Symmetric Relations and Geometric Characterizations about Standard Normal Distribution by Circle and Square

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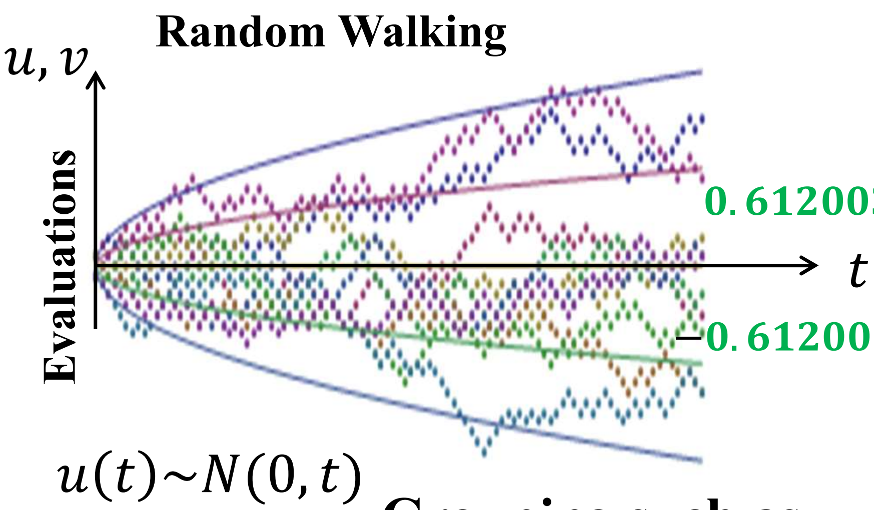
Please remember the following values.

Pearson's finding probability point:
$\lambda = 0.612003$

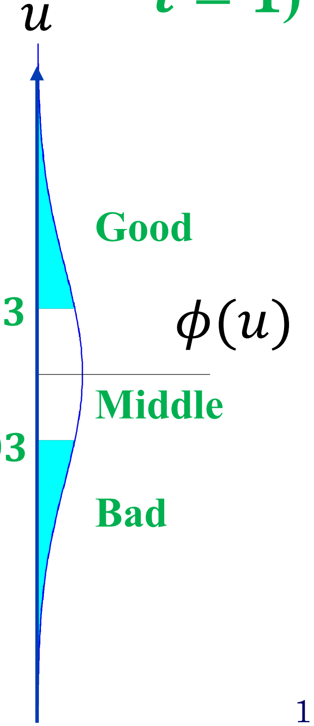
Its cumulative distribution probability:
$\Phi(-\lambda) = 0.2702678$

Kelley's formulation as 27 percent rule:
$\phi(\lambda) = 2\lambda\Phi(-\lambda) = 0.3308$



Grouping such as the proposal by Cox

Standard Normal Distribution (Aspect Ratio: 1.0, $t = 1$)



Aims and Viewpoints about Our Research

1. Reasons why Pearson's finding probability point, 0.612003,
   is important.

2. Symmetric and Geometric Proposals
   of Two types of Differential Equations
   between Standard Normal Distribution and Inverse Mills Ratio

3. These drawing methods with Circle and Square
   between Winners, Losers, and their Banker.

Nakanishi's Website:
http://www.oit.ac.jp/center/~nakanishi/english/

Winners Maximal Profit are equilibrium to its Fee by a Banker at t=1 and σ=1

$$U_W(\alpha,t) = \max_{0\le t\le\tau} E(Y_W(\alpha,t)) = \lambda\underline{\Phi(-\lambda)}\sqrt{t} = 0.612003 \times \underline{0.2702678}\sqrt{t}$$

$\Phi(-\lambda)$ 0.2702678

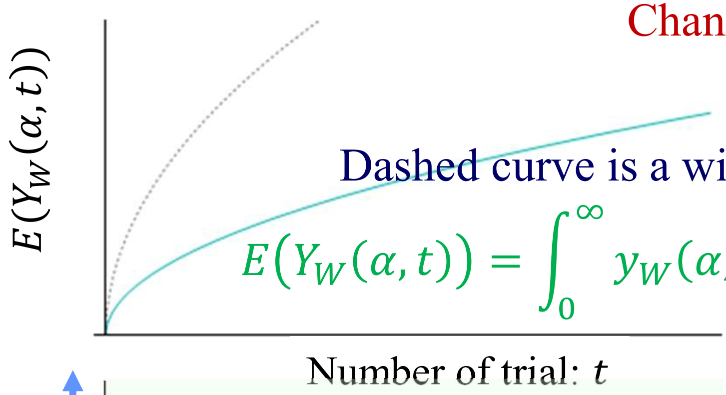
The maximal profit for winners is a **parabola** based on the winners probability **0.2702678** constantly.

Changing the fee α under the condition

$$\alpha = 0.012 + 0.1k \quad (k = 0,\cdots,20)$$

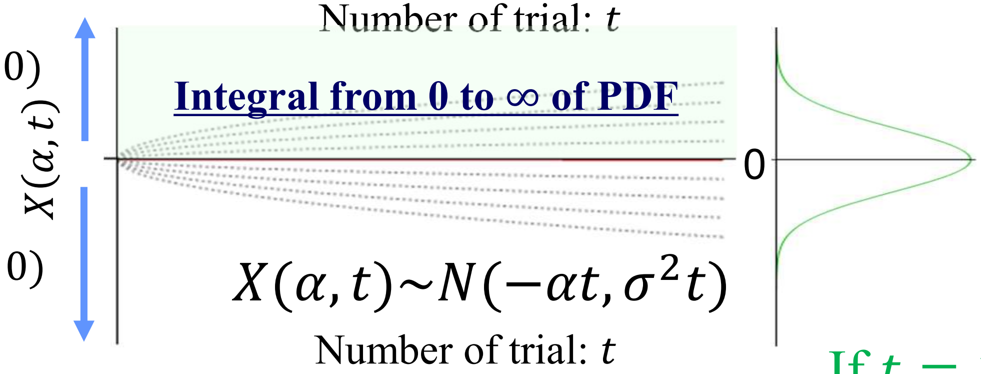
Dashed curve is a winner's expected profit which is greater than 0 :

$$E\left(Y_W(\alpha,t)\right) = \int_0^\infty y_W(\alpha,t)\cdot\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{1}{2}\frac{(y_W(\alpha,t)+\alpha t)^2}{\sigma^2 t}\right)dy_W(\alpha,t)$$



