

Visualizations of Sinusoidal Spirals, Limacons of Pascal, and Conic Curves using Equiangular Spirals of Secondary Metallic Ratios

Shingo NAKANISHI at Osaka Institute of Technology, Japan,

© ORSJ Spring and Fall Annual Meetings in 2021, 2022, 2023.
© European Conference on Operational Research in 2021.
© Annual Meeting of Japan Society for Graphical Science in 2022, 2023.
© OIT (Osaka Institute of Technology) Innovation Days in 2021, 2022, 2023.
© International Society for Geometry and Graphics (ISGG) in 2024.

Acknowledgements:
The author, Shingo NAKANISHI, would like to show my grateful to many research members at above academic societies.

Original References:
Operations Research Society of Japan (ORSJ),
European Conference on Operational Research (EURO2021),
Japan Society for Graphical Science (JSGS),
Osaka Institute of Technology in Japan (OIT).
URL: <http://www.oit.ac.jp/center/~nakanishi/english/>



Geometric Characterizations and Symmetric Relations between Standard Normal Distribution and Inverse Mills Ratio based on Pythagorean Theorem

Shingo NAKANISHI at Osaka Institute of Technology, Japan,
Masamitsu OHNISHI at Osaka University, Japan.

© 2019, 2-5, September, Belfast, UK, RSS (Royal Statistical Society) Annual Conference 2019
© 2019, Poster Presentation and Design: Shingo NAKANISHI, Osaka Institute of Technology, Japan

Acknowledgements: The first author, Shingo NAKANISHI, would like to show my grateful to many research members at Operations Research Society of Japan, Kansai-tiku Koryukai of The Securities Analysts Association of Japan, Osaka Institute of Technology in Japan, and Osaka University in Japan.

Original References: Operations Research Society of Japan (ORSJ),
Research Institute for Mathematical Sciences, Kyoto University (RIMS Kōkyūroku, 2078-10),
The 15th International Symposium on Econometric Theory and Applications (SETA2019).

URL: <http://www.oit.ac.jp/center/~nakanishi/english/>



Secondary Metallic Ratios

Golden ratio (Fibonacci ratio): $\phi = 1 + \frac{1}{\phi}$

Harmonies between cardioids and specific right triangles

Jacobsthal sequence: $\lambda_{(1,2)}^{j+2} = \lambda_{(1,2)}^{j+1} + 3\lambda_{(1,2)}^j$

Fibonacci sequence: $\lambda_{(1,1)}^{j+2} = \lambda_{(1,1)}^{j+1} + \lambda_{(1,1)}^j$

Kepler triangles (n=1): $\lambda_{(1,2)}^{j+2} = \lambda_{(1,2)}^{j+1} + 2\lambda_{(1,2)}^j$

Isosceles right triangles (n=2): $\lambda_{(1,2)}^{j+2} = \lambda_{(1,2)}^{j+1} + \lambda_{(1,2)}^j$

Quadrants and Squares based on n = 1, 2, 3, ..., 12

Harmonies about weighted Pythagorean theorem and generalized Fibonacci sequences using related right triangles of secondary metallic ratios

Pythagorean theorem using generalized Fibonacci or Lucas sequences from \ominus to \oplus

$\lambda_{(1,n)}^{j+2} = \lambda_{(1,n)}^{j+1} + n \cdot \lambda_{(1,n)}^j = F_{(1,n),j+1} + n \cdot F_{(1,n),j}$

$\sqrt{1 + 4n\lambda_{(1,n)}^{j+1}} = \sqrt{1 + 4n\lambda_{(1,n)}^j} + n \cdot \sqrt{1 + 4n\lambda_{(1,n)}^{j-1}} = L_{(1,n),j+1} + n \cdot L_{(1,n),j}$

This study aims to investigate the geometry of secondary metallic ratios proposed by de Spinadel in the poster of this conference. Equiangular spirals and related right triangles using the secondary metallic ratios indicate various fundamental shapes based on sinusoidal spirals coincidentally. The relation between the order of sinusoidal spirals and x-th power of the equiangular spirals should be verified as a simple unique equation precisely. Conic curves such as parabolas, hyperbolas, and ellipses with two types of equiangular spirals create geometrically beautiful harmonies with some sinusoidal spirals such as cardioids, lemniscates, Cayley's sextics, and Tschirnhausen cubics from artistic viewpoint respectively. Similarly, limacons of Pascal including cardioids can be illustrated concisely using the reverses of related right triangles based on equiangular spirals. Even though we explain that in the field of the plane geometry, we can also display the lemniscates with toruses and hyperbolas with cones as 3-dimensional graphics to confirm the concepts using the equiangular spirals and related right triangles more theoretically.

