi(x)=1-\Phi(-x)=0.5$
$\Phi(-x)=0.5$
Proportion of right triangle=1:2:$\sqrt{5}$
$\Phi(-x):\Phi(x)=0.5:0.5=1:1$

Probability point : $x=0.67449$
$\Phi(x)=1-\Phi(-x)=0.75$
$\Phi(-x)=0.25$
Proportion of right triangle=3:4:5
$\Phi(-x):\Phi(x)=0.25:0.75=1:3$

All points whithin the circle and square are converging at one point practically because their pobability points are 0.

From 大阪工業大学大宮キャンパス

南東側：生駒山

北西側：六甲山

注 1：左側の白枠中は，大阪工業大学イノベーションデイズでの掲載作品です．一部は OR 学会 2019 年秋季研究発表会や 2019 年王立統計学会国際会議でも発表しています．
右側は 2019 年の王立統計学会主催の国際会議ポスタープレゼンテーションで発表した掲載作品です．円や曲線を含むポスターのデザインは大阪の夜空に打ち上がる「花火の色彩と背景」を表現しています．

注 2：ポスターの背景のデザインの色合いは本学のスクールカラー「紺青，大阪工大ブルー」，コミュニケーションマークのカラー「シアン」，
淀川や大阪湾も含めて世界へ発信できる「水の都　大阪」をイメージして作成しています．

Geometric Characterizations and Symmetric Relations between Standard Normal Distribution and Inverse Mills Ratio based on Pythagorean Theorem

Shingo NAKANISHI at Osaka Institute of Technology, Japan,
Masamitsu OHNISHI at Osaka University, Japan.

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Original References: Operations Research Society of Japan (ORSJ),
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The 15th International Symposium on Econometric Theory and Applications (SETA2019).

URL: http://www.oit.ac.jp/center/~nakanishi/english/

0. Background

We present the geometric characterizations and symmetric relations between standard normal distribution and inverse Mills ratio by circles and squares from the viewpoint with considering the height of densities such as ancient Egyptian drawing styles and using the Greek Pythagorean Theorem.

First, we can clarify the integral forms of various cumulative distribution functions including standard normal distribution based on the aspect ratio (=1.0, see 1).

Second, we reconsider what the several times of standard deviations multiplied by the square root of the time mean with their positive and negative expectations multiplied by the time under the condition the aspect ratio=1.0 (see 2). At this time, we can understand the following things.

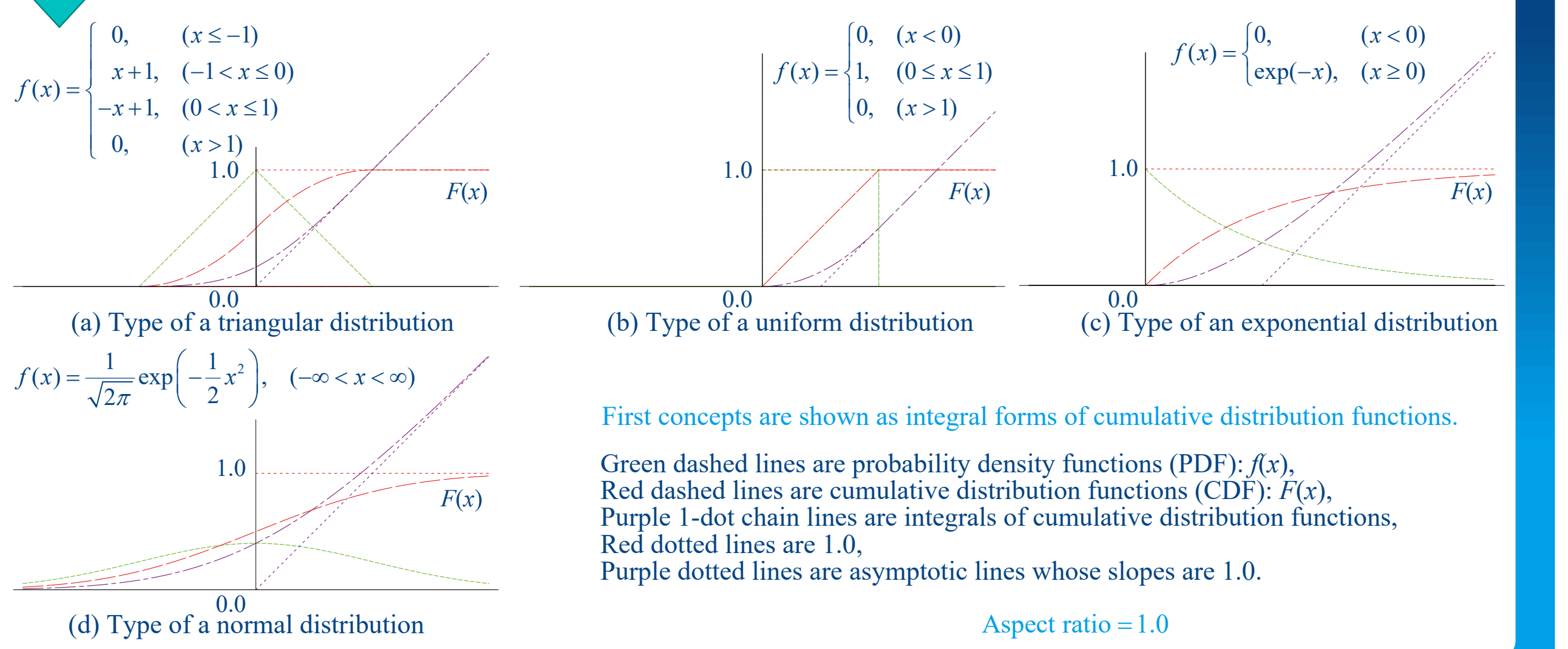
(1) We can find the equilibrium formulation at any real numbers of the time t (see 2).
(2) Its constant number, 0.612003, was found by Karl Pearson about 100 years ago.
(3) Sir David Roxbee Cox reconfirmed the value to cluster the normal distribution.
(4) Truman Lee Kelley also proposed the 27 percent rule formulation.

