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Symmetric Relations and Geometric Characterizations about Standard Normal Distribution by Circle and Square

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1. Introduction

Since ancient Babylonian, one of the human ancestors who had their civilizations by their own, found the way to measure or manage something for their communities to keep good conditions, their knowledge have been developed as one of the measuring methods with primitive mathematics and statistics [1, 2]. One was to use some digits or their squared numbers. After that, they draw the figures and pictures about the meaning of digits with some rules based on mathematics and geometry in ancient Egyptian civilization. Ancient Greeks also developed their cultural and artistic works. And they built many temples with beautiful architectural styles by using Pythagorean theorem [3] and some vivid proportions. For a long period including Medieval and Renaissance era, although they admitted various diversities and errors about something important under the uncertainties, the normal distribution had not been known yet because many famous works by researchers such as De Moivre in 1733, Laplace in 1812, Legendre in 1805, and Gauss in 1809 [4, 5] were unpublished. At that time of Renaissance era, it is said that da Vinci was interested in the relations about squares and circles [6] since these relations were very attractive and mysterious. And there was a question about squaring the circle [7] and were various religious cultures like Mandalas [8] during that time all over the world.

By the way, since our human had our cultures to survive somehow, we have also understood that we experienced various opportunities for winning and losing about our economic environments. Bernstein explained to us about the markets as one of them. [2]. According to Ellis's advice in his textbook [9], we should reduce our transaction cost not to lose the chances because over 1-year investigation brings us 60 percent of funds underperform, that over 10 years shows 70 percent underperform. And that over 20 years also shows 80 percent underperform for the chosen benchmarks.

We think that there should be reduced a transaction cost and so on about almost of all things between winners and losers by a banker to get better utilities than they worry about unlucky results under the uncertainties. We also think that this is one modeling of the minus-sum games with thinking about the cost. This is what we proposed in our previous works like repetition games of coin tossing with their fees [10-14]. That is, we proposed that the sum-total of maximal profit for winners was equal to the cost by their banker based on about the probability 27 percent of standard normal distribution [10].

Based on the background we mentioned above, we have dealt with various kinds of characterizations [10-14] of a standard normal distribution at Pearson's finding probability point $\lambda = 0.612003$ [15]. And its cumulative probability is $\Phi(-\lambda) = 0.2702678$ on a standard normal distribution. Kelley's proposal, $\phi(\lambda) = 2\lambda\Phi(-\lambda)$ about $\lambda = 0.612003$, is called as 27 percent rule [16-21]. Especially, Sclove [21], Sclove and Johari [22] informed us of Cox's proposal [23] about $\lambda = 0.612003$ as the important clustering of normal distribution. Nakamori *et. al.* [24] also mentioned that Cox's study should be one of the original papers of K-means Algorisms. Our approaches [10-14] about $\lambda = 0.612003$ are also included as some of them. We found the other characterizations about 0.612003 from the parabola of the cost of the repetitions game of coin tossing [10] and the square on a standard normal distribution with several differential equations [11, 13, 14]. One of our proposals is about integral forms of a cumulative distribution of standard normal distribution [13, 14] in section 2. Another is related to both Mills ratio and inverse Mills ratio [13, 14] in section 3.

In section 2, we would like to explain two types of the variable coefficient second order differential equations about standard normal distribution [13, 14, 25, 26] without mentioning $\lambda = 0.612003$. We reconsider integral forms of cumulative distribution about standard normal distribution [13, 14] and Mills ratio about truncated normal distribution [25-31]. In section 3, we would like to clarify that Bernoulli differential equations of standard normal distribution [13, 14, 26] are geometrically related to $\lambda = 0.612003$. Especially, we would like to show several characterizations based on their general solutions in section 2 and 3 are tied to a mathematical formulation [14].

In section 4, we would like to consider that the way such as ancient Egyptian drawing styles without imagining their depth. The height of inverse Mills ratios and integral forms of cumulative distribution of standard normal distribution are illustrated as various symmetric relations and geometric characterizations by using the drawing method. We clarify that the geometric characterizations of general solutions both differential equations [13, 14, 26] about inverse Mills ratios [31-33] and that about integral forms of cumulative distribution on standard normal distributions inform us of crucial symmetric relations. At this time, we can show that $\lambda = 0.612003$ by Pearson [15], Kelley [16, 17], and Cox [23] should also play an important role in not only their symmetric relations but also the relation as the formulations for winners, losers and, a banker. Then, we can emphasize that the tangent lines on variable coefficient second order linear homogeneous differential equations [13, 14] are equal to these probabilities on a standard normal distribution by using Pythagorean theorem [3]. We would like to introduce the other attractive probability points we found instead of $\lambda(= 0.612003)$ throughout some illustrated figures concisely. From these examples, we can define that the probabilities of a standard normal distribution are two slopes of the tangent lines of the integral form of a cumulative distribution. We would like to show that it is an essential key for solving the relations about standard normal distribution by circle and square geometrically as one of the theoretical approaches for winners, losers and their banker in the field of econometrics.

2. Integral Forms of Cumulative Probability of Standard Normal Distribution and Reconfirmation about two types of variable coefficient second order homogeneous differential equations

If we think of a differential equation and its initial conditions as follows [13, 14]

$$h_P''(u) + uh_P'(u) - h_P(u) = 0, \tag{2.1}$$

$$h_P(0) = \phi(0) = \frac{1}{\sqrt{2\pi}}, \qquad h_P'(0) = \frac{1}{2},$$

we can show you in the general solution with constants C_{P1} and C_{P2}, where $\phi(\cdot)$ is a probability density function of standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function.

