# MODIFIED PASCAL'S TRIANGLES AND MATRICES FOR FEXIBLY CHANGING INITIAL CONSTANTS ABOUT WEIGHTED SKIPPED FIBONACCI, LUCAS, PELL, JACOBSTHAL, OR RELATED SEQENCES

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*Abstract* This paper has been under peer review since January16, 2024. Therefore, I will publish this as the handout including the locations of the modified misspellings. I also informed the academic society about this, but I have not received the peer review results.

This study aims to investigate Fibonacci or Lucas sequence with flexibly changing initial constants called Gibonacci sequence. We consider k-Pell or k-Jacosthal sequences as weighted Fibonacci sequences in this paper. Weighted Lucas sequences mean original Lucas sequence contained k-Pell-Lucas or k-Jacobsthal-Lucas sequences. First, we think weighted Fibonacci sequences using addition theorem of original Fibonacci sequence as the weighted skipped Fibonacci sequences using the difference between skipped and originally initial constants. Second, for adopting flexibly changing the initial constants, we call that the flexible Fibonacci sequence instead of Gibonacci sequence. If we transform that into the extended sequences using above describing addition theorem, these findings provide a route to display the modified Pascal's triangles calculated by using binomial theorem or some metrices. We present how to make modified Pascal's triangles based on the weighted, skipped, and flexible Fibonacci or Lucas sequences. Similarly, we also create extended Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacob-Lucas sequences as weighted skipped flexible sequences precisely. Negative numbers of sequences can be obtained by the modified Pascal's matrices introduced several computing techniques. These findings also demonstrate visualizing flexible Padovan or Perrin sequences.

**Keywords**: Algorithm, metallic ratio, addition theorem, binomial theorem, Padovan sequence, Perrin sequence

## 1. Introduction

Pascal's triangle [15, 7] is well known for the beautiful harmony related to Fibonacci sequence. Green has published that and mentioned to the triangles about Lucas sequence [7] and Pell sequence with duplex triangle [8]. Koshy also has informed us of that included Jacobsthal sequence [11, 12, 13]. About original Padovan sequence and k-Padovan sequences, these are proposed by Willson [27] or Giuseppina, et.al. [6] respectively. The author presents the relations of several modified Pascal's triangles and skipped sequences included negative numbers [16, 17, 18, 19]. Generally, it seems to be admitted the descriptions about modified Pascal's triangle of Lucas sequence by Koshy [11, 12, 13]. However, there appears not to be considered the negative numbers of these sequences even if we can apply Newton's negative binomial theorem to that properly. This study focuses on creating the positive and negative weighted Fibonacci or Lucas sequences using that precisely. The ideas allow to visualize the flexibly changing initial constants and several skipped sequences effectively.

About above mentioned of generalizations of sequences, there seems to be several streams throughout the survey. One is that changing initial constants flexibly appears to be called weighted Fibonacci sequence or Gibonacci sequence [2]. Other is that the first and second coefficients should be fixed $a$ and $b$ for the weighted summation of the terms of sequences. The later manner has been widely connected with k-Pell or k-Jacobsthal sequences [9, 25] respectively. It is likely to be called the Fibonacci sequence with the only first coefficient $k$ k-Pell sequence [21] and the second coefficient $k$ k-Jacobsthal sequence [26]. Increasing the number of terms for the summation with skipped or step orders might be admitted as some of generalizations of the sequences [10]. From above reasons, we stop naming generalizations about the title of this paper to clarify the meaning of generalization of the sequences concisely.

First, we define that the sequences with two coefficients $a$ and $b$ as the weighted sequences from original Fibonacci or Lucas sequences such as one of generalizations. Second, we reconsider the skipped orders of the sequences such as Padovan or Perrin sequences for that of Fibonacci or Lucas sequences. It is indispensable for considering that with coefficients adopting the transformed weights from $a$ and $b$ based on the addition theorem of original Fibonacci sequence. From the weighted and skipped expansions of the sequences, we can name the sequences of the title of this paper the weighted skipped sequences instead of using the generalized sequences. Since $a$ and $b$ are simply used to explain the characterizations instead of the pair $k$ and 1, we need not to call some of the sequences k-Pell or k-Jacobsthal sequences distinguishably. Third, we plan to introduce the method of changing initial constants according to the weighted skipped sequences flexibly as weighted skipped Gibonacci sequences. After investigating that, we can understand the ideas should be also effective to clarify the modified Pascal's triangles systematically. Having displayed the changing initial constants on the modified Pascal's triangles, we can imagine the flows of the initial constants are moving on the conveyor belts like conveyor belt sushi shops to create various modified Pascal's triangles and these related sequences specifically. This is why we aim to explain that in detail from next sections.

Similarly, the Pascal's triangle can be illustrated by using the matrix [4, 5]. We suggest that the above findings enable us to present the various matrices and these visualizations to display the sequences effectively. In the same manner, we have also applied the method to that of Padovan or Perrin sequences [23, 20, 22, 28] with flexibly changing initial constants. We focus on describing these algorithms throughout the following sections.