Third, we can show the parabola for maximal profits including fees. And that is equal to its fee based on the 27 percent probability. In addition to these tendencies, we can clarify the relations between winners, losers, and their banker (see 3).

Fourth, we can understand two types of ordinary differential equations to explain that with circles and squares (see 4). One is the Bernoulli differential equation of inverse Mills ratio. The other is the second order linear differential equation of the integral of cumulative normal distribution function (see 1). From these tendencies, we can also get the modified intercept forms geometrically and symmetrically for maximal profits of winners, these losses of losers, and their banker's fee. We can understand that these equations should be changing the probability points and these probabilities based on Pythagorean Theorem correctly. If the probability point is that found by Karl Pearson, we can show you that it is the special point of standard normal distribution such as Sir Cox's proposal. Other point on the right triangle with 3:4:5 is also the third quantile of standard normal distribution.

Finally, we can also realize there are many similar tendencies close to the relations between circles and squares such as Vitruvian man by Da Vinci and various Mandalas although there might not be related to normal distribution directly and historically. The ancient Egyptian drawing styles enable us to illustrate the geometric characterizations and symmetric relations between standard normal distribution and inverse Mills ratio with circle and square based on the Pythagorean theorem in the ancient Greece. We think that our ideas shall be contributed in the statistical modelling and these evaluation fields since our suggested figures should be much more easily understood and powerful than we thought.

1. First concept as integrals of various cumulative distributions



(a) Type of a triangular distribution　(b) Type of a uniform distribution　(c) Type of an exponential distribution