$$h_P(u) = C_{P1}\big(\phi(u) + u\Phi(u)\big) + C_{P2}u. \tag{2.2}$$

From equation (2.2), the first derivative function is

$$h_P'(u) = C_{P1}\Phi(u) + C_{P2}. \tag{2.3}$$

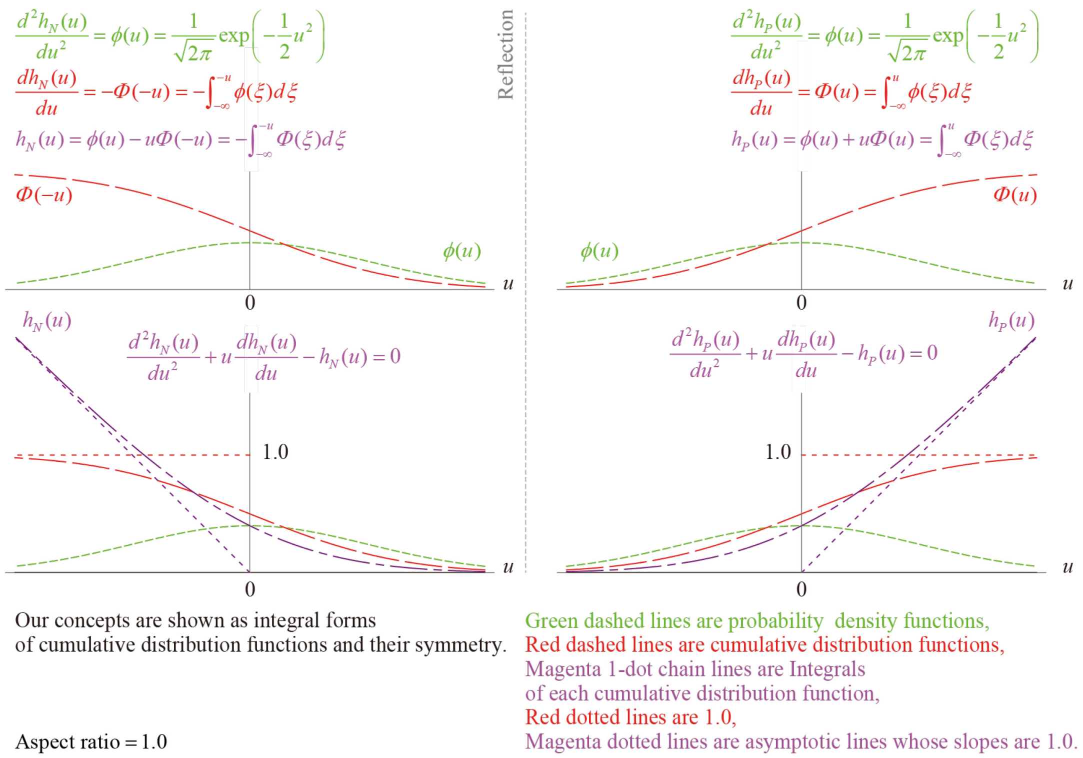


Figure 1 Concepts of integral forms of cumulative distribution function of a standard normal distribution and visualizations about their variable coefficient second order differential equations symmetrically (Original Ref. [13]).

Therefore, since we understand that $C_{P1} = 1.0$ and $C_{P2} = 0.0$, we can estimate the solution [13, 14]

$$h_P(u) = \phi(u) + u\Phi(u). \tag{2.4}$$

This equation is shown as 1-dot chain lines in the right of the bottom of Figure 1.

Similarly, if we think of another initial condition as follows

$$h_N''(u) + uh_N'(u) - h_N(u) = 0, \tag{2.5}$$

$$h_N(0) = \phi(0) = \frac{1}{\sqrt{2\pi}}, \qquad h_N'(0) = -\frac{1}{2},$$

we can also confirm the following equation [13, 14]

$$h_N(u) = \phi(u) - u\Phi(-u). \tag{2.6}$$

This is also shown symmetrically in the left of the bottom of Figure 1. Equations (2.4) and (2.6) are also described as

$$h_P(u) = \int_{-\infty}^{u} h_P'(\xi)d\xi = \int_{-\infty}^{u} \Phi(\xi)d\xi, \tag{2.7}$$

$$h_N(u) = -\int_{-\infty}^{-u} h_N'(\xi)d\xi = -\int_{-\infty}^{-u} \Phi(-\xi)d\xi. \tag{2.8}$$

By the way, although we can get the solutions $h_P(u) = \phi(u) + u\Phi(u)$ and $h_N(u) = \phi(u) - u\Phi(u)$ mentioned above, we would like to consider another type of differential equation [25, 26] as follows.

$$m_P''(u) - um_P'(u) - m_P(u) = 0,$$

$$m_P(0) = \frac{\sqrt{2\pi}}{2}, \qquad m_P'(0) = 1.0. \tag{2.9}$$

From Equation (2.9), we can show you the general solution with constants C_{P1} and C_{P2} as follows

$$m_P(u) = \frac{C_{P1}\Phi(u) + C_{P2}}{\phi(u)}. \tag{2.10}$$

Then, the first derivative function is shown as

$$m_P'(u) = \frac{C_{P1}\big(\phi(u) + u\Phi(u)\big) + C_{P2}u}{\phi(u)}. \tag{2.11}$$

Since we can get the values $C_{P1} = 1.0$ and $C_{P2} = 0.0$, we can display the following solution as Mills ratio [14, 25, 26].

$$m_P(u) = \frac{\Phi(u)}{\phi(u)}. \tag{2.12}$$

In the same way, if we have another condition [14]

$$m_N''(u) - um_N'(u) - m_N(u) = 0,$$

$$m_N(0) = \frac{\sqrt{2\pi}}{2}, \qquad m_N'(0) = -1.0, \tag{2.13}$$

we can also display the following solution [14]

$$m_N(u) = \frac{1 - \Phi(u)}{\phi(u)} = \frac{\Phi(-u)}{\phi(u)}. \tag{2.14}$$

About two types of the differential equations, we would like to reconsider the relations mathematically as follows [14].