## 2. Description of Weighted Fibonacci Sequence and Flexibly Changing the Initial Constants

### 2.1. Original and flexible Fibonacci or Lucas numbers instead of Gibonacci sequence

Generally, we can confirm the original Fibonacci numbers as the following equation

$$F_{(1,1),0} = 0, \quad F_{(1,1),1} = 1, \quad F_{(1,1),j} = F_{(1,1),j-1} + F_{(1,1),j-2} \quad (j \geq 2). \tag{2.1}$$

This study aims to describe (2.1) as the other following equation. That is

$$\begin{aligned} P^{(F_{(1,1),0},F_{(1,1),1})}_{(1,1),0} &= 0, \\ P^{(F_{(1,1),0},F_{(1,1),1})}_{(1,1),1} &= 1, \\ P^{(F_{(1,1),0},F_{(1,1),1})}_{(1,1),j} &= P^{(F_{(1,1),0},F_{(1,1),1})}_{(1,1),j-1} + P^{(F_{(1,1),0},F_{(1,1),1})}_{(1,1),j-2} \quad (j \geq 2) \end{aligned} \tag{2.2}$$

where the subscript $(1,1), j$ means the weighted coefficient of the first term as $F_{(1,1),2} = 1$, that of the second term as $F_{(1,1),1} = 1$, and $j$-th order of the Fibonacci numbers, $F_{(1,1),j}$. The superscript $(F_{(1,1),0}, F_{(1,1),1})$ indicates the first initial number, $F_{(1,1),0} = 0$, and the second initial number, $F_{(1,1),1} = 1$, as the initial constants clearly in this sequence. (2.2) has the

same meaning of (2.1) concisely.
The original Lucas numbers are also shown as

$$L_{(1,1),0} = 2, \quad L_{(1,1),1} = 1, \quad L_{(1,1),j} = L_{(1,1),j-1} + L_{(1,1),j-2} \quad (j \geq 2). \tag{2.3}$$

or the other description as the following equation,

$$\begin{aligned} &P_{(1,1),0}^{(L_{(1,1),0},L_{(1,1),1})} = 2, \\ &P_{(1,1),1}^{(L_{(1,1),0},L_{(1,1),1})} = 1, \\ &P_{(1,1),j}^{(L_{(1,1),0},L_{(1,1),1})} = P_{(1,1),j-1}^{(L_{(1,1),0},L_{(1,1),1})} + P_{(1,1),j-2}^{(L_{(1,1),0},L_{(1,1),1})} \quad (j \geq 2), \end{aligned} \tag{2.4}$$

under the same description of (2.2). If we think that the first and second initial constants are flexibly integer numbers ($g_1$ and $g_2$) $\in \mathbb{Z}$ , we can call that the original Gibonacci sequence [2] shown as the following equation

$$G_{(1,1),0} = g_1, \quad G_{(1,1),1} = g_2, \quad G_{(1,1),j} = G_{(1,1),j-1} + G_{(1,1),j-2} \quad (j \geq 2). \tag{2.5}$$

This study plans to change the first and second initial numbers of the original Fibonacci sequence into $i$-th order and $(i-1)$-th order of (2.1) for showing the modified Pascal's triangles in the later sections respectively. We focus on naming that the flexible Fibonacci sequence instead of (2.5) simply. Similarly, we can also describe the flexible Lucas numbers of (2.3) as one of (2.5).
If we consider the first initial number of (2.5) as $F_{(1,1),i-1}$ and the second initial number as $F_{(1,1),i}$, we can describe the flexible Fibonacci sequence as the following equation

$$\begin{aligned} &G_{(1,1),0} = F_{(1,1),i-1}, \quad G_{(1,1),1} = F_{(1,1),i}, \\ &G_{(1,1),j} = F_{(1,1),2} \cdot G_{(1,1),j-1} + F_{(1,1),1} \cdot G_{(1,1),j-2} \quad (j \geq 2). \end{aligned} \tag{2.6}$$

or the other following description such as (2.2). That is

$$\begin{aligned} &P_{(1,1),0}^{(F_{(1,1),i-1},F_{(1,1),i})} = F_{(1,1),i-1}, \\ &P_{(1,1),1}^{(F_{(1,1),i-1},F_{(1,1),i})} = F_{(1,1),i}, \\ &P_{(1,1),j}^{(F_{(1,1),i-1},F_{(1,1),i})} = P_{(1,1),j-1}^{(F_{(1,1),i-1},F_{(1,1),i})} + P_{(1,1),j-2}^{(F_{(1,1),i-1},F_{(1,1),i})} \quad (j \geq 2). \end{aligned} \tag{2.7}$$

Similarly, we can indicate that the flexible Lucas numbers are displayed as

$$\begin{aligned} &G_{(1,1),0} = L_{(1,1),i-1}, \quad G_{(1,1),1} = L_{(1,1),i}, \\ &G_{(1,1),j} = F_{(1,1),2} \cdot G_{(1,1),j-1} + F_{(1,1),1} \cdot G_{(1,1),j-2} \quad (j \geq 2) \end{aligned} \tag{2.8}$$

or the other following description

$$\begin{aligned} &P_{(1,1),0}^{(L_{(1,1),i-1},L_{(1,1),i})} = L_{(1,1),i-1}, \\ &P_{(1,1),1}^{(L_{(1,1),i-1},L_{(1,1),i})} = L_{(1,1),i}, \\ &P_{(1,1),j}^{(L_{(1,1),i-1},L_{(1,1),i})} = P_{(1,1),j-1}^{(L_{(1,1),i-1},L_{(1,1),i})} + P_{(1,1),j-2}^{(L_{(1,1),i-1},L_{(1,1),i})} \quad (j \geq 2). \end{aligned} \tag{2.9}$$

based on (2.4).