Profit of a Winner :
$Y_W(\alpha,t) = \max(X(\alpha,t),0)$

Loss of a Loser :
$Y_L(\alpha,t) = \min(X(\alpha,t),0)$



Equilibrium formulation :

$$\left(\frac{\alpha}{\sigma}\right)\sqrt{t} = \lambda = Const.$$

$$\because \lambda = 0.612003$$

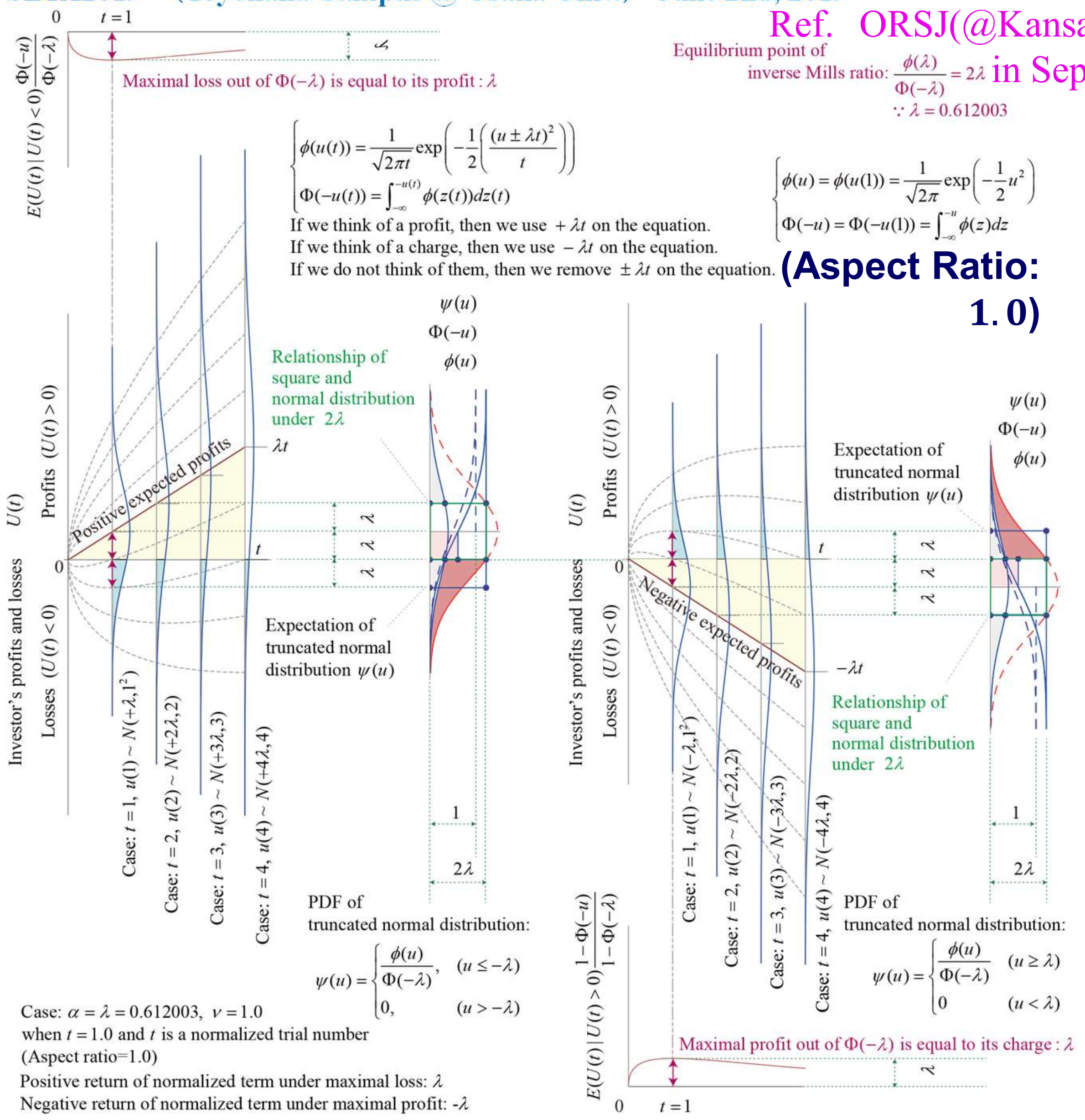
If $t = 1$ and $\sigma = 1$, $\alpha = \lambda$

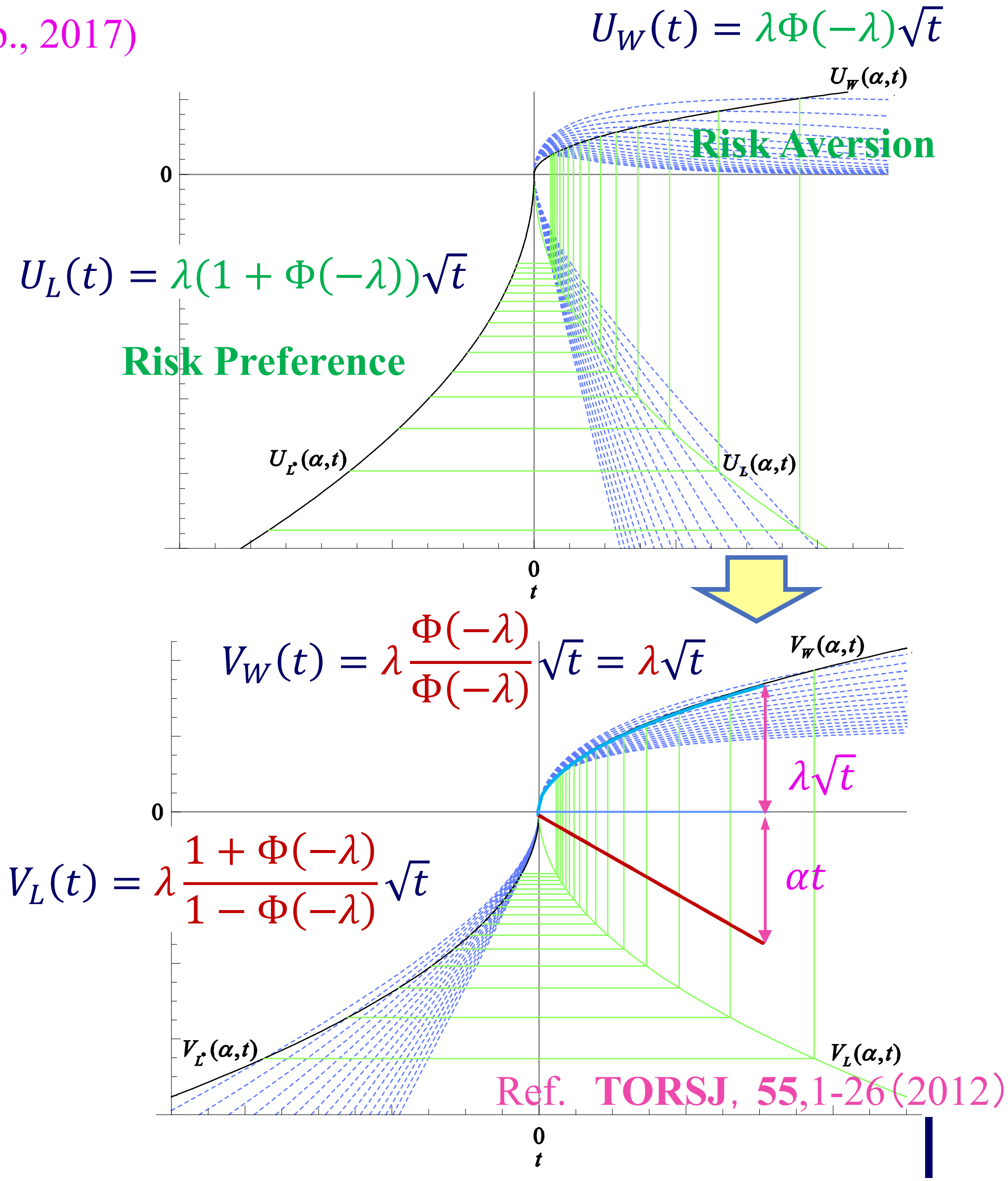
The fee αt is equal to $\lambda(=0.612)$ times of $\sigma\sqrt{t}$ based on $\Phi(-\lambda) \cong 0.2702678$