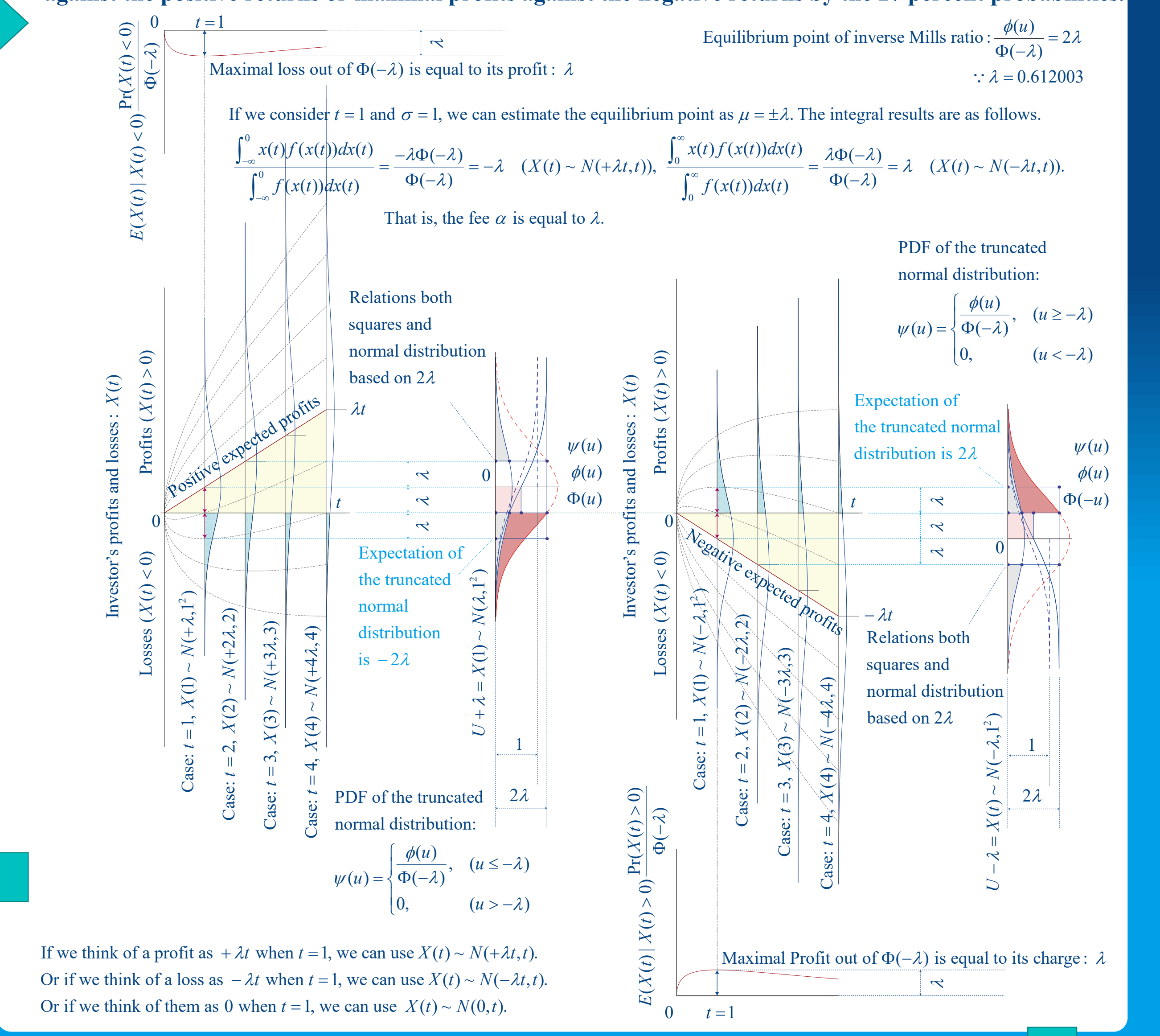
(d) Type of a normal distribution

First concepts are shown as integral forms of cumulative distribution functions.

Green dashed lines are probability density functions (PDF): $f(x)$,
Red dashed lines are cumulative distribution functions (CDF): $F(x)$,
Purple 1-dot chain lines are integrals of cumulative distribution functions,
Red dotted lines are 1.0,
Purple dotted lines are asymptotic lines whose slopes are 1.0.

Aspect ratio = 1.0

2. Second concept as distances of probability points based on the time t. There should be maximal losses against the positive returns or maximal profits against the negative returns by the 27 percent probabilities.



Equilibrium point of inverse Mills ratio : $\frac{\phi(u)}{\Phi(-\lambda)}=2\lambda$
$\because \lambda=0.612003$

If we consider $t=1$ and $\sigma=1$, we can estimate the equilibrium point as $\mu=\pm\lambda$. The integral results are as follows.

$$\frac{\int_{-\infty}^{0}x(t)f(x(t))dx(t)}{\int_{-\infty}^{0}f(x(t))dx(t)}=\frac{-\lambda\Phi(-\lambda)}{\Phi(-\lambda)}=-\lambda\quad(X(t)\sim N(+\lambda t,t)),\quad \frac{\int_{0}^{\infty}x(t)f(x(t))dx(t)}{\int_{0}^{\infty}f(x(t))dx(t)}=\frac{\lambda\Phi(-\lambda)}{\Phi(-\lambda)}=\lambda\quad(X(t)\sim N(-\lambda t,t)).$$

That is, the fee α is equal to λ.

PDF of the truncated normal distribution:

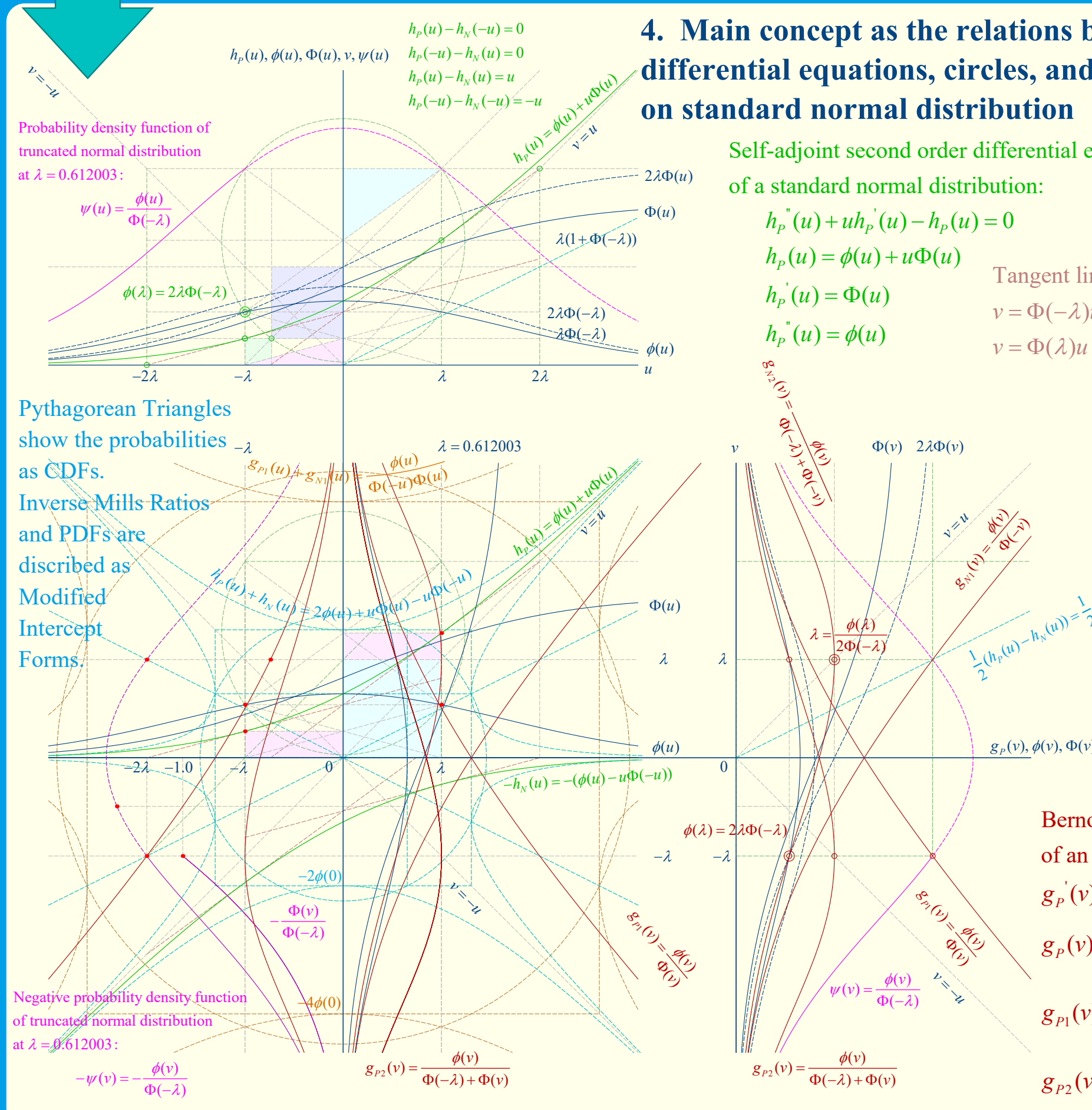
$$\psi(u)=\begin{cases}\frac{\phi(u)}{\Phi(-\lambda)}, & (u\le -\lambda)\\ 0, & (u>-\lambda)\end{cases}$$