First, an inverse of $\phi(u)$ multiplied by Equation (2.4) is written as

$$\mathcal{L}h_P(u) = \frac{1}{\phi(u)}\left(\frac{d^2h_P(u)}{du^2} + u\frac{dh_P(u)}{du} - h_P(u)\right) = 0. \tag{2.15}$$

It is transformed into

$$\mathcal{L}h_P(u) = \left(\frac{h_P'(u)}{\phi(u)}\right)' - \left(\frac{h_P(u)}{\phi(u)}\right) = 0. \tag{2.16}$$

On the other hand, we can also rewrite Equation (2.15) as

$$\mathcal{M}h_P(u) = \frac{d^2}{du^2}\left\{\frac{h_P(u)}{\phi(u)}\right\} - \frac{d}{du}\left\{\frac{uh_P(u)}{\phi(u)}\right\} - \left\{\frac{h_P(u)}{\phi(u)}\right\} = 0$$
$$= \left\{\frac{h'_P(u)}{\phi(u)}\right\}' - \left\{\frac{h_P(u)}{\phi(u)}\right\} = 0. \tag{2.17}$$

Second, we can consider $\phi(u)$ multiplied by Equation (2.12) as

$$\mathcal{L}m_P(u) = \phi(u)\left(\frac{d^2 m_P(u)}{du^2} - u\frac{dm_P(u)}{du} - m_P(u)\right) = 0. \tag{2.18}$$

It is also transformed into

$$\mathcal{L}m_P(u) = \left(\phi(u)m'_P(u)\right)' - \left(\phi(u)m_P(u)\right) = 0. \tag{2.19}$$

$$\mathcal{M}m_P(u) = \frac{d^2}{du^2}\{\phi(u)m_P(u)\} + \frac{d}{du}\{-u\phi(u)m_P(u)\} - \{\phi(u)m_P(u)\} = 0$$
$$= \{\phi(u)m'_P(u)\}' - \{\phi(u)m_P(u)\} = 0. \tag{2.20}$$

Therefore, we can confirm Equations (2.16) and (2.19) are self-adjoint differential equations about standard normal distribution symmetrically. From Equations (2.12) and (2.14), a probability density of standard normal distribution is shown as

$$\phi(u) = \frac{1}{m_P(u) + m_N(u)}. \tag{2.21}$$

From Equations (2.4) and (2.12), we would like to confirm the following relation.

$$\frac{h_P(u)}{\phi(u)} = \frac{\phi(u) + u\Phi(u)}{\phi(u)} = 1 + u\frac{\Phi(u)}{\phi(u)} = m'_P(u). \tag{2.22}$$

$$\frac{h'_P(u)}{\phi(u)} = \frac{\Phi(u)}{\phi(u)} = m_P(u) \tag{2.23}$$

Then, we can insert equations (2.22) and (2.23) into (2.16) as follows.

$$\left(m_P(u)\right)' - m'_P(u) = 0. \tag{2.24}$$

Thus, Equation (2.24) is correct. Moreover, we can find the relations

$$h_P(u) = \phi(u)m'_P(u) \tag{2.25}$$

and

$$h'_P(u) = \phi(u)m_P(u). \tag{2.26}$$

In the same way, Equation (2.6) and (2.14), we can clarify the following relation.

$$\frac{h_N(u)}{\phi(u)} = -\frac{\phi(u) - u\Phi(-u)}{\phi(u)} = -1 + u\frac{\Phi(-u)}{\phi(u)} = -m'_N(u). \tag{2.27}$$

$$\frac{h'_N(u)}{\phi(u)} = \frac{\Phi(-u)}{\phi(u)} = -m_N(u) \tag{2.28}$$

From equations (2.27) and (2.28), we can transform

$$\left(\frac{h'_N(u)}{\phi(u)}\right)' - \left(\frac{h_N(u)}{\phi(u)}\right) = 0. \tag{2.29}$$

into

$$\left(-m_N(u)\right)' - \left(-m'_N(u)\right) = 0. \tag{2.30}$$

Thus, Equation (2.30) is also correct. Moreover, we can find the relation

$$h'_P(u) - h'_N(u) = 1 \tag{2.31}$$

and

$$h'_N(u) = -\phi(u)m_N(u) \text{ and } h_N(u) = -\phi(u)m'_N(u)\,. \tag{2.32}$$

3. Bernoulli Differential Equations for Inverse Mills Ratio

In section 2, we considered that two types of the second order differential equations about a standard normal distribution. one is described in Figure 1 as the integral forms of a cumulative distribution function [13, 14]. The other is that about Mills Ratio [13, 14]. In this section, we would like to focus on the later type of differential equations and these relations based on $\lambda = 0.612003$. From Maddala's explanation [27] and that of Johnson and Kotz [28], we can know that there are important censored and truncated normal distribution theories in these fields such as Tobit [29] and Heckman's models [30]. We have already proposed the following formulations as Bernoulli differential equation with a standard normal distribution [13, 14].

$$\frac{dg_P(u)}{du} + ug_P(u) + g_P(u)^2 = 0, \tag{3.1}$$

$$\frac{dg_N(u)}{du} + ug_N(u) - g_N(u)^2 = 0. \tag{3.2}$$

These general solutions [13] are solved as

$$g_P(u) = \frac{\phi(u)}{\Phi(u) + C}. \tag{3.3}$$

$$g_N(u) = \frac{\phi(u)}{\Phi(-u) + C}. \tag{3.4}$$

If we reconsider the tendencies of an inverse Mills ratio and a hazard function [29] as the special cases on $\lambda = 0.612003$, we can illustrate the tendencies in Figure 2. From Figure 2, we can clarify that there are several relations between Equation (3.3) and (3.4) about the squares [11, 12] at λ on standard normal distribution if $C = 0$.

On the other hand, we can also display that maximal value $\phi(\lambda)/\Phi(-\lambda) = 2\lambda$ (in case of $u = \pm\lambda$) and $\mp\lambda$ which is the symmetric point on $\pm\lambda$ about 0 is gotten as $2\lambda\Phi(-\lambda)\big(= \phi(\lambda)\big)$ shown in Figure 2 if $C = \Phi(-\lambda)$ in Equation (3.3) and (3.4). Therefore, we can understand that the formulation [11, 16, 17] should be