Ref. **TORSJ**, **55**,1-26(2012)

Ref. ORSJ(@Kansai Univ. in Sep., 2017)

(Aspect Ratio: 1.0)





Ref. TORSJ, 55,1-26(2012)

Relations between Inverse Mills Ratio, Conditional Expectation, and $\lambda = 0.612$

Ref. RIMS2078-10（@Kyoto Univ. in Nov., 2017）

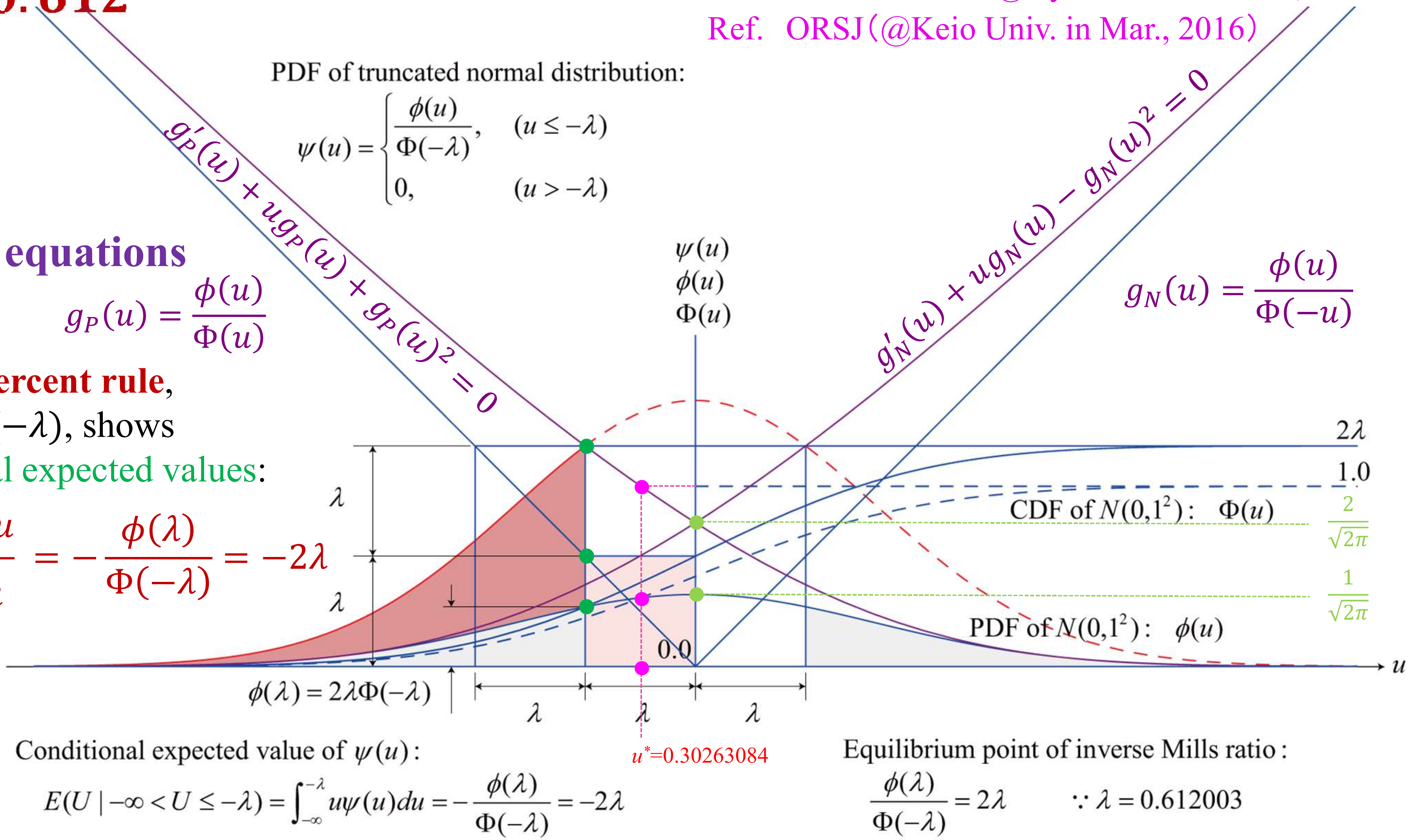
Ref. ORSJ（@Keio Univ. in Mar., 2016）

Bernoulli differential equations

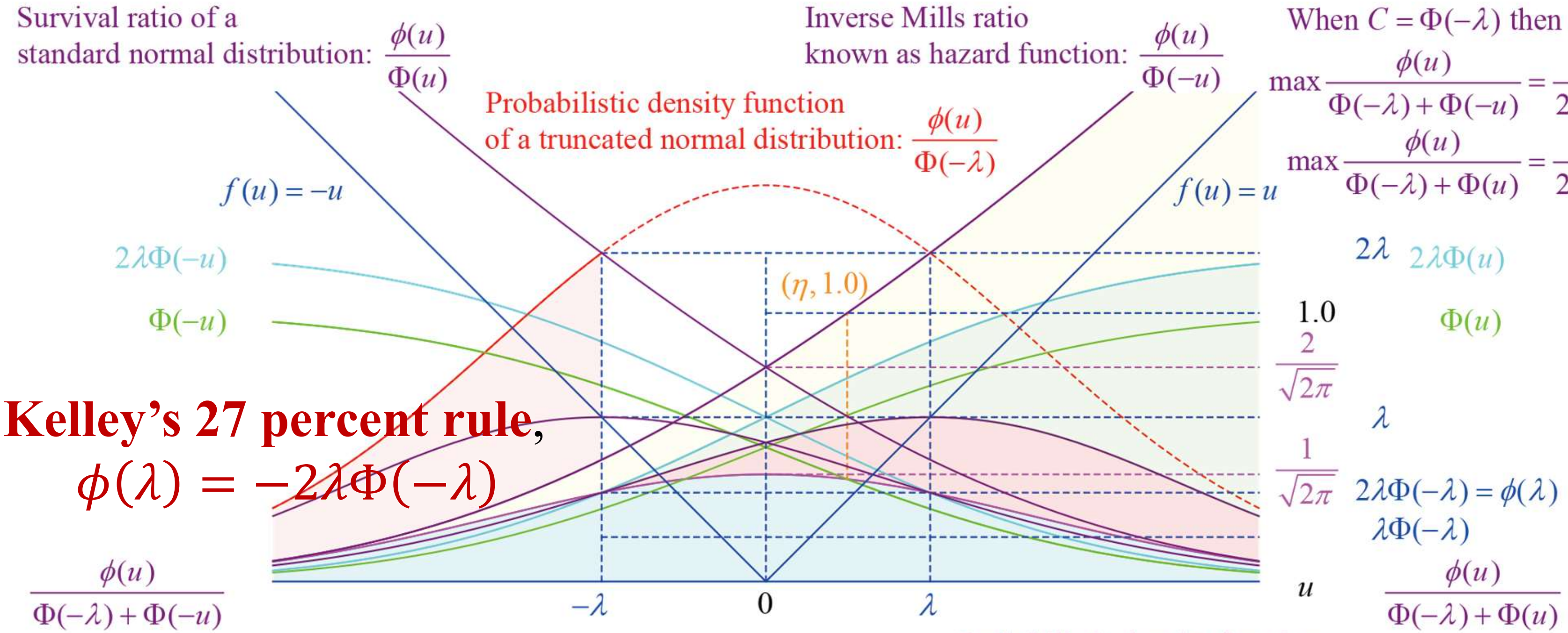
$g_P(u) = \dfrac{\phi(u)}{\Phi(u)}$

Kelley's 27 percent rule, $\phi(\lambda) = 2\lambda\Phi(-\lambda)$, shows the conditional expected values:

$$\frac{\int_{-\infty}^{-\lambda} u\phi(u)du}{\int_{-\infty}^{-\lambda} \phi(u)du} = -\frac{\phi(\lambda)}{\Phi(-\lambda)} = -2\lambda$$