PDF of the truncated normal distribution:

$$\psi(u)=\begin{cases}\frac{\phi(u)}{\Phi(-\lambda)}, & (u\ge -\lambda)\\ 0, & (u<-\lambda)\end{cases}$$

If we think of a profit as $+\lambda t$ when $t=1$, we can use $X(t)\sim N(+\lambda t,t)$.
Or if we think of a loss as $-\lambda t$ when $t=1$, we can use $X(t)\sim N(-\lambda t,t)$.
Or if we think of them as 0 when $t=1$, we can use $X(t)\sim N(0,t)$.

4. Main concept as the relations between Pythagorean theorem, differential equations, circles, and squares on standard normal distribution



Pythagorean Triangles show the probabilities as CDFs. Inverse Mills Ratios and PDFs are discribed as Modified Intercept Forms.

Self-adjoint second order differential equation of a standard normal distribution:

$$h_P''(u)+uh_P'(u)-h_P(u)=0$$
$$h_P(u)=\phi(u)+u\Phi(u)$$
$$h_P'(u)=\Phi(u)$$
$$h_P''(u)=\phi(u)$$

Equilibrium point between a banker, winners and losers : $\phi(\lambda)=2\lambda\Phi(-\lambda)$
$\lambda=0.612003$

Tangent line of $h_P(u)$

$$v=\Phi(-\lambda)u+\phi(\lambda)=\Phi(-\lambda)u+2\lambda\Phi(\lambda)\quad(u=-\lambda)$$
$$v=\Phi(\lambda)u+\phi(\lambda)=\Phi(\lambda)u+2\lambda\Phi(\lambda)\quad(u=\lambda)$$

Modified intercept form of linear equation for winners :

$$-\frac{1}{\frac{\phi(\lambda)}{\Phi(-\lambda)}}u+\frac{1}{\phi(\lambda)}v=1$$

Modified intercept form of linear equation for losers :

$$-\frac{1}{\frac{\phi(\lambda)}{\Phi(\lambda)}}u+\frac{1}{\phi(\lambda)}v=1$$

Modified intercept form of linear equation for their banker :

$$-\frac{1}{\frac{\phi(\lambda)}{\Phi(-\lambda)+\Phi(\lambda)}}u+\frac{1}{\phi(\lambda)}v=1$$

Intercept form of linear equation for winners :

$$-\frac{1}{\lambda}u+\frac{1}{\lambda\Phi(-\lambda)}v=1$$

Intercept form of linear equation for losers :

$$-\frac{1}{\lambda\left(\frac{1+\Phi(-\lambda)}{1-\Phi(-\lambda)}\right)}u+\frac{1}{\lambda(1+\Phi(-\lambda))}v=1$$

Intercept form of linear equation for their banker :

$$-\frac{1}{\lambda}u+\frac{1}{\lambda}v=1$$

Bernouilli differential equation of an inverse Mills ratio :

$$g_P'(v)+ug_P(v)+g_P(v)^2=0$$
$$g_P(v)=\frac{\phi(v)}{\Phi(v)+C}$$
$$g_{P1}(v)=\frac{\phi(v)}{\Phi(v)},\quad(C=0,\ \text{inverse Mills ratio})$$
$$g_{P2}(v)=\frac{\phi(v)}{\Phi(-\lambda)+\Phi(v)},\quad(C=\Phi(-\lambda))$$