### 2.2. Weighted Fibonacci or Lucas sequences with the coefficients $F_{(a,b),2} = a$ and $b \cdot F_{(a,b),1} = b$ and flexibly changing the initial constants

In the previous subsection, having mentioned the case of $a = b = 1$ simply may consider Fibonacci numbers as the original Fibonacci sequence. Strictly, Lucas numbers are not same as Lucas sequence [1, 14] because of $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ if we consider the integer sequences [3] carefully. We attempt to think the original Lucas sequence as the weighted Lucas sequence such as the same naming of weighted Fibonacci sequence to compare various conditions later in this paper. We would also like to consider (2.1) and (2.3) as the following equations with the weighted coefficient $a$ and $b$ in Figures 1 and 2 respectively. These are shown as

$$F_{(a,b),0} = 0, \quad F_{(a,b),1} = 1, \quad F_{(a,b),j} = a \cdot F_{(a,b),j-1} + b \cdot F_{(a,b),j-2} \quad (j \geq 2), \tag{2.10}$$

$$L_{(a,b),0} = 2, \quad L_{(a,b),1} = a, \quad L_{(a,b),j} = a \cdot L_{(a,b),j-1} + b \cdot L_{(a,b),j-2} \quad (j \geq 2) \tag{2.11}$$

or the other following descriptions

$$\begin{aligned} P_{(a,b),0}^{(F_{(a,b),0}, F_{(a,b),1})} &= 0, \\ P_{(a,b),1}^{(F_{(a,b),0}, F_{(a,b),1})} &= 1, \\ P_{(a,b),j}^{(F_{(a,b),0}, F_{(a,b),1})} &= a \cdot P_{(a,b),j-1}^{(F_{(a,b),0}, F_{(a,b),1})} + b \cdot P_{(a,b),j-2}^{(F_{(a,b),0}, F_{(a,b),1})} \quad (j \geq 2), \end{aligned} \tag{2.12}$$

$$\begin{aligned} P_{(a,b),0}^{(L_{(a,b),0}, L_{(a,b),1})} &= 2, \\ P_{(a,b),1}^{(L_{(a,b),0}, L_{(a,b),1})} &= a, \\ P_{(a,b),j}^{(L_{(a,b),0}, L_{(a,b),1})} &= a \cdot P_{(a,b),j-1}^{(L_{(a,b),0}, L_{(a,b),1})} + b \cdot P_{(a,b),j-2}^{(L_{(a,b),0}, L_{(a,b),1})} \quad (j \geq 2). \end{aligned} \tag{2.13}$$

We can deal with the weighted Fibonacci sequence under the condition with the first coefficient $a$ and the second coefficient $b$ as $(F_{(a,b),j}$ or $P_{(a,b),j}^{(F_{(a,b),0}, F_{(a,b),1})}) \in \mathbb{R}$. In the same thinking, we can use the weighted Lucase sequence with $(L_{(a,b),j}$ or $P_{(a,b),j}^{(L_{(a,b),0}, L_{(a,b),1})}) \in \mathbb{R}$ in this paper. If we think of the case $j < 0$ shown in Figures 1 and 2, we can apply the

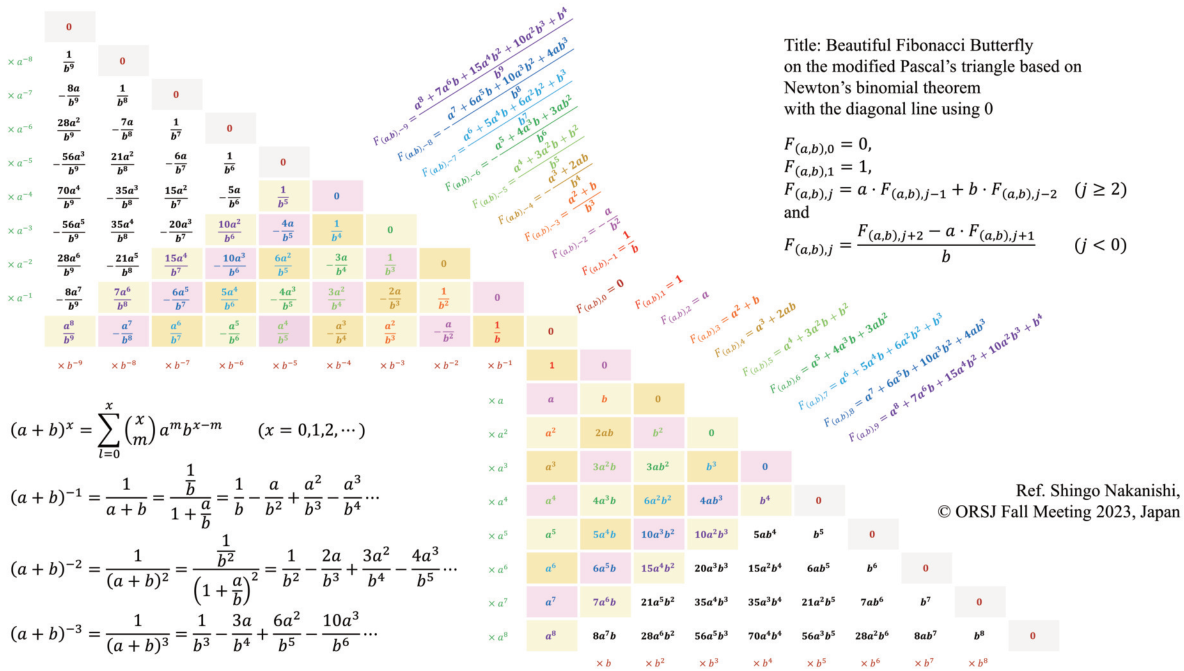


Figure 1: Visualization of the weighted Fibonacci sequences using the modified Pascal's triangle [16]