Limacon of Pascal, cardioid, and circle with right triangles

Golden Ratio: $\frac{1}{\phi^2} + \frac{1}{\phi} + \frac{\lambda_{(1,1)} - 1}{\lambda_{(1,1)}} = 1$

Kepler triangles with cardioids (n=1)

Nested Radicals?

Continued Fractions?

New Symmetries of Golden Ratio

Pythagorean Triangles show the probabilities as CDFs. Inverse Mills Ratios and PDFs are described as Modified Intercept Forms.

Self-adjoint second order differential equation of a standard normal distribution: $h_p''(u) + u h_p'(u) - h_p(u) = 0$

Equilibrium point between a banker, winners and losers: $\Phi(\lambda) = 2\lambda\Phi(-\lambda)$, $\lambda = 0.612003$

0. Background

We present the geometric characterizations and symmetric relations between standard normal distribution and inverse Mills ratio by circles and squares from the viewpoint with considering the height of densities such as ancient Egyptian drawing styles and using the Greek Pythagorean Theorem.

First, we can clarify the integral forms of various cumulative distribution functions including standard normal distribution based on the aspect ratio (=1.0, see 1).

Second, we reconsider what the several times of standard deviations multiplied by the square root of the time mean with their positive and negative expectations multiplied by the time under the condition the aspect ratio=1.0 (see 2). At this time, we can understand the following things.

- (1) We can find the equilibrium formulation at any real numbers of the time t (see 2).
- (2) Its constant number, 0.612003, was found by Karl Pearson about 100 years ago.
- (3) Sir David Roxbee Cox reconfirmed the value to cluster the normal distribution.
- (4) Truman Lee Kelley also proposed the 27 percent rule formulation.

Third, we can show the parabola for maximal profits including fees. And that is equal to its fee based on the 27 percent probability. In addition to these tendencies, we can clarify the relations between winners, losers, and their banker (see 3).

Fourth, we can understand two types of ordinary differential equations to explain that with circles and squares (see 4). One is the Bernoulli differential equation of inverse Mills ratio. The other is the second order linear differential equation of the integral of cumulative normal distribution function (see 1). From these tendencies, we can also get the modified intercept forms geometrically and symmetrically for maximal profits of winners, these losses of losers, and their banker's fee. We can understand that these equations should be changing the probability points and these probabilities based on Pythagorean Theorem correctly. If the probability point is that found by Karl Pearson, we can show you that it is the special point of standard normal distribution such as Sir Cox's proposal. Other point on the right triangle with 3-4-5 is also the third quartile of standard normal distribution.

Finally, we can also realize there are many similar tendencies close to the relations between circles and squares such as Vitruvian man by Da Vinci and various Mandalas although there might not be related to normal distribution directly and historically. The ancient Egyptian drawing styles enable us to illustrate the geometric characterizations and symmetric relations between standard normal distribution and inverse Mills ratio with circle and square based on the Pythagorean theorem in the ancient Greece. We think that our ideas shall be contributed in the statistical modelling and these evaluation fields since our suggested figures should be much more easily understood and powerful than we thought.

1. First concept as integrals of various cumulative distributions

(a) Type of a triangular distribution

(b) Type of a uniform distribution

(c) Type of an exponential distribution

(d) Type of a normal distribution

First concepts are shown as integral forms of cumulative distribution functions.

Green dashed lines are probability density functions (PDF), $f(x)$.
Red dashed lines are cumulative distribution functions (CDF), $F(x)$.
Purple 1-dot chain lines are integrals of cumulative distribution functions.
Red dotted lines are 1.0.