$$\phi(\lambda) = 2\lambda\Phi(-\lambda). \tag{3.5}$$

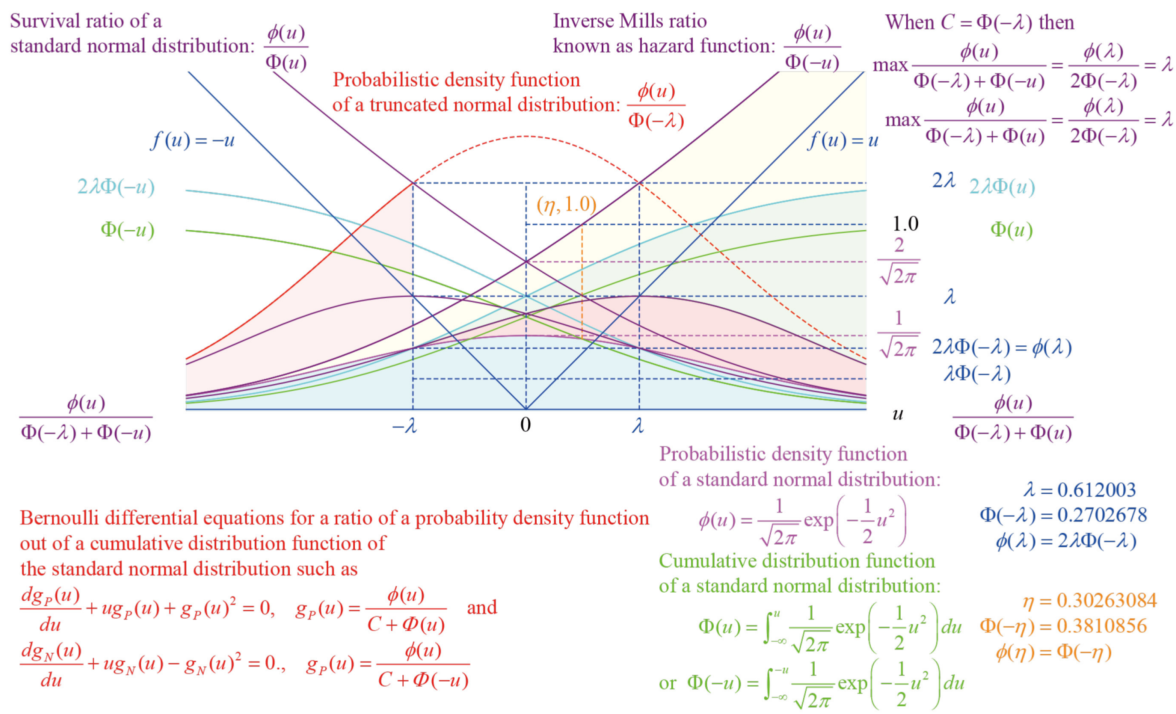


Figure 2 Bernoulli differential equations of the ratios of probability density function of standard normal distribution out of its cumulative distribution function with thinking of its truncated normal distribution at the probabilistic point $\lambda = 0.612003$ and $\eta = 0.30263084$ (Original Ref. [13]).

This Equation (3.5) plays a vital important role about the truncated normal distribution about $\pm\lambda$ symmetrically and geometrically.

By the way, the probability point, $\eta = 0.30263084$, is also a crucial important probability point because the value of cumulative distribution function is equal to that of probability density function at η in Figure 2. We can find that it is another important relation since the inverse Mills ratio is equal to 1.0 in this case. Under the condition at $u = 0$, we find the inverse Mills ratio is $2\phi(0)(= 2/\sqrt{2\pi})$ as 2 times of $\phi(0)$. These geometric tendencies about Equations (3.3) and (3.4) separated by $u = 0$ are also illustrated symmetrically in Figure 2. In section 4, we would like to explain that it is what we should emphasize about our proposals by using Pythagorean theorem in more detail.

However, the equation such as Bernoulli differential equation with standard normal distribution is also mentioned by Hald [31]. Since we have been interested in the geometric characterizations of them and their relations with other differential equations in this section, we think of them as follows.

First, the first order derivatives are shown as

$$g'_N(u) = \frac{-u\phi(u)(\Phi(-u) + C) + \phi(u)^2}{(\Phi(-u) + C)^2}, \tag{3.6}$$

$$g'_P(u) = \frac{-u\phi(u)(\Phi(u) + C) - \phi(u)^2}{(\Phi(u) + C)^2}. \tag{3.7}$$

If $C = 0$, the first order derivatives are

$$g'_N(u) = \frac{-u\phi(u)\Phi(-u) + \phi(u)^2}{\Phi(-u)^2} = \frac{\phi(u)}{\Phi(-u)}\frac{\left(\phi(u) - u\Phi(-u)\right)}{\Phi(-u)} = -\frac{g_N(u)h_N(u)}{h'_N(u)} = -\frac{h_N(u)}{m_N(u)h'_N(u)}, \tag{3.8}$$

$$g'_P(u) = \frac{-u\phi(u)\Phi(u) - \phi(u)^2}{\Phi(u)^2} = -\frac{\phi(u)}{\Phi(u)}\frac{\left(\phi(u) + u\Phi(u)\right)}{\Phi(u)} = -\frac{g_P(u)h_P(u)}{h'_P(u)} = -\frac{h_P(u)}{m_P(u)h'_P(u)}. \tag{3.9}$$

Therefore, we can understand the general solutions of three types of differential equations are tied as an equation mathematically. Finally, we can define the relations about $h_P(u), m_P(u), and\ g_P(u)$ or $h_N(u), m_N(u), and\ g_N(u)$ as the following crucial important formulations

$$-\frac{g'_P(u)}{g_P(u)} = \frac{h_P(u)}{h'_P(u)} = \frac{m'_P(u)}{m_P(u)}, \tag{3.10}$$

$$-\frac{g'_N(u)}{g_N(u)} = \frac{h_N(u)}{h'_N(u)} = \frac{m'_N(u)}{m_N(u)}. \tag{3.11}$$

4. Right Triangle for Probability of Standard Normal Distribution and its Truncated Normal Distribution

In section 2, we mentioned some variable coefficient second order differential equations and their relations. In section 3, their differential equations and Bernoulli differential equations are connected by the fractions of Equation (3.10) about their primitive functions and first order derivatives.

Since we have also found that the tangent lines of Equation (2.4) has important probabilistic characterizations shown in Figure 3 geometrically [13, 14], we would like to focus on our reconsiderations in section 2 as the integral forms of a cumulative distribution of a standard normal distribution. We have searched for their several characterizations such as Mills ratio [31-33], standard normal distribution [31, 34-35], their differential equations [36, 37], and the tendencies for winners, losers and a banker [2, 9, 10, 38]. From our investigations, we can propose that the general solutions of the variable coefficient second order differential equation in Figure 4 instead of our misspelled proposal [13, 14] inform us of several correct characterizations between statistics and geometry cooperatively as follows.