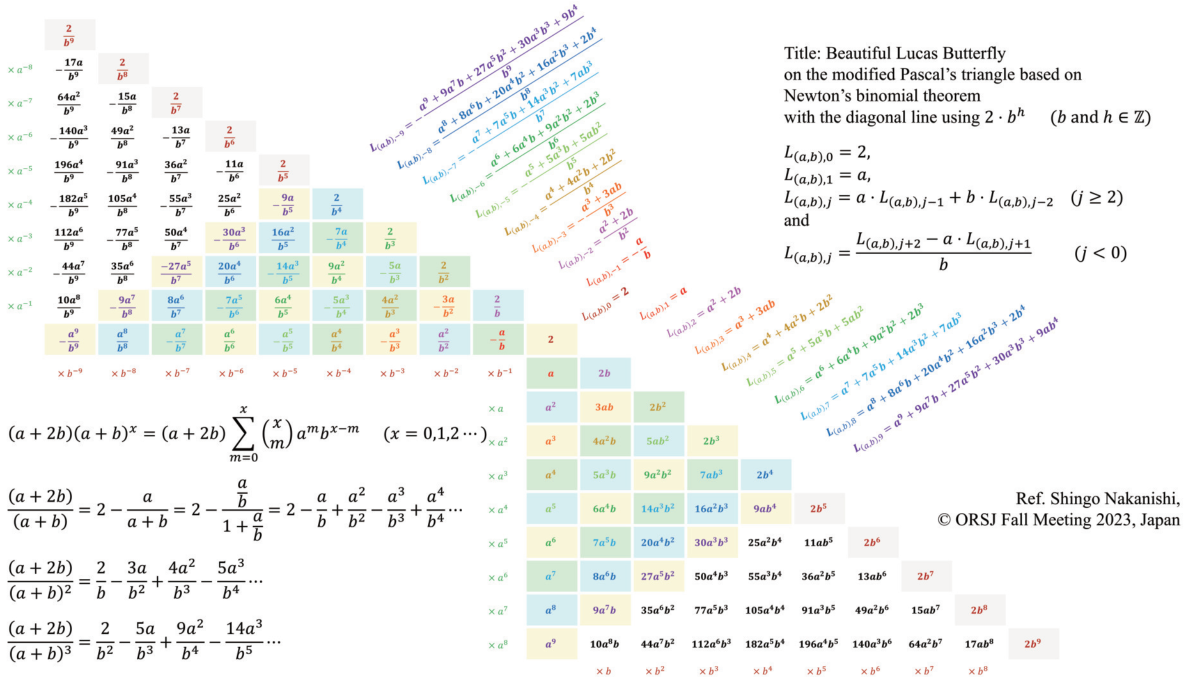


Figure 2: Visualization of the weighted Lucas sequences using the modified Pascal's triangle [16]

binomial theorem by Newton's proposal to these series such as

$$\begin{aligned}
(a+b)^x &= \sum_{m=0}^{x} a^m b^{x-m} \qquad (x=0,1,2,\cdots),\\
(a+b)^{-1} &= \frac{1}{b} - \frac{a}{b^2} + \frac{a^2}{b^3} - \frac{a^3}{b^4} + \frac{a^4}{b^5}\cdots,\\
(a+b)^{-2} &= \frac{1}{b^2} - \frac{2a}{b^3} + \frac{3a^2}{b^4} - \frac{4a^3}{b^5}\cdots,\\
(a+b)^{-3} &= \frac{1}{b^3} - \frac{3a}{b^4} + \frac{6a^2}{b^5} + \frac{10a^3}{b^6}\cdots,\\
&\vdots
\end{aligned} \tag{2.14}$$

for the weighted negative Fibonacci sequence in Figure 1 or

$$\begin{aligned}
(a+2b)(a+b)^x &= (a+2b)\sum_{m=0}^{x} a^m b^{x-m} \qquad (x=0,1,2,\cdots),\\
(a+2b)(a+b)^{-1} &= 2 - \frac{a}{b} + \frac{a^2}{b^2} - \frac{a^3}{b^3} + \frac{a^4}{b^4}\cdots,\\
(a+2b)(a+b)^{-2} &= \frac{2}{b} - \frac{3a}{b^2} + \frac{4a^2}{b^3} - \frac{5a^3}{b^4}\cdots,\\
(a+2b)(a+b)^{-3} &= \frac{2}{b^2} - \frac{5a}{b^3} + \frac{9a^2}{b^4} - \frac{14a^3}{b^5}\cdots,\\
&\vdots
\end{aligned} \tag{2.15}$$

for the weighted negative Lucas sequence in Figure 2 .

It is natural to use Newton's negative binomial theorem properly for visualizations of modified Pascal's triangles since the sequences are proceeding for both positive and negative numbers of that. The approach using the negative Lucas sequence and related Pascal's triangle in Figure 2 has not been known. Because of this, we focus on adding the equations of the sequences as some of the beautiful arts shown in Figures 1 and 2. From (2.10) and (2.11), we can redefine (2.5) as the following weighted equation

$$G_{(a,b),0} = g_1, \quad G_{(a,b),1} = g_2, \quad G_{(a,b),j} = a \cdot G_{(a,b),j-1} + b \cdot G_{(a,b),j-2} \quad (j \geq 2). \tag{2.16}$$

By using (2.12) and (2.13), (2.16) can be rewritten as

$$\begin{aligned}
P_{(a,b),0}^{(F_{(a,b),i-1},F_{(a,b),i})} &= F_{(a,b),i-1},\\
P_{(a,b),1}^{(F_{(a,b),i-1},F_{(a,b),i})} &= F_{(a,b),i},\\
P_{(a,b),j}^{(F_{(a,b),i-1},F_{(a,b),i})} &= a \cdot P_{(a,b),j-1}^{(F_{(a,b),i-1},F_{(a,b),i})} + b \cdot P_{(a,b),j-2}^{(F_{(a,b),i-1},F_{(a,b),i})} \quad (j \geq 2),
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
P_{(a,b),0}^{(L_{(a,b),i-1},L_{(a,b),i})} &= L_{(a,b),i-1},\\
P_{(a,b),1}^{(L_{(a,b),i-1},L_{(a,b),i})} &= L_{(a,b),i},\\
P_{(a,b),j}^{(L_{(a,b),i-1},L_{(a,b),i})} &= a \cdot P_{(a,b),j-1}^{(L_{(a,b),i-1},L_{(a,b),i})} + b \cdot P_{(a,b),j-2}^{(L_{(a,b),i-1},L_{(a,b),i})} \quad (j \geq 2).
\end{aligned} \tag{2.18}$$

We focus on calling (2.17) or (2.18) the weighted Fibonacci or Lucas sequences with flexibly changing the initial constants respectively in this paper.