Inverse Mills Ratio, Standard Normal Distribution, and Bernoulli Differential Equations



When $C = \Phi(-\lambda)$ then

$$\max \frac{\phi(u)}{\Phi(-\lambda)+\Phi(-u)} = \frac{\phi(\lambda)}{2\Phi(-\lambda)} = \lambda$$

$$\max \frac{\phi(u)}{\Phi(-\lambda)+\Phi(u)} = \frac{\phi(\lambda)}{2\Phi(-\lambda)} = \lambda$$

Kelley's 27 percent rule,

$$\phi(\lambda) = -2\lambda\Phi(-\lambda)$$

Ref. **RIMS2078-10**（Kyoto Univ. in Nov., 2017）

Bernoulli differential equations for a ratio of a probability density function out of a cumulative distribution function of the standard normal distribution such as

$$\frac{dg_P(u)}{du} + ug_P(u) + g_P(u)^2 = 0, \quad g_P(u) = \frac{\phi(u)}{C+\Phi(u)} \quad \text{and}$$

$$\frac{dg_N(u)}{du} + ug_N(u) - g_N(u)^2 = 0., \quad g_P(u) = \frac{\phi(u)}{C+\Phi(-u)}$$

Probabilistic density function of a standard normal distribution:

$$\phi(u) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)$$

Cumulative distribution function of a standard normal distribution:

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)du$$

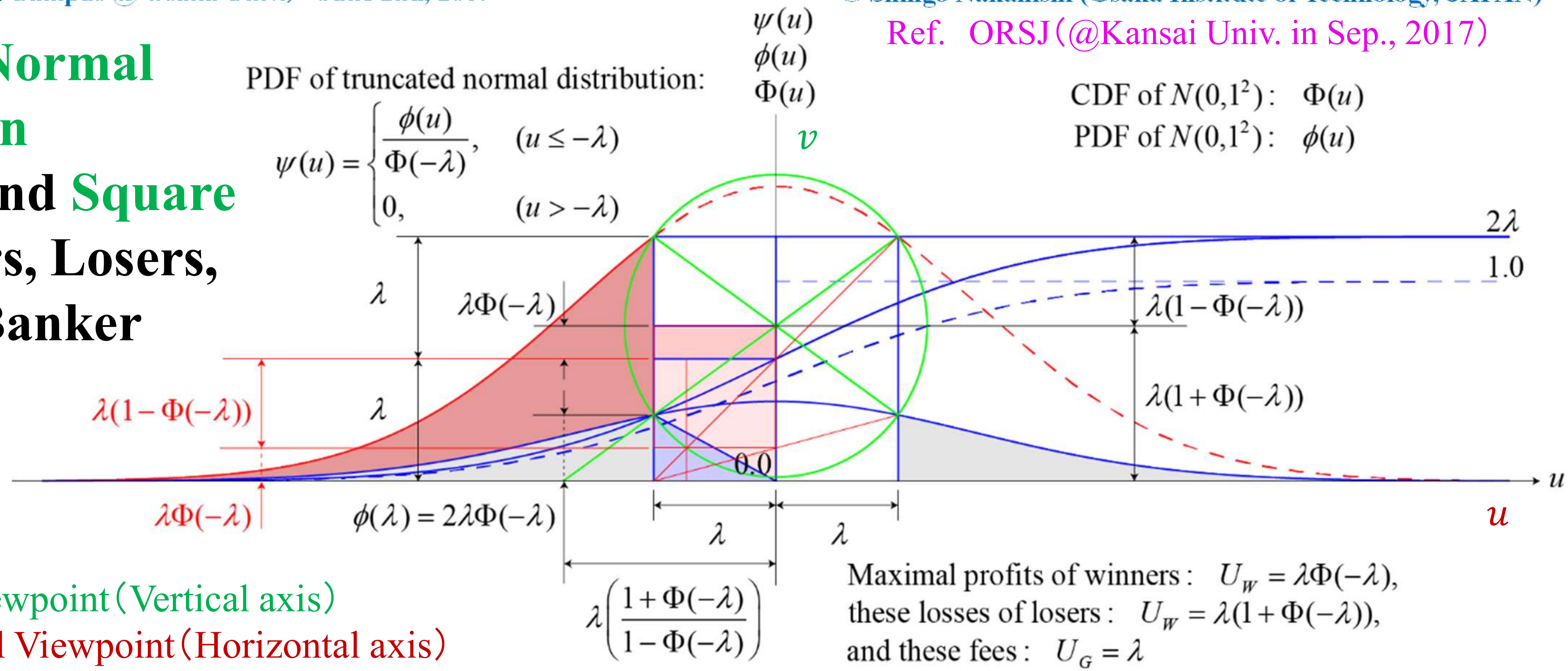
$$\text{or } \Phi(-u) = \int_{-\infty}^{-u} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)du$$

$\lambda = 0.612003$
$\Phi(-\lambda) = 0.2702678$
$\phi(\lambda) = 2\lambda\Phi(-\lambda)$

$\eta = 0.30263084$
$\Phi(-\eta) = 0.3810856$
$\phi(\eta) = \Phi(-\eta)$

Ref. ORSJ（@Kansai Univ. in Sep., 2017）

Standard Normal Distribution by Circle and Square for Winners, Losers, and their Banker



Maximal profits of winners : $U_W=\lambda\Phi(-\lambda)$,
these losses of losers : $U_W=\lambda(1+\Phi(-\lambda))$,
and these fees : $U_G=\lambda$
based on the conditions: fees=λ, standard deviation=1.0

v : Entire Viewpoint（Vertical axis）
u : Individual Viewpoint（Horizontal axis）

Intercept form of a linear equation for winners :