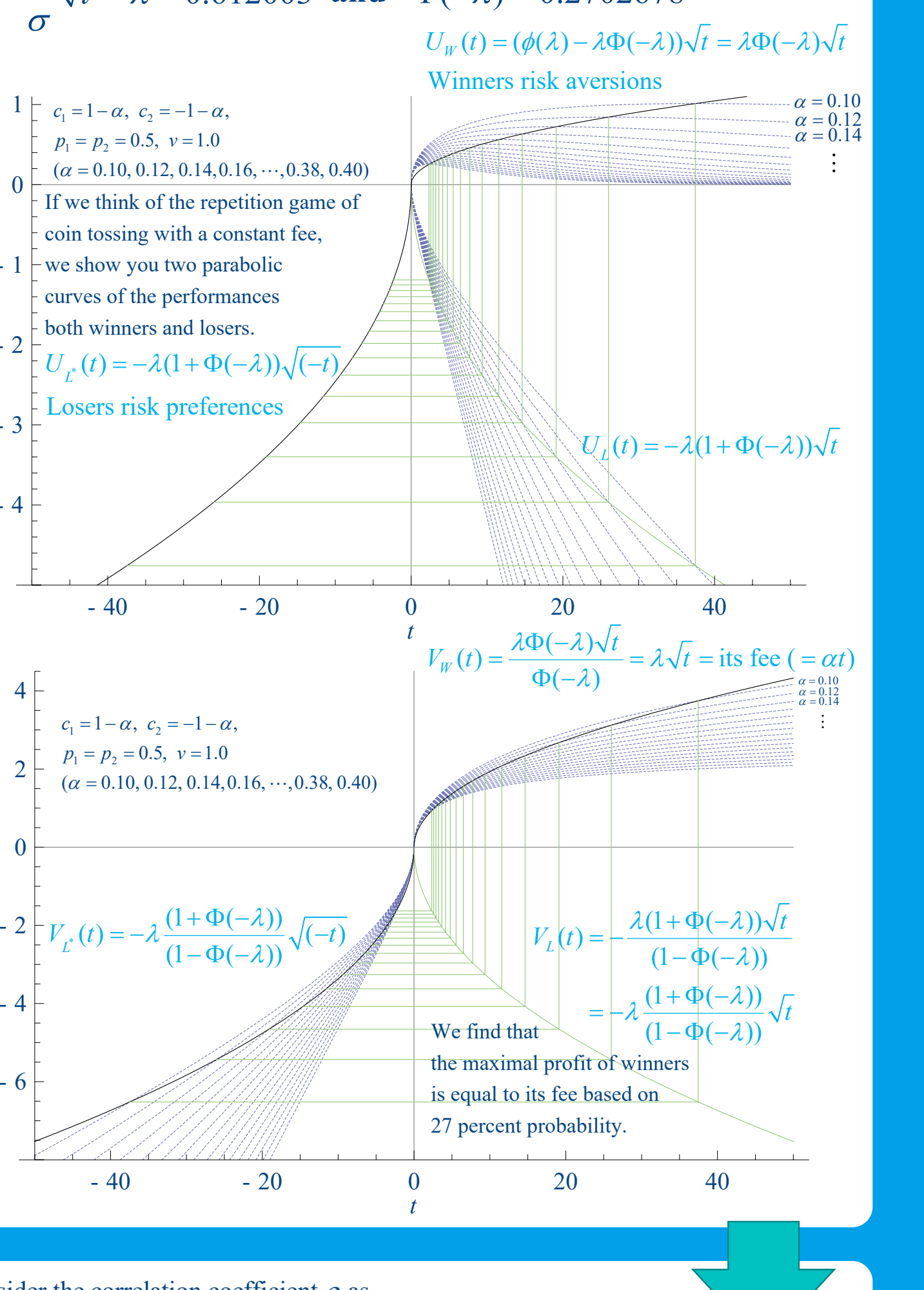
$$Z_W(\alpha,\sigma,t)=\frac{E(Y_W(\alpha,\sigma,t))}{\Pr(X(\alpha,\sigma,t)>0)}=\frac{\int_{-\infty}^{\infty}y_W(\alpha,\sigma,t)\cdot\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(y_W(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dy_W(\alpha,\sigma,t)}{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(y_W(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dy_W(\alpha,\sigma,t)}=\frac{\int_{0}^{\infty}x(\alpha,\sigma,t)\cdot\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(x(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dx(\alpha,\sigma,t)}{\int_{0}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(x(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dx(\alpha,\sigma,t)}$$

$$Z_L(\alpha,\sigma,t)=\frac{E(Y_L(\alpha,\sigma,t))}{\Pr(X(\alpha,\sigma,t)<0)}=\frac{\int_{-\infty}^{\infty}y_L(\alpha,\sigma,t)\cdot\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(y_L(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dy_L(\alpha,\sigma,t)}{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(y_L(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dy_L(\alpha,\sigma,t)}=\frac{\int_{-\infty}^{0}x(\alpha,\sigma,t)\cdot\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(x(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dx(\alpha,\sigma,t)}{\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\sigma^2t}}\exp\left(-\frac{1}{2}\frac{(x(\alpha,\sigma,t)+\alpha t)^2}{\sigma^2t}\right)dx(\alpha,\sigma,t)}$$