## 3. Description of Skipped Weighted Fibonacci or Lucas Sequences and Flexibly Changing the Initial Constants

### 3.1. Description of skipped weighted Fibonacci or Lucas sequences

In subsection 2.2, it is dealt with the weighted Fibonacci or Lucas sequences with the coefficients $a$ and $b$. In this subsection, we require to use the addition theorem of the Fibonacci sequence for creating skipped sequences. If we think that the addition theorem of Fibonacci or Lucas sequences are shown as the following equation

$$F_{(a,b),j} = (F_{(a,b),k}) \cdot F_{(a,b),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot F_{(a,b),j-k} \quad (j \geq k) \tag{3.1}$$

and

$$L_{(a,b),j} = (F_{(a,b),k}) \cdot L_{(a,b),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot L_{(a,b),j-k} \quad (j \geq k), \tag{3.2}$$

we can describe the following equations such as (3.1) and (3.2). Those are rewritten as

$$\begin{aligned} &P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j} \\ &= (F_{(a,b),k}) \cdot P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-k} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j} \\ &= (F_{(a,b),k}) \cdot P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-k} \end{aligned} \tag{3.4}$$

where the subscript of $P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j}$ means the first or second coefficients, the superscript the initial constants from $F_{(a,b),0}$ to $F_{(a,b),k-1}$. Similarly, the subscript of $P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j}$ means the first or second coefficients, the superscript the initial constants from $L_{(a,b),0}$ to $L_{(a,b),k-1}$. About (3.3), we focus on describing the first coefficient $(F_{(a,b),k})$ and the second coefficient $(F_{(a,b),k-1})$ by using both the left and right parentheses respectively because this method is an indispensable tool for creating the skipped sequences and these modified Pascal's triangles. From (3.1) and (3.3), we can define the $(k-2)$ skipped weighted Fibonacci sequence as follows. That is

$$\begin{aligned} &P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),0} = F_{(a,b),0} = 0, \\ &P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),1} = F_{(a,b),1} = 1, \\ &\quad \vdots \\ &P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),k-1} = F_{(a,b),k-1}, \\ &P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j} \\ &= (F_{(a,b),k}) \cdot P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot P^{(F_{(a,b),0},\cdots,F_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-k}, \\ &\qquad (k \geq 2, j \geq k). \end{aligned} \tag{3.5}$$

Similarly, from (3.4), we can also suggest the $(k-2)$ skipped weighted Lucas sequence. That is

$$P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),0} = L_{(a,b),0} = 2,$$
$$P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),1} = L_{(a,b),1} = a,$$
$$\vdots$$
$$P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),k-1} = L_{(a,b),k-1},$$
$$P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j}$$
$$= (F_{(a,b),k}) \cdot P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-(k-1)} + (b \cdot F_{(a,b),k-1}) \cdot P^{(L_{(a,b),0},\cdots,L_{(a,b),k-1})}_{(F_{(a,b),k},b\cdot F_{(a,b),k-1}),j-k}, \quad (k \geq 2, j \geq k). \tag{3.6}$$

(3.5) and (3.6) illustrate the one or two skipped types of weighted Fibonacci or Lucas sequences shown in Figures 3, 4, and, 5 precisely for using the identifiers $\alpha$ and $\beta$ to make the various modified Pascal's triangles. Figure 3 indicates the illustrated Fibonacci or Lucas sequences in the case of $a = 1, b = 1$. Figure 4 shows the illustrated Pell or Pell-Lucas sequences in the case of $a = 2, b = 1$. From Figure 5, we can obtain the illustrated Jacobsthal or Jacobsthal-Lucas sequences concisely ($a = 1, b = 2$). At this time, we can understand the relatedly modified Pascal's triangles are verified by using the following relations. These are the modified Pascal's triangles for the skipped weighted Fibonacci sequences such as

$$1 \cdot (F_{(a,b),2} \cdot \alpha + F_{(a,b),1} \cdot \beta)^x \quad \text{for the no skipped weighted Fibonacci sequence,}$$
$$(F_{(a,b),1} \cdot \alpha + F_{(a,b),2} \cdot \beta)$$
$$\cdot (F_{(a,b),3} \cdot \alpha + F_{(a,b),2} \cdot \beta)^x \quad \text{for the one skipped weighted Fibonacci sequence,}$$
$$(F_{(a,b),1} \cdot \alpha^2 + F_{(a,b),2} \cdot \alpha\beta + F_{(a,b),3} \cdot \beta^2)$$
$$\cdot (F_{(a,b),4} \cdot \alpha + F_{(a,b),3} \cdot \beta)^x \quad \text{for the two skipped weighted Fibonacci sequence,}$$
$$\vdots$$
$$(F_{(a,b),1} \cdot \alpha^{k-2} + F_{(a,b),1} \cdot \alpha^{k-3}\beta + \cdots + F_{(a,b),k-2} \cdot \alpha\beta^{k-3} + F_{(a,b),k-1} \cdot \beta^{k-2})$$
$$(F_{(a,b),k} \cdot \alpha + F_{(a,b),k-1} \cdot \beta)^x \quad \text{for the } (k-2) \text{ skipped weighted Fibonacci sequence.} \tag{3.7}$$