$$-\frac{1}{\lambda}u+\frac{1}{\lambda\Phi(-\lambda)}v=1$$

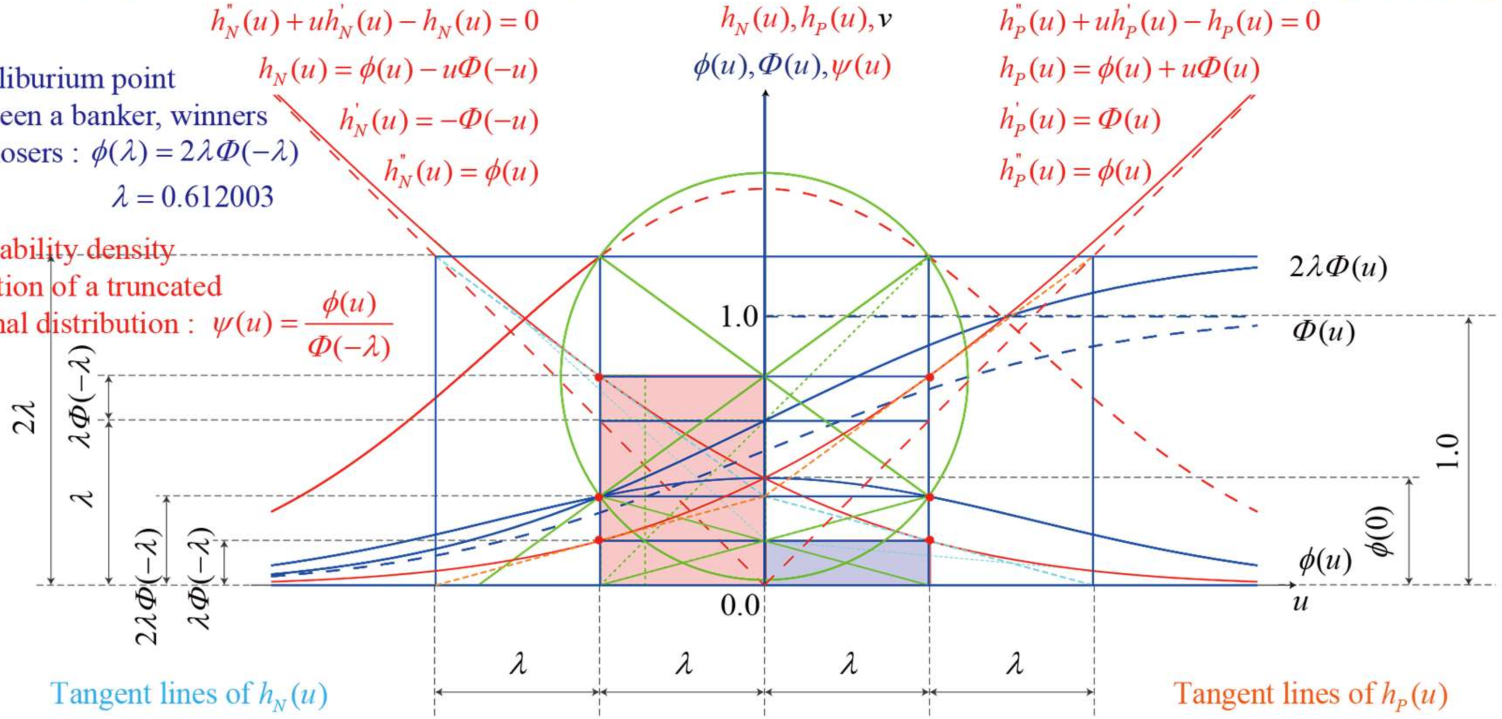
Intercept form of a linear equation for losers :

$$-\frac{1}{\lambda\left(\dfrac{1+\Phi(-\lambda)}{1-\Phi(-\lambda)}\right)}u+\frac{1}{\lambda\left(1+\Phi(-\lambda)\right)}v=1$$

Intercept form of a linear equation for their banker :

$$-\frac{1}{\lambda}u+\frac{1}{\lambda}v=1$$

Variable coefficient type Second-Order Linear Differential Equations about Integrals of CDF of Standard Normal Distribution



Tangent lines of $h_N(u)$

$v = -\Phi(\lambda)u + \phi(\lambda) = -\Phi(\lambda)u + 2\lambda\Phi(-\lambda) \quad (u = -\lambda)$

$v = -\Phi(-\lambda)u + \phi(\lambda) = -\Phi(-\lambda)u + 2\lambda\Phi(-\lambda) \quad (u = \lambda)$

Tangent lines of $h_P(u)$

$v = \Phi(-\lambda)u + \phi(\lambda) = \Phi(-\lambda)u + 2\lambda\Phi(-\lambda) \quad (u = -\lambda)$

$v = \Phi(\lambda)u + \phi(\lambda) = \Phi(\lambda)u + 2\lambda\Phi(-\lambda) \quad (u = \lambda)$

Probability density function of a standard normal distribution : $\phi(u)$
Cumulative distribution function of a standard normal distribution : $\Phi(u)$

Equiliburium point of an inverse Mills ratio : $\dfrac{\phi(\lambda)}{\Phi(-\lambda)} = 2\lambda$

Intercept form of a linear equation for winners : $-\dfrac{1}{\lambda}u + \dfrac{1}{\lambda\Phi(-\lambda)}v = 1$

Intercept form of a linear equation for losers : $-\dfrac{1}{\lambda\left(\dfrac{1+\Phi(-\lambda)}{1-\Phi(-\lambda)}\right)}u + \dfrac{1}{\lambda(1+\Phi(-\lambda))}v = 1$

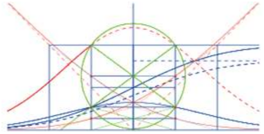
Intercept form of a linear equation for a banker : $-\dfrac{1}{\lambda}u + \dfrac{1}{\lambda}v = 1$

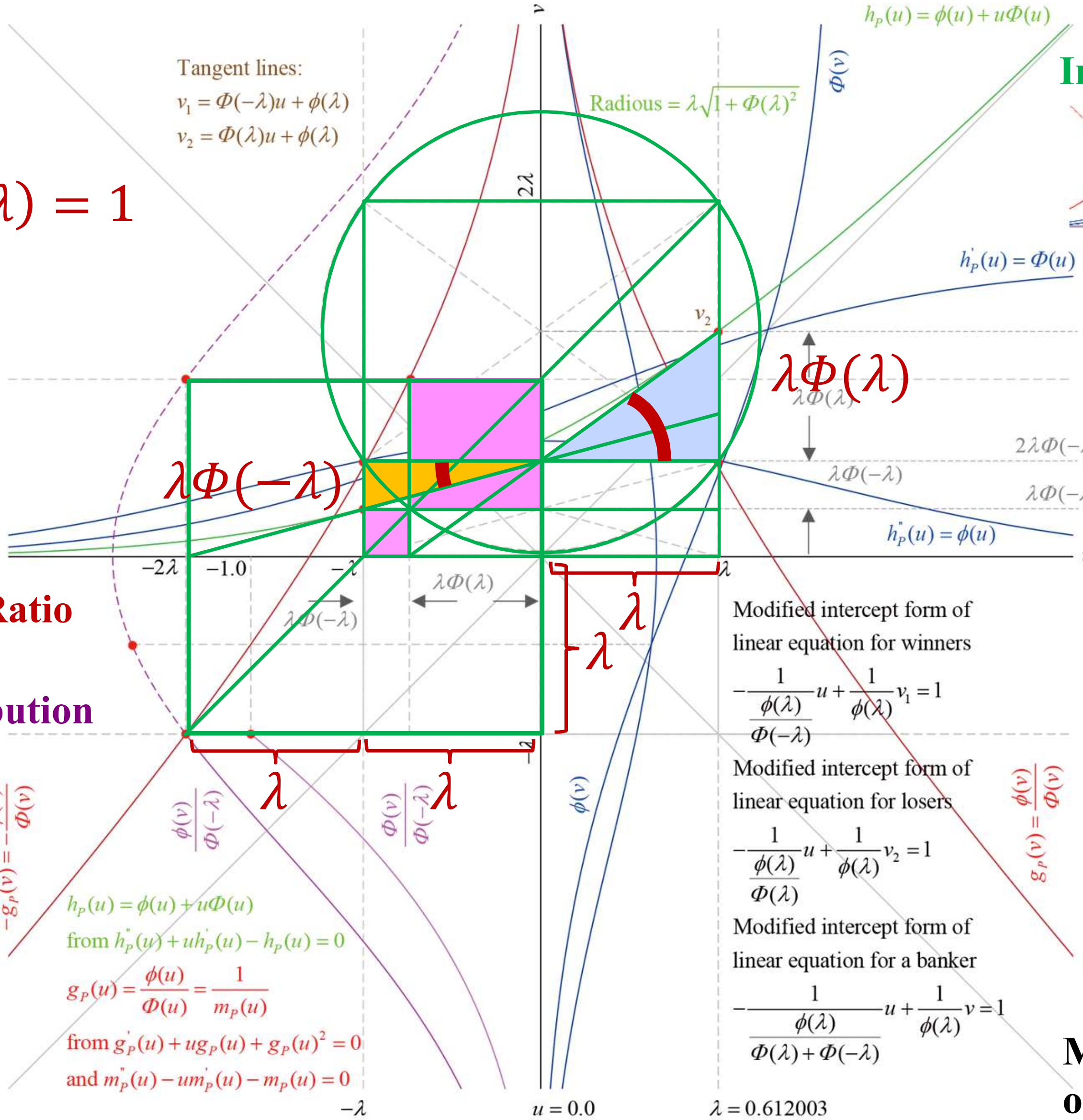
Utility function for winners :
$U_W(t) = (\phi(\lambda) - \lambda\Phi(-\lambda))\sqrt{t} = \lambda\Phi(-\lambda)\sqrt{t}$