$U_W(\alpha,\sigma,t^*)=\max_{0\le t\le T}E(Y_W(\alpha,\sigma,t))$
$U_L(\alpha,\sigma,t^*)=E(Y_L(\alpha,\sigma,t^*))$
$\because Y_W(\alpha,\sigma,t)=\max[X(\alpha,\sigma,t),0]$
$\because Y_L(\alpha,\sigma,t)=\min[X(\alpha,\sigma,t),0]$

3. Relations between winners, losers, and their banker by the equilibrium formulation

$$\frac{\alpha}{\sigma}\sqrt{t}=\lambda=0.612003\ \ \textbf{and}\ \ \Phi(-\lambda)=0.2702678$$



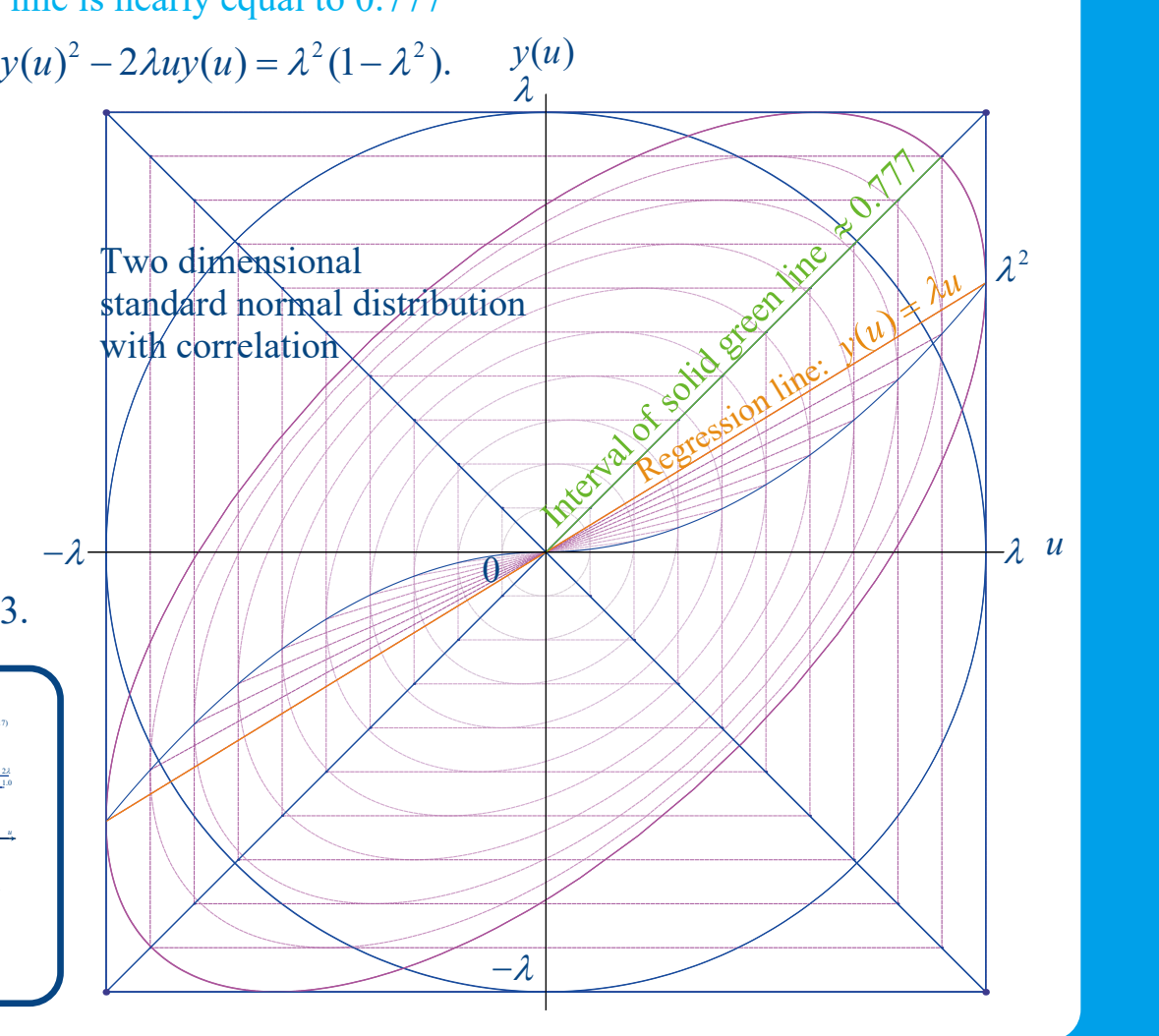
If we think of the repetition game of coin tossing with a constant fee, we show you two parabolic curves of the performances both winners and losers.