From Figure 3, we can reconfirm the proportion for winners about per unit and per entire is equal to $\lambda : \lambda\Phi(-\lambda)$ by our previous works [13, 14]. Its intercept form of a linear equation for winners [13] is shown as

$$-\frac{1}{\lambda}u + \frac{1}{\lambda\Phi(-\lambda)}v = 1. \tag{4.1}$$

In the same way, we can define the intercept form of a linear equation for losers [13].

$$-\frac{1}{\lambda\left(\frac{1+\Phi(-\lambda)}{1-\Phi(-\lambda)}\right)}u + \frac{1}{\lambda\left(1+\Phi(-\lambda)\right)}v = 1. \tag{4.2}$$

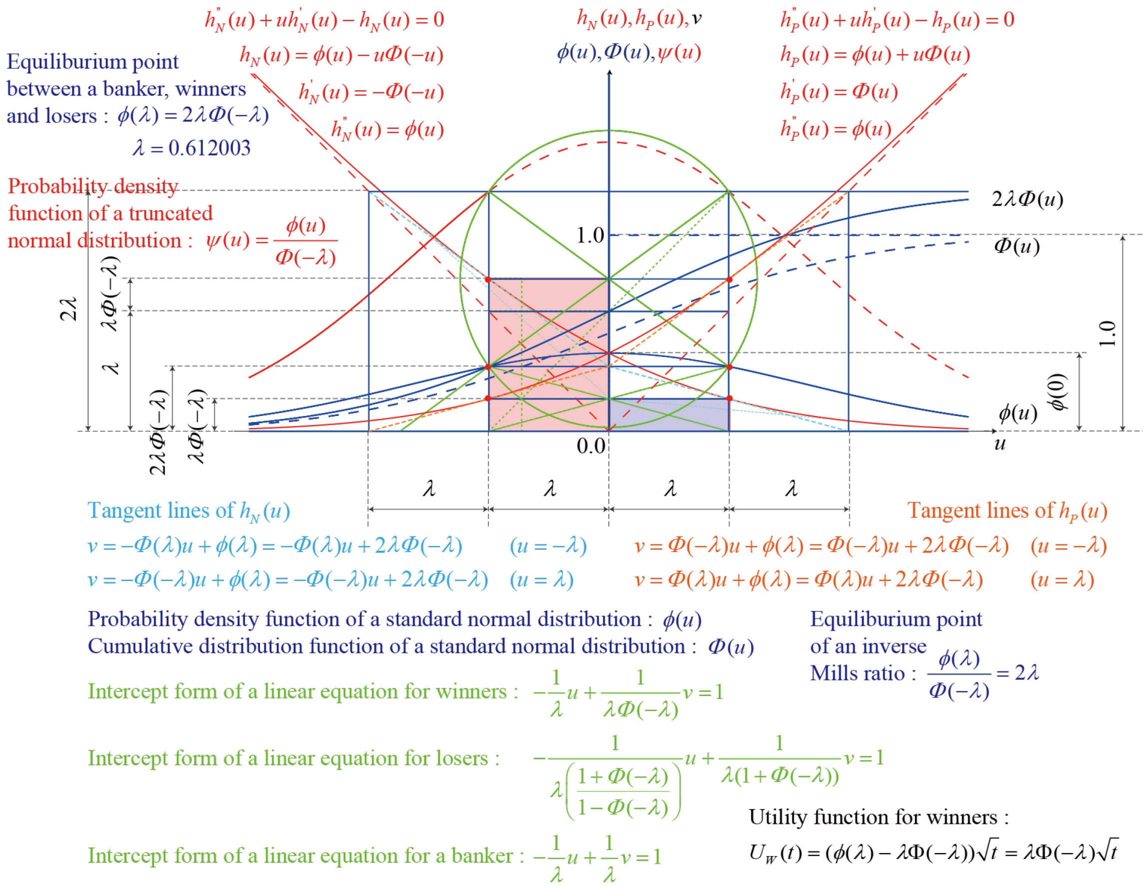


Figure 3 Geometric relationships between second order differential equations and standard normal distribution with a circle, squares, meaningful rectangles, their diagonals, and special tangent lines at $\lambda = 0.612003$ (Original Ref. [13]).

Its proportion is equal to $(1 + \Phi(-\lambda))/(1 - \Phi(-\lambda)) : (1 + \Phi(-\lambda))$. From Equations (4.1) and (4.2), the intercept form for a banker [13] is also estimated as the following equation because of the proportion $1 : 1$. The meaning of per unit is that of per entire since the number of a banker should remain unit in this case. That is

$$-\frac{1}{\lambda}u + \frac{1}{\lambda}v = 1. \tag{4.3}$$

As described Equations (4.1) to (4.3), we can understand there are three intercept forms shown in Figure 3. At the same time, we can also confirm the crucial important tangent lines of $h_P(u)$ at the probability points $\pm\lambda$ as follows

$$v_1 = \Phi(-\lambda)u + \phi(\lambda) \ \left(= \Phi(-\lambda)u + 2\lambda\Phi(-\lambda)\right),$$
$$v_2 = \Phi(\lambda)u + \phi(\lambda) \ \left(= \Phi(\lambda)u + 2\lambda\Phi(-\lambda)\right). \tag{4.4}$$

Therefore, we can notice that $\phi(\lambda) = 2\lambda\Phi(-\lambda)$ by Equation (3.5) is intercept of Equation (4.4) at $u = 0$ shown in Figure 3. We can transform Equation (4.4) into the following intercept forms as our special proposals. Moreover, if we consider that both charts shown in Figures 2 and 3 as one picture such as Egyptian drawing styles without imaging their depth in Figure 4, we can also propose several modified intercept forms for Equations (4.1) to (4.3).