In the same way, we can get the modified Pascal's triangles for the skipped weighted Lucas sequences as follows

$$
\begin{aligned}
&(L_{(a,b),1}\cdot\alpha + b\cdot L_{(a,b),0}\cdot\beta)\\
&\cdot(F_{(a,b),2}\cdot\alpha + F_{(a,b),1}\cdot\beta)^x \quad \text{for the no skipped weighted Lucas sequence,}\\
&(L_{(a,b),1}\cdot\alpha + b\cdot L_{(a,b),0}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha + F_{(a,b),2}\cdot\beta)\\
&\cdot(F_{(a,b),3}\cdot\alpha + F_{(a,b),2}\cdot\beta)^x \quad \text{for the one skipped weighted Lucas sequence,}\\
&(L_{(a,b),1}\cdot\alpha + b\cdot L_{(a,b),0}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha^2 + F_{(a,b),2}\cdot\alpha\beta + F_{(a,b),3}\cdot\beta^2)\\
&\cdot(F_{(a,b),4}\cdot\alpha + F_{(a,b),3}\cdot\beta)^x \quad \text{for the two skipped weighted Lucas sequence,}\\
&\vdots\\
&(L_{(a,b),1}\cdot\alpha + b\cdot L_{(a,b),0}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha^{k-2} + F_{(a,b),1}\cdot\alpha^{k-3}\beta + \cdots + F_{(a,b),k-2}\cdot\alpha\beta^{k-3} + F_{(a,b),k-1}\cdot\beta^{k-2})\\
&(F_{(a,b),k}\cdot\alpha + F_{(a,b),k-1}\cdot\beta)^x \quad \text{for the } (k-2) \text{ skipped weighted Lucas sequence.} \qquad (3.8)
\end{aligned}
$$

(3.7) and (3.8) show the modified Pascal's triangles in Figure 3 as one or two skipped Fibonacci or Lucas sequences in the case of $((k-2) = 1 \text{ or } 2, a = 1, \text{ and } b = 1)$. In the same manner, we can confirm the similar tendencies in Figure 4 about one or two skipped Pell or Pell-Lucas sequences in the case of $((k-2) = 1 \text{ or } 2, a = 2, \text{ and } b = 1)$, and in Figure 5 as that of Jacobsthal or Jacobsthal-Lucas sequences in the case of $((k-2) = 1 \text{ or } 2, a = 1, \text{ and } b = 2)$ respectively These concepts can be illustrated in Figures 6 and 7

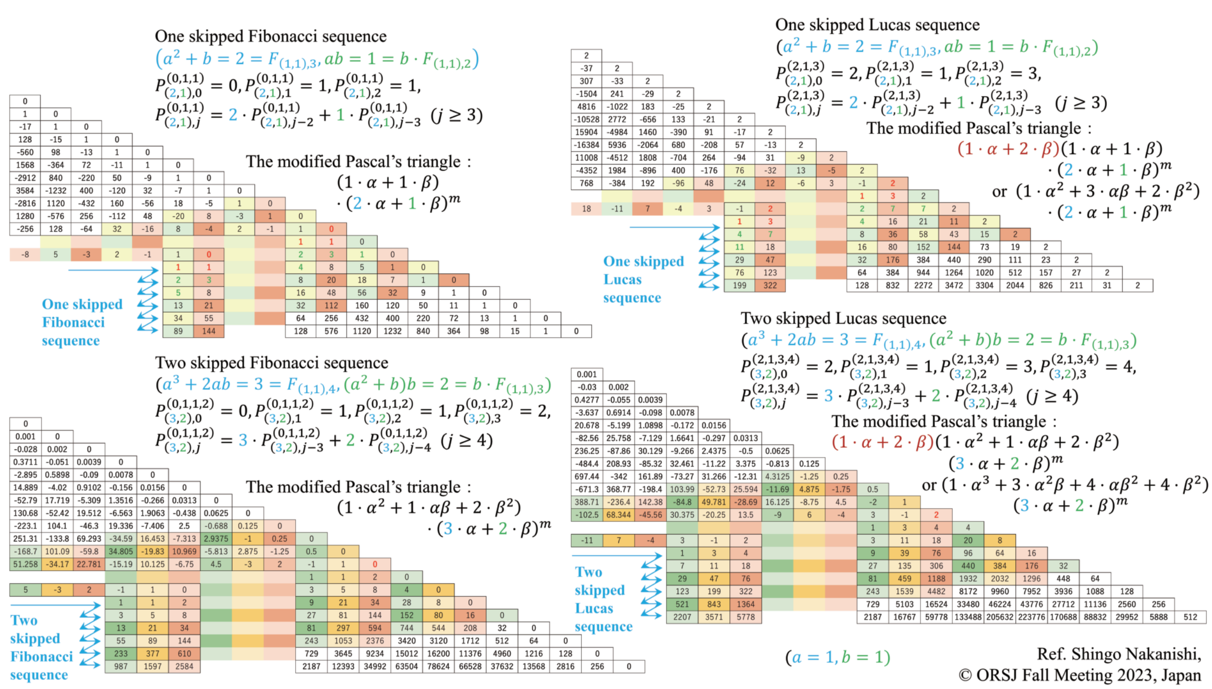


Figure 3: Visualization of the one or two skipped Fibonacci or Lucas sequences using the modified Pascal's triangles [16, 17]

to undestand that easily. First, we can set the initial constants, $(i = 0, \cdots, k-1)$, shown in