Ref. RIMS Modified Version 2078-10（See Research Gate 2019）
Ref. ORSJ Workshop(@National Graduate Institute for Policy Studies, in Nov., 2018.)

$\Phi(-\lambda) + \Phi(\lambda) = 1$

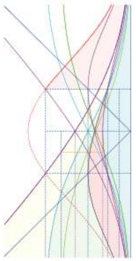
Integrals of CDF





Negative Inverse Mills Ratio

Truncated Normal Distribution

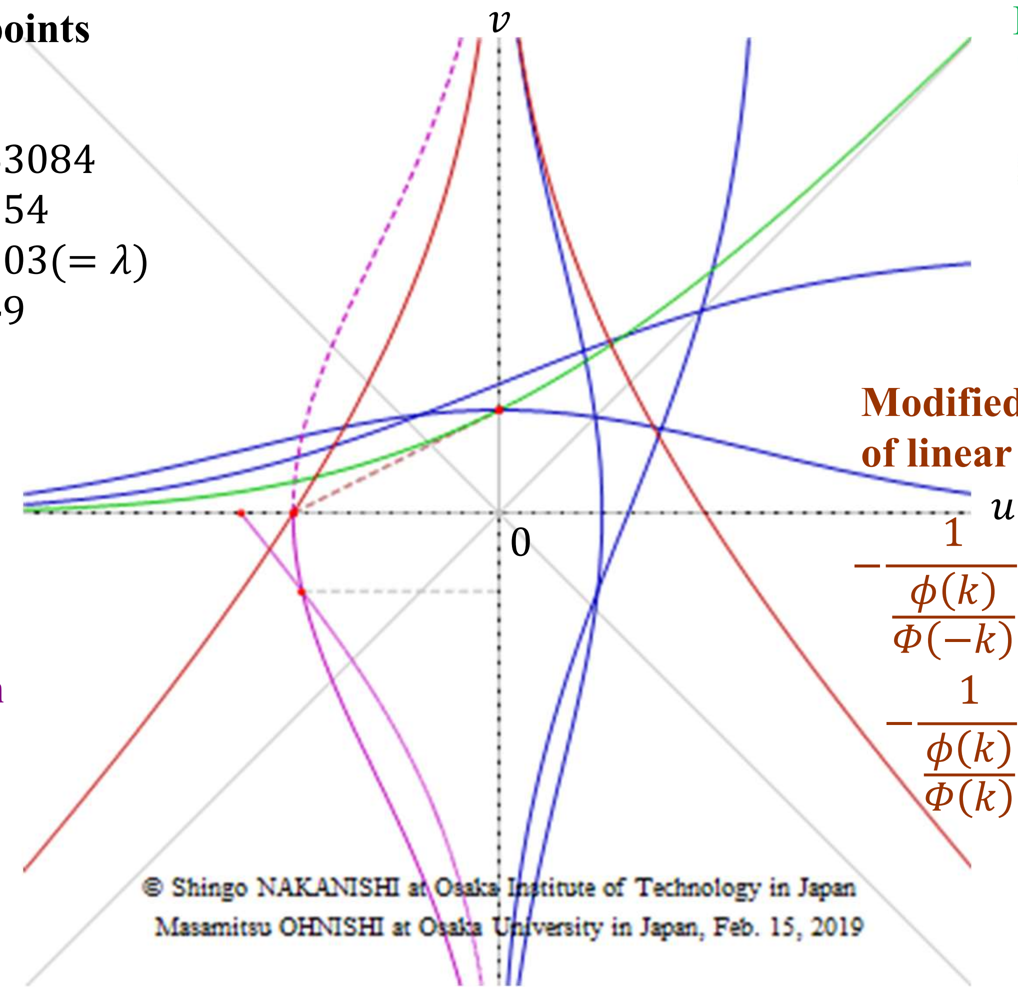


Inverse Mills Ratio

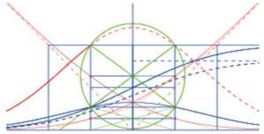
Modified Intercept forms of linear equations

k =Probability points

1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 (= \lambda)$
5. $k = \pm 0.67449$



Integrals of CDF



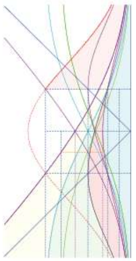
Modified intercept forms of linear equations according to k