That is, a modified intercept form of linear equation for a negative probability point $-x(= -\lambda)$ in Figure 4 is rewritten as

$$-\frac{1}{\frac{\phi(x)}{\Phi(-x)}}u + \frac{1}{\phi(x)}v_1 = 1. \tag{4.5}$$

Another modified intercept form of linear equation for a positive probability point $x(= \lambda)$ in Figure 4 is also rewritten as

$$-\frac{1}{\frac{\phi(x)}{\Phi(x)}}u + \frac{1}{\phi(x)}v_2 = 1. \tag{4.6}$$

A modified intercept form of linear equation for probabilities both above negative probability point $-x(= -\lambda)$ and positive probability point $x(=\lambda)$ in Figure 4 are shown as

$$-\frac{1}{\dfrac{\phi(x)}{\Phi(x)+\Phi(-x)}}u+\frac{1}{\phi(x)}v=1 \text{ or } -\frac{1}{\phi(x)}u+\frac{1}{\phi(x)}v=1. \tag{4.7}$$

Thus, we can clarify that these intercept forms from Equations (4.5) to (4.7) are composed of probability density function $\phi(x)$ at x and its inverse Mills ratio $\phi(u)/\Phi(x)$ symmetrically.

If we consider the conditions, we also confirm that the special curve in Figure 4 passes through several fundamental points with $\lambda\Phi(-\lambda)$ and $\lambda(1+\Phi(-\lambda))$ [13, 14]. At the same time, we find these points also have two of the crucial tangent lines shown in Figure 4 as slopes of intercept forms for winners and losers. Moreover, we can illustrate a true circle with a square about standard normal distribution shown in Figure 4 [13, 14] as meaningful tendencies for winners, losers and their banker. The center of this circle is an intersection of the rectangle diagonals. Without passing through the intersection, we find the other important line whose slope is equal to 1.0 to distinguish into the separated squares by the proportion $\Phi(-\lambda) : \Phi(\lambda)$ at the probability points $x = \pm\lambda$ [13]. This message is the most important proposal for us to show our approach as geometrically correct.

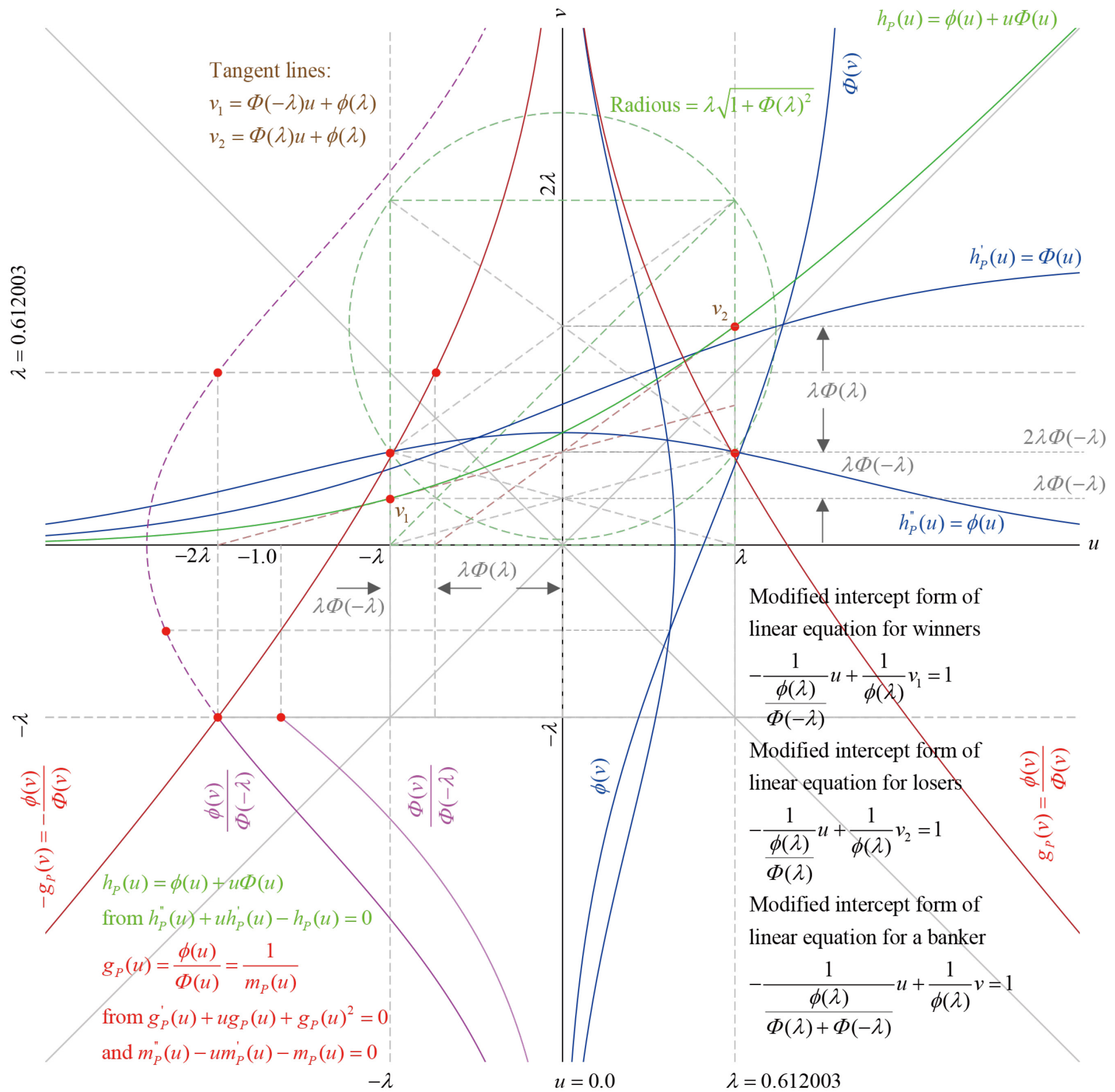


Figure 4 Geometric relations between second order differential equation and Bernoulli differential equation on standard normal distribution with a circle, squares, meaningful special tangent lines and their intercept forms based on $\lambda = 0.612003$.