Aspect Ratio = 1.0

4. Main concept as the relations between Pythagorean theorem, differential equations, circles, and squares on standard normal distribution

Equilibrium point between a banker, winners and losers: $\Phi(\lambda) = 2\lambda\Phi(-\lambda)$, $\lambda = 0.612003$

Modified intercept form of linear equation for winners: $\frac{1}{\lambda}u + \frac{1}{\Phi(-\lambda)}v = 1$

Modified intercept form of linear equation for losers: $\frac{1}{\lambda}u + \frac{1}{\Phi(\lambda)}v = 1$

Modified intercept form of linear equation for their banker: $\frac{1}{\lambda}u + \frac{1}{\Phi(\lambda)}v = 1$

Bernoulli differential equation of an inverse Mills ratio: $g_p(v) + u g_p'(v) + g_p(v)^2 = 0$

$g_p(v) = \frac{\Phi(v)}{\Phi(v) + C}$ (C=0, inverse Mills ratio)

$g_{p1}(v) = \frac{\Phi(v)}{\Phi(-\lambda) + \Phi(v)}$ (C = $\Phi(-\lambda)$)

$g_{p2}(v) = \frac{\Phi(v)}{\Phi(-\lambda) + \Phi(v)}$ (C = $\Phi(-\lambda)$)

2. Second concept as distances of probability points based on the time t. There should be maximal losses against the positive returns or maximal profits against the negative returns by the 27 percent probabilities.

Maximal loss out of $\Phi(-\lambda)$ is equal to its profit: λ

Equilibrium point of inverse Mills ratio: $\frac{\Phi(u)}{\Phi(-\lambda)} = 2\lambda$, $\lambda = 0.612003$

If we consider $t = 1$ and $\sigma = 1$, we can estimate the equilibrium point as $\mu = \pm\lambda$. The integral results are as follows.

That is, the fee α is equal to λ .

Expectation of the truncated normal distribution is 2λ

Expectation of the truncated normal distribution is -2λ

PDF of the truncated normal distribution: $\psi(u) = \frac{\Phi(u)}{\Phi(-\lambda)}$ ($u \geq -\lambda$), $\psi(u) = \frac{\Phi(u)}{\Phi(-\lambda)}$ ($u < -\lambda$)

Expectation of the truncated normal distribution is 2λ

Expectation of the truncated normal distribution is -2λ

Relations both squares and normal distribution based on 2λ

Relations both squares and normal distribution based on 2λ

Maximal Profit out of $\Phi(-\lambda)$ is equal to its charge: λ

If we think of a profit as $+\lambda r$ when $t = 1$, we can use $X(t) = N(+\lambda, t)$.
Or if we think of a loss as $-\lambda r$ when $t = 1$, we can use $X(t) = N(-\lambda, t)$.
Or if we think of them as 0 when $t = 1$, we can use $X(t) = N(0, t)$.

3. Relations between winners, losers, and their banker by the equilibrium formulation

$\sigma\sqrt{t} = \lambda = 0.612003$ and $\Phi(-\lambda) = 0.2702678$

Winners risk aversions: $U_w(t) = \Phi(\lambda) - 2\Phi(-\lambda)\sqrt{t}$

Losers risk preferences: $U_l(t) = -\lambda(1 + \Phi(-\lambda))\sqrt{t}$

Banker's utility: $U_b(t) = \Phi(\lambda) - 2\Phi(-\lambda)\sqrt{t}$

Winners risk aversions: $\alpha = 0.10$, $\beta = 0.5$, $\nu = 1.0$

Losers risk preferences: $\alpha = 0.10$, $\beta = 0.5$, $\nu = 1.0$

Banker's utility: $\alpha = 0.10$, $\beta = 0.5$, $\nu = 1.0$

Sinusoidal Spirals

The dark blue equiangular spirals: $r_{\phi}(s) = \sqrt{\lambda_{(1,0)}^2 \cos^2(\arccos(\frac{1}{\lambda_{(1,0)}}x) + s \sin(\arccos(\frac{1}{\lambda_{(1,0)}}x))}$