$$-\frac{1}{\frac{\phi(k)}{\Phi(-k)}}u + \frac{1}{\phi(k)}v = 1$$

$$-\frac{1}{\frac{\phi(k)}{\Phi(k)}}u + \frac{1}{\phi(k)}v = 1$$

Negative Inverse Mills Ratio

Truncated Normal Distribution



Inverse Mills Ratio

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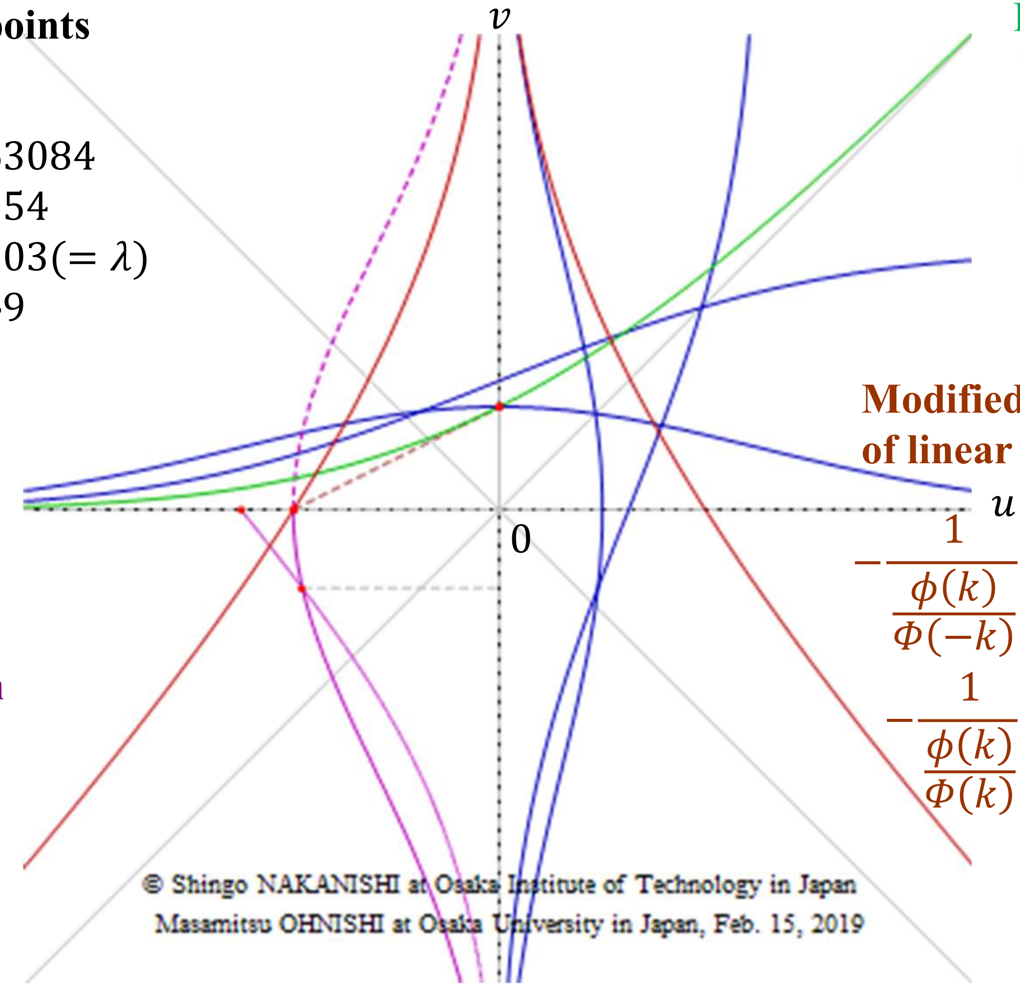
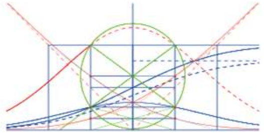


Integrals of CDF



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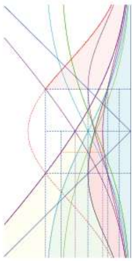
Negative Inverse Mills Ratio

Truncated Normal Distribution



Inverse Mills Ratio

k =**Probability points**

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2. $k = \pm 0.30263084$
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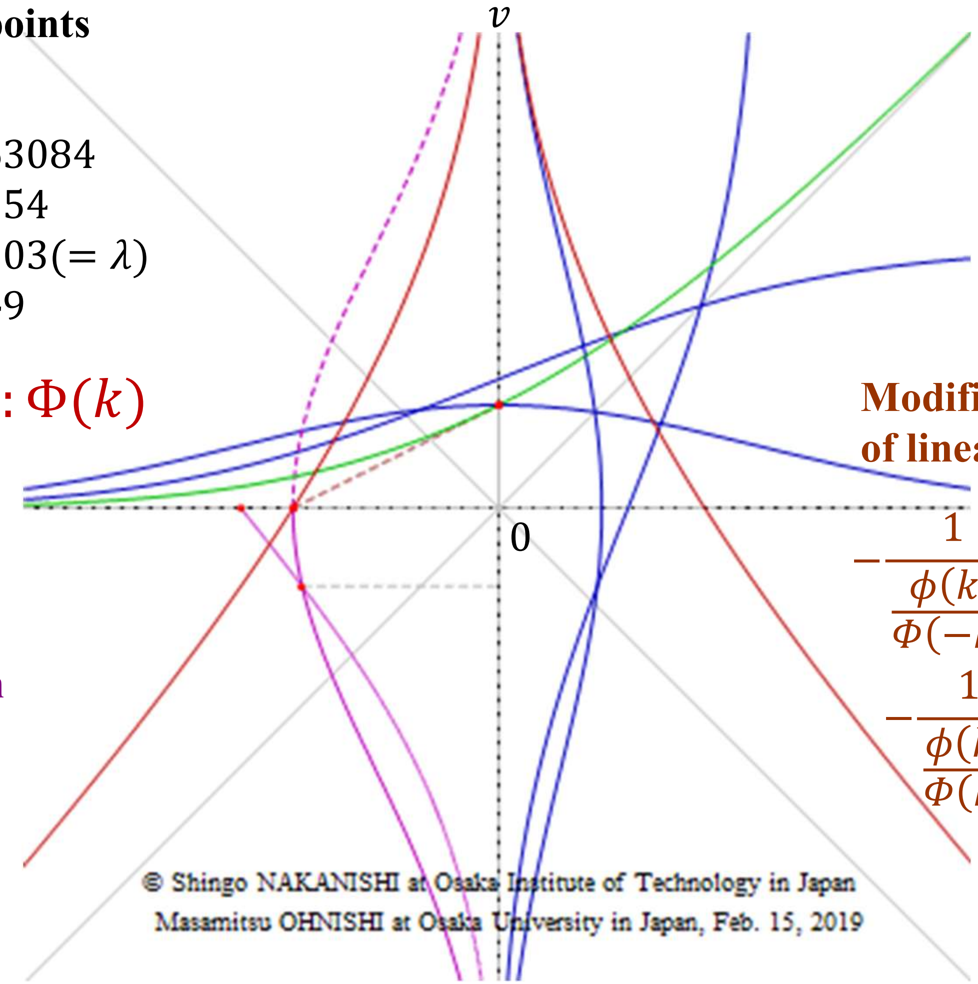
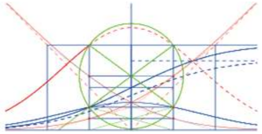
4. $k = \pm 0.612003 (= \lambda)$
5. $k = \pm 0.67449$

The proportion : $\Phi(-k) : \Phi(k)$



Integrals of CDF



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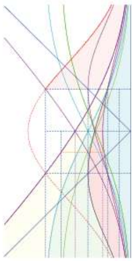
Negative Inverse Mills Ratio