Furthermore, we would like to illustrate other probability points such as $x = 0.0$ $(\Phi(-0.0) = 0.5,\ \Phi(0.0) = 0.5)$ and $x = \pm 0.67449$ $(\Phi(-0.67449) = 0.25,\ \Phi(0.67449) = 0.75)$. These values $x (= \pm 0.67449)$ are upper and lower quantiles [21]. They are also special points to explain our approach concretely shown in the top of Figure 5 since we can find that both the slopes of the right triangles at probability points $\pm x$ by using Pythagorean theorem are equal to these probabilities of standard normal distribution.

If we estimate that $x = \pm 0.67449$, x brings us the right triangle whose proportion is $3:4:5$ by using Pythagorean theorem correctly and geometrically shown in Figure 5.

$$\Phi(-x) : \Phi(x) : \Phi(-x) + \Phi(x) = 1/4 : 3/4 : 4/4 = 1:3:4, \qquad \because x = 0.67449. \tag{4.8}$$

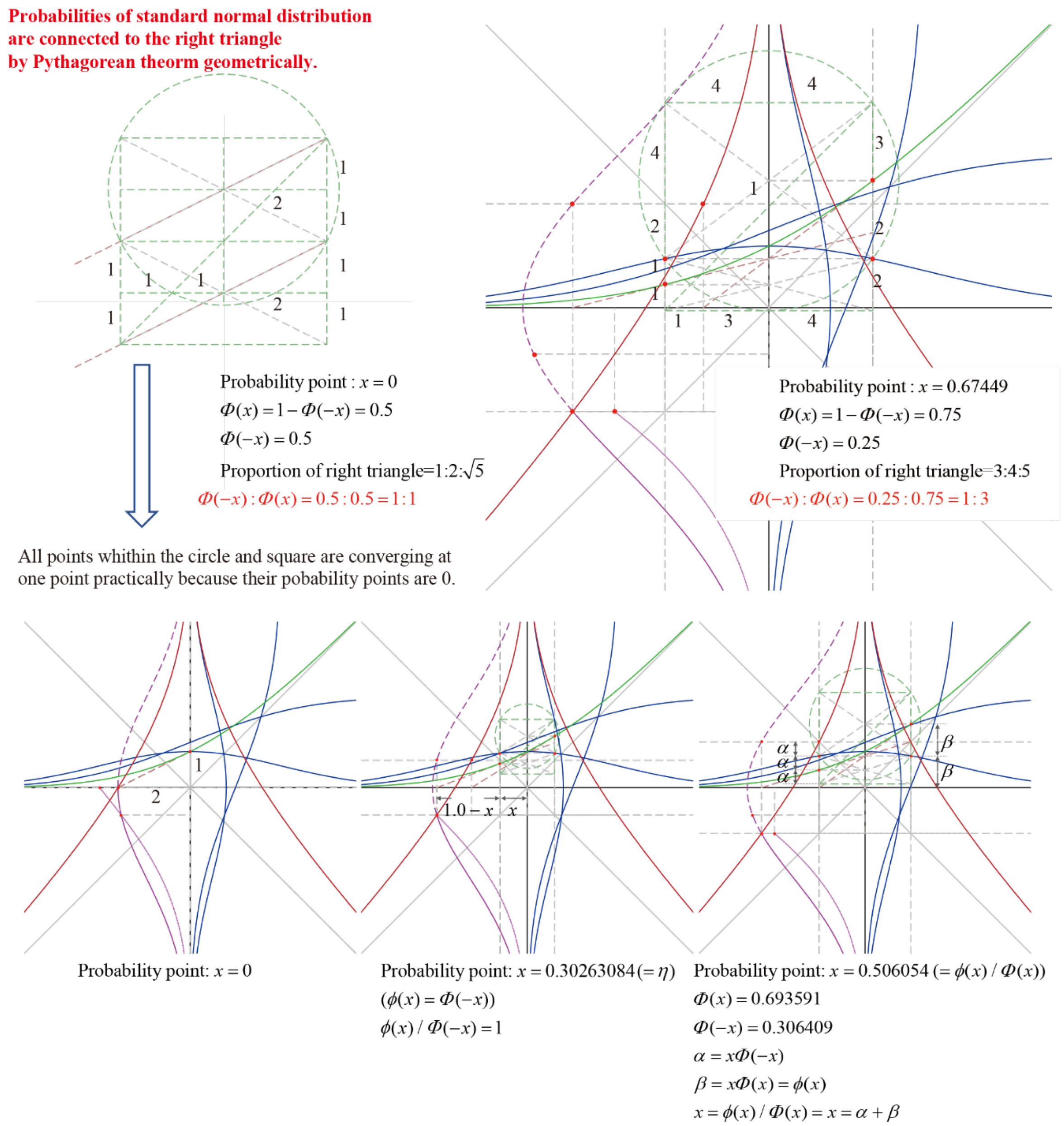
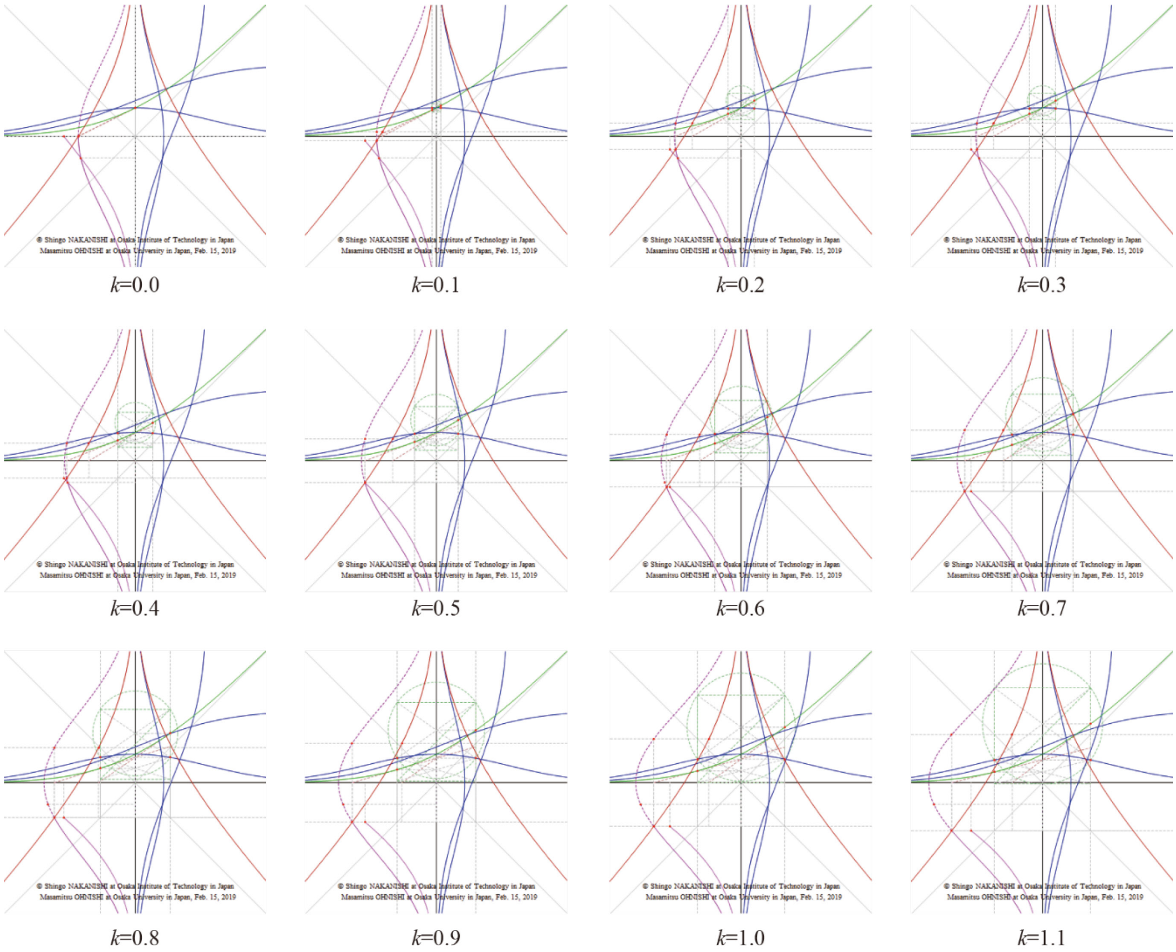