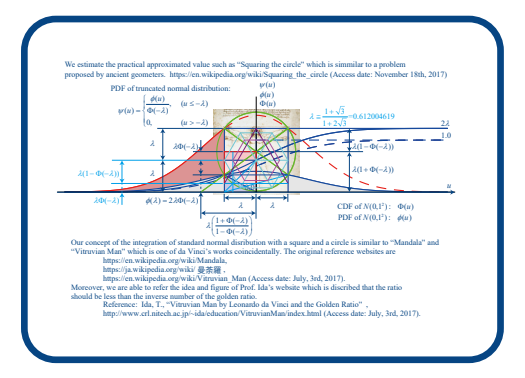
We find that the maximal profit of winners is equal to its fee based on 27 percent probability.

If we consider the correlation coefficient ρ as Karl Pearson's another finding value 0.612003, the interval of solid green line is nearly equal to 0.777 under the condition: $u^2+y(u)^2-2\lambda uy(u)=\lambda^2(1-\lambda^2)$.

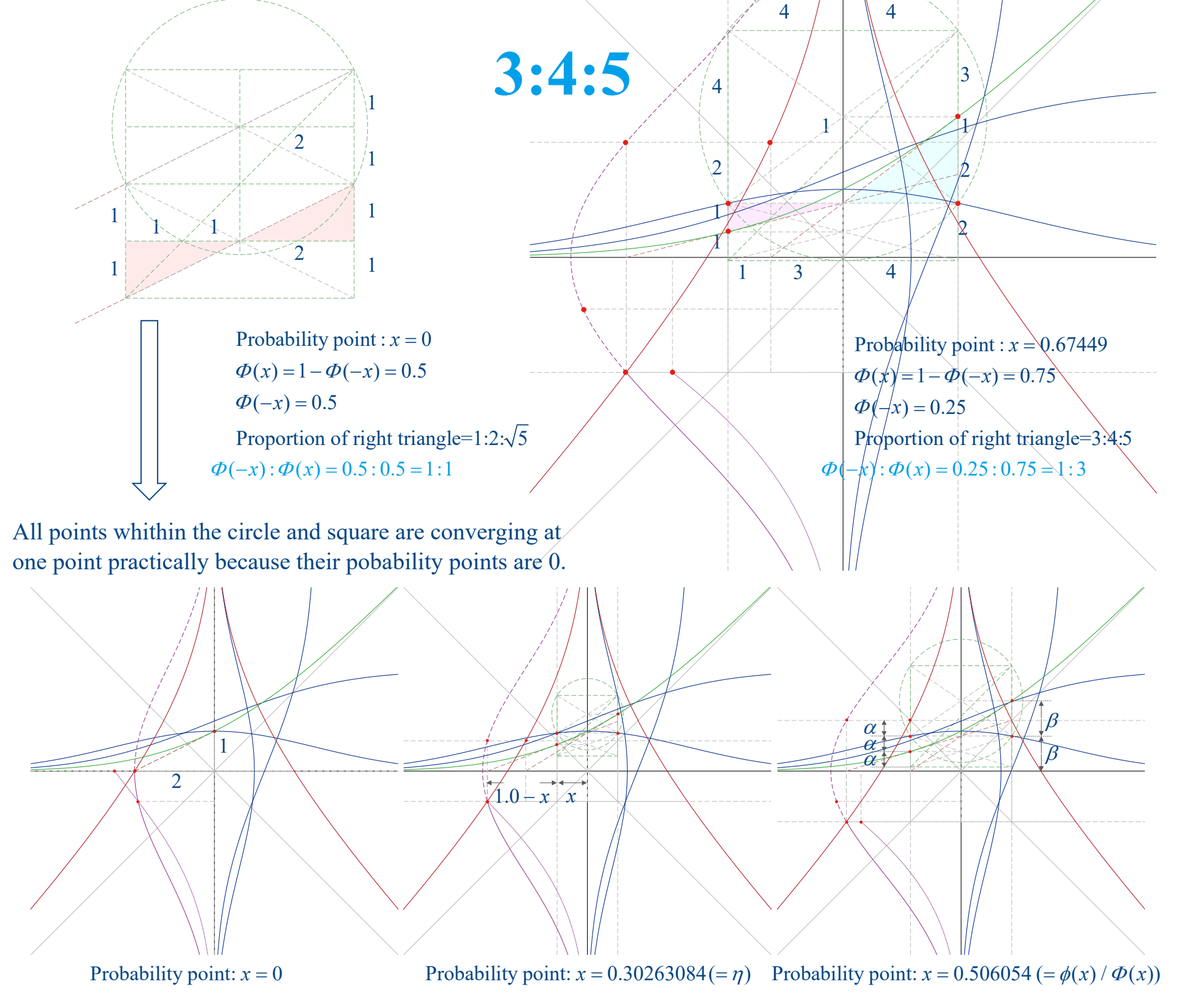
Radius of the circle:
$\lambda(=0.612003)$,
Correlation coefficient:
$\rho=\lambda$.

Relations between regression analysis, principal component analysis, and $\lambda=0.612003$.





Slopes as Probabilities of standard normal distribution are connected to the right triangle by Pythagorean theorem geometrically.