Truncated Normal Distribution



Inverse Mills Ratio

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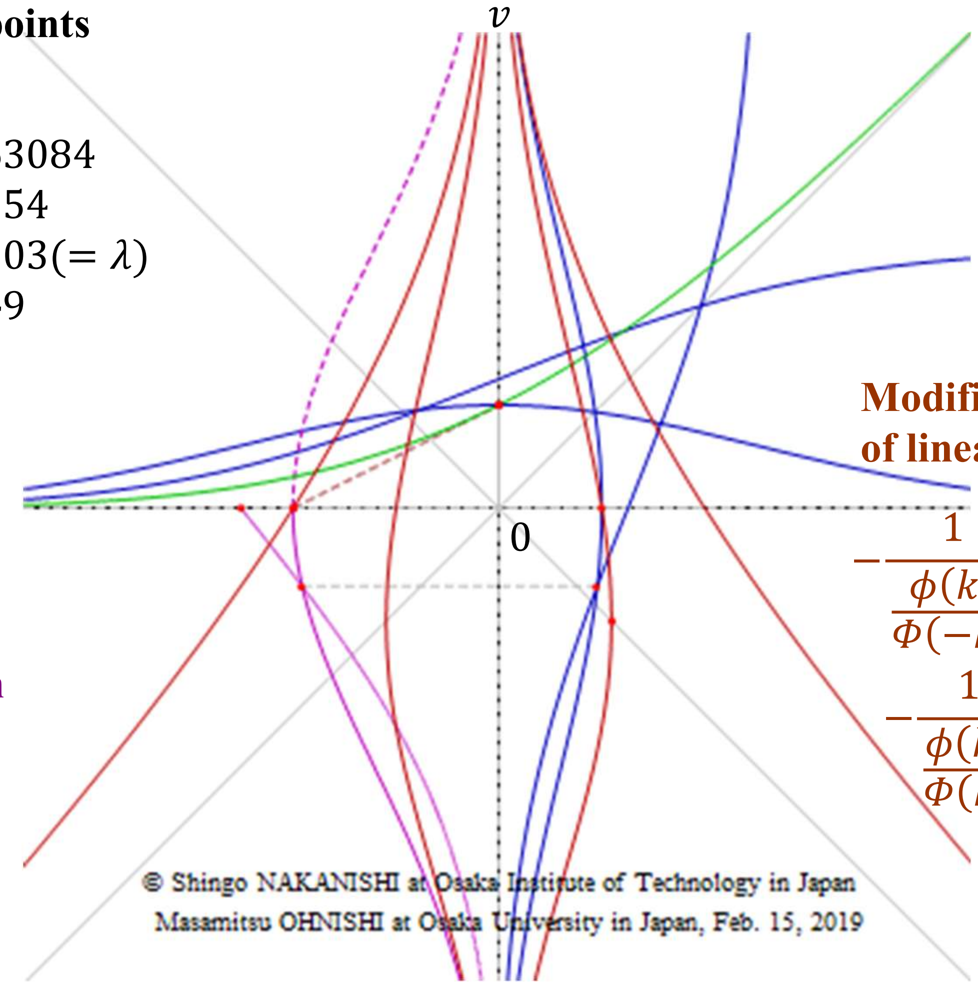

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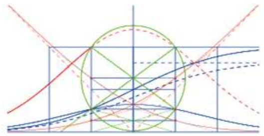
Integrals of CDF



Modified intercept forms of linear equations according to k

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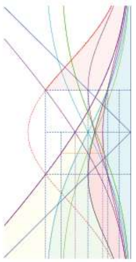
Negative Inverse Mills Ratio

Truncated Normal Distribution



Inverse Mills Ratio

$h_P(u) = \phi(u) + u\Phi(u)$,

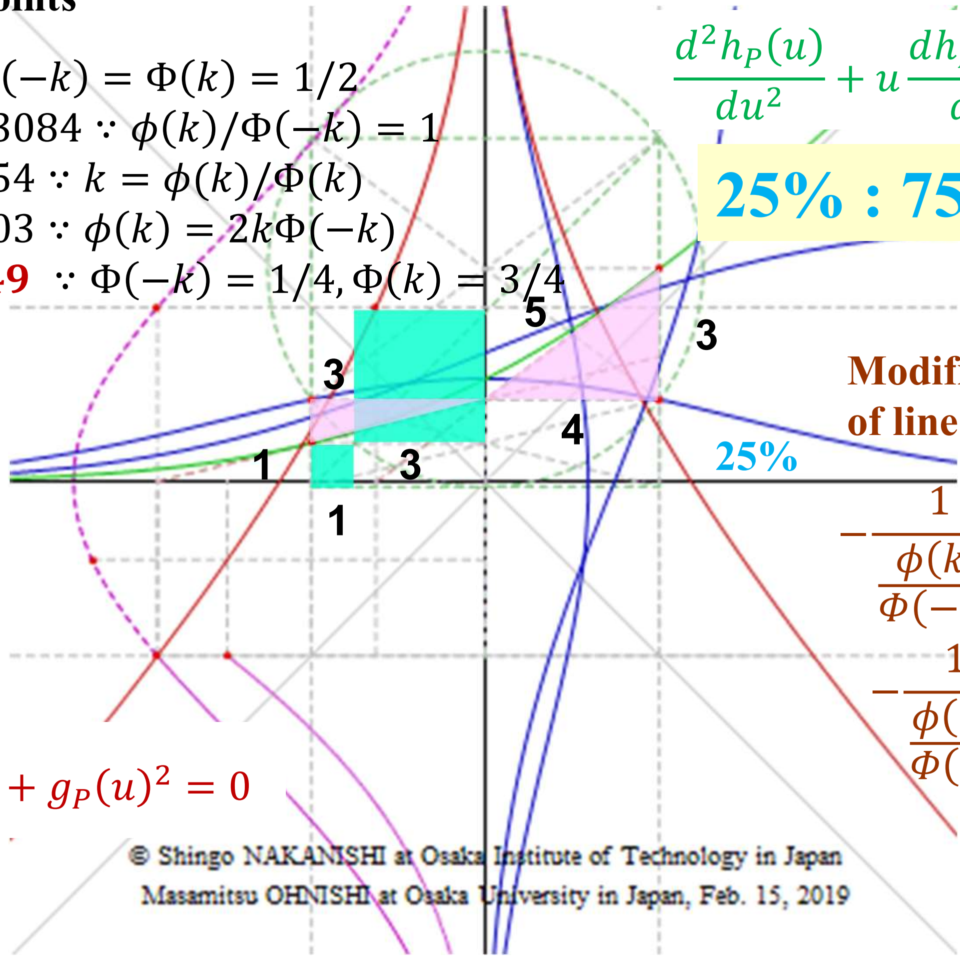
Integrals of CDF

$$\frac{d^2 h_P(u)}{du^2} + u\frac{dh_P(u)}{du} - h_P(u) = 0$$

k =**Probability points**

1. $k = \pm 0.0 \because \Phi(-k) = \Phi(k) = 1/2$
2. $k = \pm 0.30263084 \because \phi(k)/\Phi(-k) = 1$
3. $k = \pm 0.506054 \because k = \phi(k)/\Phi(k)$
4. $k = \pm 0.612003 \because \phi(k) = 2k\Phi(-k)$
5. $k = \pm \mathbf{0.67449} \because \Phi(-k) = 1/4, \Phi(k) = 3/4$

25% : 75%=1/4 : 3/4



$$g_P(u) = \frac{\phi(u)}{\Phi(u)}$$

Negative Inverse Mills Ratio

Truncated Normal Distribution

$$\frac{dg_P(u)}{du} + ug_P(u) + g_P(u)^2 = 0$$

Modified intercept forms of linear equations according to k

$$-\frac{1}{\frac{\phi(k)}{\Phi(-k)}}u + \frac{1}{\phi(k)}v = 1$$

$$-\frac{1}{\frac{\phi(k)}{\Phi(k)}}u + \frac{1}{\phi(k)}v = 1$$

Inverse Mills Ratio

Pythagorean Triangles show the probability as CDF.
Inverse Mills Ratios and PDFs are discribed as Modified Intercept Forms.

Thank you
for your kind attention.

Concluding Remarks

1. Importance of
   Pearson's finding probability point, 0.612003.
2. Symmetric Relations and Geometric Characterizations
   about Two types of Differential Equations
   between Standard Normal Distribution and Inverse Mills Ratio.
3. Greek Pythagorean Theorem about CDF
   and Ancient Egyptian Drawing Styles with Circle and Square.
4. Proposals of Modified Intercept Forms
   for Winners, Losers, and Their Banker.

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