Figure 5 Various geometric characterizations between right triangles and probabilities on standard normal distribution by Pythagorean theorem. (Pythagorean theorem makes slopes and probabilities of standard normal distribution equally according to circle, square, and tangent lines.)



$u = k\lambda, \quad k = 0.0, 0.1, 0.2, \cdots, 1.0, 1.1, \quad \lambda = 0.612003$

Blue solid lines are probability density functions or cumulative distribution functions of standard normal distribution.
Green solid curves are integrals of cumulative distribution function of standarad normal distribution.
Red solid curves are plus and minum inverse functions of inverse mills ratios of standard normal distribution.
Purple solid inverse functions of curves are probability density functions or
cumulative distribution functions of truncated normal distribution.
Orange dashed lines are tangent lines on the probability density based on both probability points.
Green dashed curves are circles and green dased lines related squares and their important lines.

Figure 6 k times of $\lambda = 0.612003$ on the standard normal distribution with circle and square.

In the same way, we can consider that the probability point $x = 0.0$ gets the proportion $1:1:2$ since all points in the left of the top of Figure 5 are converging at one point $x = 0.0$ such as the point shown in the left of the bottom of Figure 5. That is also described as

$$\Phi(-x) : \Phi(x) : \Phi(-x) + \Phi(x) = 1/2 : 1/2 : 2/2 = 1:1:2, \qquad \because x = 0.0. \tag{4.9}$$

Therefore, we clarify that the tangent lines whose slopes are equal to the probabilities $\Phi(-x)$ and $\Phi(x)$ show the relations between a circle and a square based on the probabilities by using Pythagorean theorem in this section if x is given as a real number shown in Figure 6. These mathematical formulations satisfy Equations (4.5) and (4.6) based on $\phi(x)$ and $\phi(x)/\Phi(x)$. From above mentioned, we can also explain similar characterizations on $x = \eta (= 0.30263084)$ and $x = 0.506054$ shown in Figure 5. We can confirm that the values $x = \eta$ which brings us the Mills ratio is equal to 1.0 and $x = 0.506054$ whose meaning shows $x = \phi(x)/\Phi(x)$. These points are also geometrically attractive shown in Figures 5 and 6.

Furthermore, we realized that a real number x as probability point of standard normal distribution brings us the symmetric relations and geometric characterizations by circle and square about standard normal distribution by using Pythagorean theorem for winners, losers, and their banker shown in Figure 6.

5. Conclusions

In this paper, we dealt with the symmetric relations and the geometric characterizations about a standard normal distribution by circle and square from the view point without imaging the height of densities such as Egyptian pictures in the ancient era.

First, we can clarify that the general solutions as the integral form of cumulative distribution functions of standard normal distribution, Mills ratio, and inverse Mills ratio are shown as a mathematical formulation in section 3. In this case, we can confirm the integrals are related to inverse of Mills ratio. Second, from these tendencies, we can also get the modified intercept forms geometrically and symmetrically. We can understand these equations for winners, losers, and their banker according to the probability points. When the bottom line of the square is located on the height $v = 0.0$, these probability points are $u = \pm\lambda = \pm 0.612003$ which are illustrated as the special case in our studies. Third, we can confirm that the integral form of a cumulative distribution function of standard normal distribution is expressed as Self-adjoint differential equation.

As described above, we can reconfirm the geometric characterizations about $\lambda = 0.612003$ with considering square, circle, and normal distribution as the special modeling throughout this study. Furthermore, we can also realize there are many similar tendencies from European through Oriental historical cultures close to the relations between circles and squares such as Vitruvian man by Da Vinci [6], Squaring the Circle [7], and Mandalas [8]. There might not be related to normal distribution directly. However, the ancient Egyptian drawing styles enable us to illustrate symmetric relations and geometric characterizations between standard normal distribution and inverse Mills ratio with circle and square by using Pythagorean theorem as one of the greatest ancient Greek mathematical tools. We would like to expect that our proposals will be useful and contributed in the other fields as well as the statistical areas since our suggested charts and figures should be much simpler and more powerful than we thought.

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