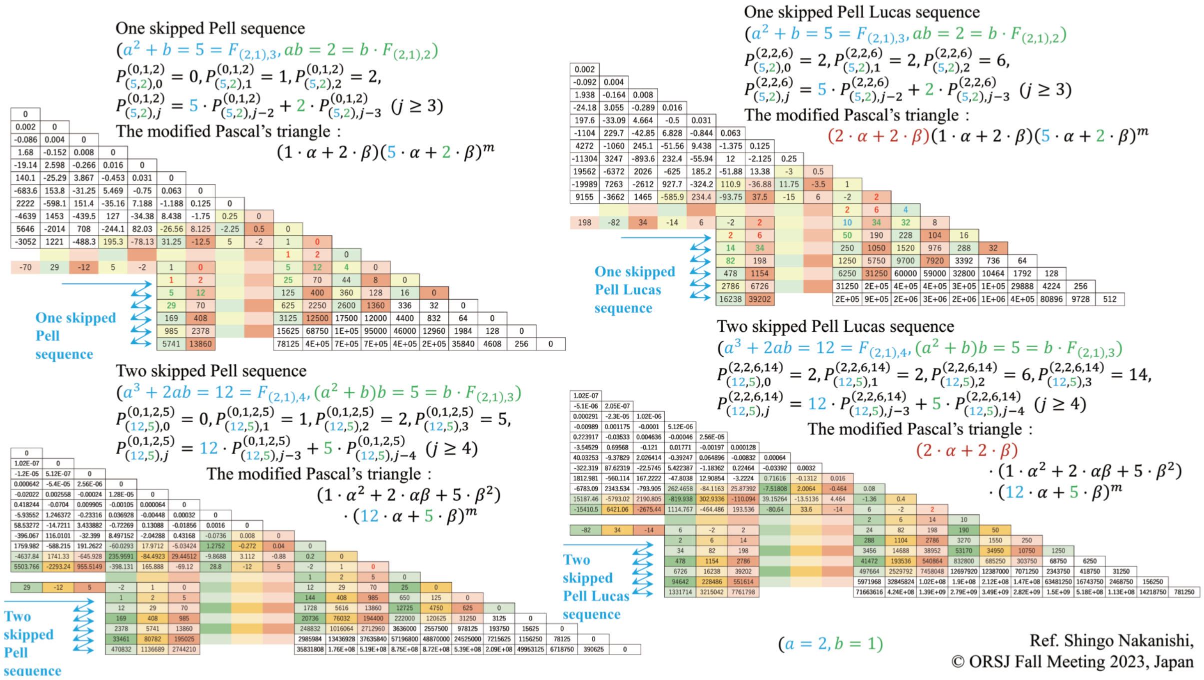


Figure 4: Visualization of the one or two skipped Pell or Pell-Lucas sequences using the modified Pascal's triangles [16, 17]

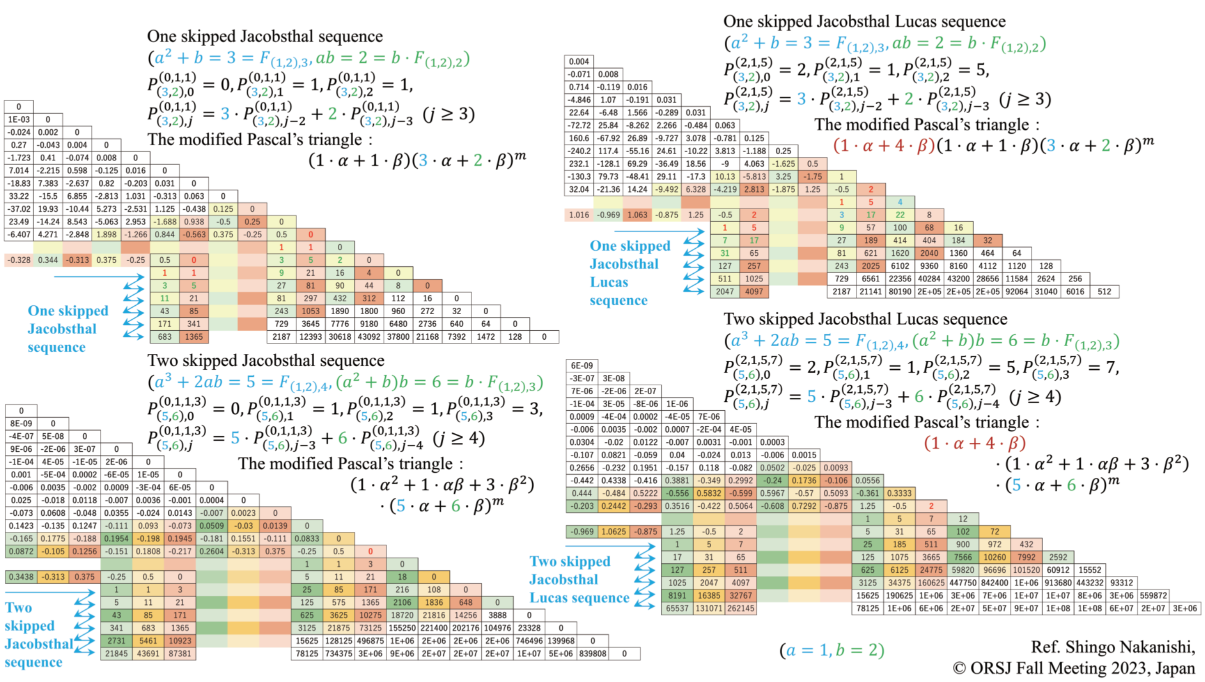


Figure 5: Visualization of the one or two skipped Jacobsthal or Jacobsthal-Lucas sequences using the modified Pascal's triangles [16, 17]

Table 1: Illustrative examples for no, one, or two skipped Fibonacci sequences [16, 17]

| Numbers | No skipped Fibonacci sequence | One skipped Fibonacci sequence | Two skipped Fibonacci sequence |
|---|---|---|---|
| 1 | $(1)\times 1+(1)\times 0$ | $(2)\times 0+(1)\times 1$ | $(3)\times 1+(2)\times -1$ |
| 2 | $(1)\times 1+(1)\times 1$ | $(2)\times 1+(1)\times 0$ | $(3)\times 0+(2)\times 1$ |
| 3 | $(1)\times 2+(1)\times 1$ | $(2)\times 1+(1)\times 1$ | $(3)\times 1+(2)\times 0$ |
| 5 | $(1)\times 3+(1)\times 2$ | $(2)\times 2+(1)\times 1$ | $(3)\times 1+(2)\times 1$ |
| 8 | $(1)\times 5+(1)\times 3$ | $(2)\times 3+(1)\times 2$ | $(3)\times 2+(2)\times 1$ |
| 13 | $(1)\times 8+(1)\times 5$ | $(2)\times 5+(1)\times 3$ | $(3)\times 3+(2)\times 2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $F_{(1,1),j}$ | $(1)F_{(1,1),j-1}+(1)F_{(1,1),j-2}$ | $(2)F_{(1,1),j-2}+(1)F_{(1,1),j-3}$ | $(3)F_{(1,1),j-3}+(2)F_{(1,1),j-4}$ |