The crimson equiangular spirals: $r_{\phi}(s) = \sqrt{\lambda_{(1,0)}^2 \cos^2(\arccos(\frac{1}{\lambda_{(1,0)}}x) - s \sin(\arccos(\frac{1}{\lambda_{(1,0)}}x))}$

Relation of x and s of sinusoidal spiral: $x = -s^2$, $r = \sqrt{\cos s\theta}$ or $r = -\sqrt{\sin s\theta}$

Rectangular hyperbola (x=1/2, s=-3)

Humbert cubic (x=1/3, s=-3)

Keiper cubic (x=1/3, s=3)

Cayley's sextic (x=3)

Tschirnhausen cubic (x=3)

Parabola (x=2)

Line (x=1)

Point (x=0)

Lemniscate of Bernoulli (x=-1/2, s=2)

Circle (x=1)

Cardioid (x=-2)

Cayley's sextic (x=3)

Conic Curves

(Important relation between x th power of equiangular spirals and the order s of sinusoidal spiral)

$x = -s^2$

$r = \sqrt{\cos s\theta}$ or $r = -\sqrt{\sin s\theta}$

The right triangles based on $\lambda_{(1,n)}^2 - \lambda_{(1,n)} - n = 0$ and $n = 1, 2, \dots, 12$

Title: They will be one flesh. Shingo Nakanishi, OIT, Japan, Ver. 2024

Conic curves, sinusoidal spirals, equiangular spirals, and Limacons of Pascal

Conic curves such as parabolas, hyperbolas, and ellipses in addition to cardioids, lemniscates, lines, circles, points, Cayley's sextics, Tschirnhausen cubics, and Limacons of Pascal.

Equiangular Spirals

Refs: OIT Innovation Days 2022 & 2023, ORSJ Fall 2022, Japan, ORSJ Spring 2023, Japan, and Annual Meeting of JSGS 2022, Japan

The circles, cardioids, and limacons of Pascal with right triangles

Case: k = 1

Case: k = 2

Case: k = 3

Case: k = 12

Comparisons of drawing methods using right triangles of similar metallic ratios about $(\frac{\cos \theta(1+k \cos \theta)}{2} + \frac{1-k}{2}, \frac{\sin \theta(1+k \cos \theta)}{2})$ and $(\frac{\cos \theta(1+k \cos \theta)}{2}, \frac{\sin \theta(1+k \cos \theta)}{2})$

Limacons of Pascal

Limacons of Pascal

The circles, cardioids, and limacons of Pascal with right triangles

Case: k = 1

Case: k = 2

Case: k = 3

Case: k = 12

5. Special case as the geometric characterizations and rotationally symmetric relations between winners, losers, and their banker based on the condition: $\lambda = 0.612003$ and $\Phi(-\lambda) = 0.2702678$.

Probability point: x=0, $\Phi(x) = 1 - \Phi(-x) = 0.5$, $\Phi(-x) = 0.5$

Probability point: x=0.67449, $\Phi(x) = 1 - \Phi(-x) = 0.75$, $\Phi(-x) = 0.25$

Probability point: x=0.596054 (= $\Phi(x)$ / $\Phi(x)$), $\Phi(x) = 0.693591$, $\Phi(-x) = 0.306409$, $\alpha = x\Phi(-x)$, $\beta = x\Phi(x) = \Phi(x)$, $x = \Phi(x) / \Phi(x) = x + \beta$

Concluding Remarks

We can understand that the slopes should be the probabilities of standard normal distribution based on the probability points. And above numbers are also the special probability points.

Special case as the geometric characterizations and rotationally symmetric relations between winners, losers, and their banker based on the condition: $\lambda = 0.612003$ and $\Phi(-\lambda) = 0.2702678$.

Radius of the circle: $\lambda = 0.612003$.

Correlation coefficient: $\rho = \lambda$.

Relations between regression analysis, principal component analysis, and $\lambda = 0.612003$.

Two dimensional standard normal distribution with correlation

Radius of the circle: $\lambda = 0.612003$.

Correlation coefficient: $\rho = \lambda$.

Relations between regression analysis, principal component analysis, and $\lambda = 0.612003$.