Probability point : $x=0$
$\Phi(x)=1-\Phi(-x)=0.5$
$\Phi(-x)=0.5$
Proportion of right triangle=1:2:$\sqrt{5}$
$\Phi(-x):\Phi(x)=0.5:0.5=1:1$

Probability point : $x=0.67449$
$\Phi(x)=1-\Phi(-x)=0.75$
$\Phi(-x)=0.25$
Proportion of right triangle=3:4:5
$\Phi(-x):\Phi(x)=0.25:0.75=1:3$

All points whithin the circle and square are converging at one point practically because their pobability points are 0.

Probability point: $x=0$

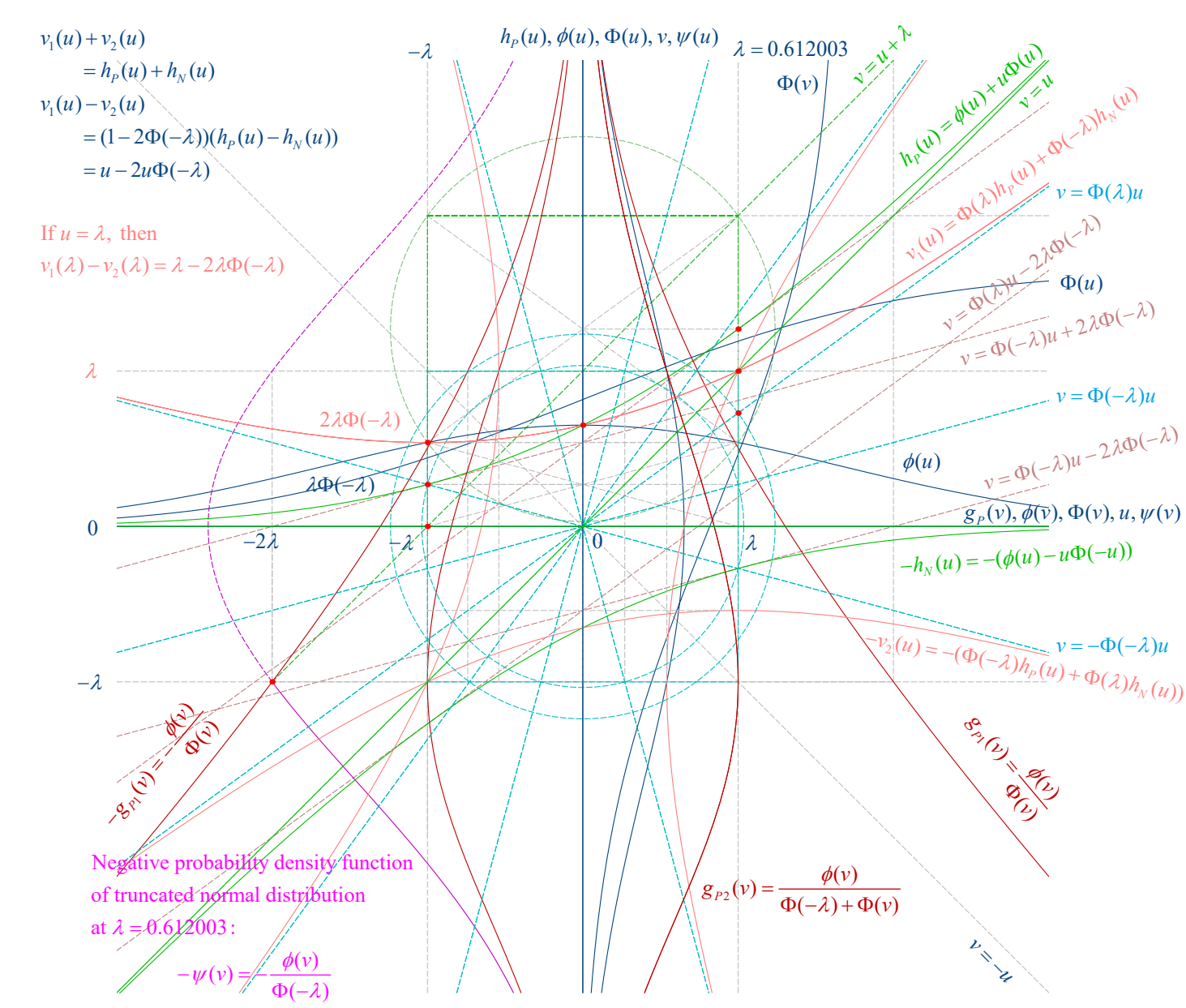
Probability point: $x=0.30263084(=\eta)$
$(\phi(x)=\Phi(-x))$
$\phi(x)/\Phi(-x)=1$

Probability point: $x=0.506054\ (=\phi(x)/\Phi(x))$
$\Phi(x)=0.693591$
$\Phi(-x)=0.306409$
$\alpha=x\Phi(-x)$
$\beta=x\Phi(x)=\phi(x)$
$x=\phi(x)/\Phi(x)=x=\alpha+\beta$

Concluding Remarks

We can understand that the slopes should be the probabilites of standard normal distribution based on the probability points. And above numbers are also the special probability points.

5. Special case as the geometric characterizations and rotationally symmetric relations between winners, losers, and their banker based on the condition: $\lambda=0.612003$ and $\Phi(-\lambda)=0.2702678$.



MP4 Visual Animation　**PowerPoint Visual Animation**