Note: $(F_{(1,1),2}=(1), F_{(1,1),1}=(1))$, $(F_{(1,1),3}=(2), F_{(1,1),2}=(1))$, $(F_{(1,1),4}=(3), F_{(1,1),3}=(2))$

Figures 6 and 7. Second, we can consider the weights of the sequences shown in Figures 6 and 7 to create the modified Pascal's triangles systematically. In case of Figure 7, we can admit the diagonal from the first initial constant. In the same way, we can confirm how to calculate some skipped Fibonacci sequences in Table 1 to reconfirm that more easily. In the next subsection, this idea should be crucial for expanding this model with flexibly changing the initial constants.

### 3.2. $(k-2)$ skipped weighted Fibonacci or Lucas sequences and flexibly changing the initial constants

Based on (3.7), we can obtain the modified Pascal's triangles for the skipped weighted Fibonacci sequences with flexibly changing the initial constants using $F_{(a,b),i-1}$ and $F_{(a,b),i}$ as follows

$$\begin{aligned}
&(F_{(a,b),i}\cdot\alpha + b\cdot F_{(a,b),i-1}\cdot\beta)\\
&\cdot(F_{(a,b),2}\cdot\alpha + F_{(a,b),1}\cdot\beta)^x \quad \text{for the no skipped weighted Fibonacci sequence,}\\
&(F_{(a,b),i}\cdot\alpha + b\cdot F_{(a,b),i-1}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha + F_{(a,b),2}\cdot\beta)\\
&\cdot(F_{(a,b),3}\cdot\alpha + F_{(a,b),2}\cdot\beta)^x \quad \text{for the one skipped weighted Fibonacci sequence,}\\
&(F_{(a,b),i}\cdot\alpha + b\cdot F_{(a,b),i-1}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha^2 + F_{(a,b),2}\cdot\alpha\beta + F_{(a,b),3}\cdot\beta^2)\\
&\cdot(F_{(a,b),4}\cdot\alpha + F_{(a,b),3}\cdot\beta)^x \quad \text{for the two skipped weighted Fibonacci sequence,}\\
&\quad\vdots\\
&(F_{(a,b),i}\cdot\alpha + b\cdot F_{(a,b),i-1}\cdot\beta)\\
&(F_{(a,b),1}\cdot\alpha^{k-2} + F_{(a,b),1}\cdot\alpha^{k-3}\beta + \cdots + F_{(a,b),k-2}\cdot\alpha\beta^{k-3} + F_{(a,b),k-1}\cdot\beta^{k-2})\\
&(F_{(a,b),k}\cdot\alpha + F_{(a,b),k-1}\cdot\beta)^x \quad \text{for the } (k-2) \text{ skipped weighted Fibonacci sequence.} \qquad (3.9)
\end{aligned}$$

Based on (3.8), we can demonstrate the modified Pascal's triangles for the skipped weighted Lucas sequences with flexibly changing the initial constants using $L_{(a,b),i-1}$ and $L_{(a,b),i}$ as follows. That is, the flexibly changing the initial constants of skipped weighted Lucas sequences $L_{(a,b),i-1}$ and $L_{(a,b),i}$ can bring us the related modified Pascal's triangles as

$$
\begin{aligned}
&(L_{(a,b),i} \cdot \alpha + b \cdot L_{(a,b),i-1} \cdot \beta) \\
&\cdot (F_{(a,b),2} \cdot \alpha + F_{(a,b),1} \cdot \beta)^x \quad \text{for the no skipped weighted Lucas sequence,} \\
&(L_{(a,b),i} \cdot \alpha + b \cdot L_{(a,b),i-1} \cdot \beta) \\
&(F_{(a,b),1} \cdot \alpha + F_{(a,b),2} \cdot \beta) \\
&\cdot (F_{(a,b),3} \cdot \alpha + F_{(a,b),2} \cdot \beta)^x \quad \text{for the one skipped weighted Lucas sequence,} \\
&(L_{(a,b),i} \cdot \alpha + b \cdot L_{(a,b),i-1} \cdot \beta) \\
&(F_{(a,b),1} \cdot \alpha^2 + F_{(a,b),2} \cdot \alpha\beta + F_{(a,b),3} \cdot \beta^2) \\
&\cdot (F_{(a,b),4} \cdot \alpha + F_{(a,b),3} \cdot \beta)^x \quad \text{for the two skipped weighted Lucas sequence,} \\
&\quad \vdots \\
&(L_{(a,b),i} \cdot \alpha + b \cdot L_{(a,b),i-1} \cdot \beta) \\
&(F_{(a,b),1} \cdot \alpha^{k-2} + F_{(a,b),1} \cdot \alpha^{k-3}\beta + \cdots + F_{(a,b),k-2} \cdot \alpha\beta^{k-3} + F_{(a,b),k-1} \cdot \beta^{k-2}) \\
&(F_{(a,b),k} \cdot \alpha + F_{(a,b),k-1} \cdot \beta)^x \quad \text{for the } (k-2) \text{ skipped weighted Lucas sequence.} \qquad (3.10)
\end{aligned}
$$

From (3.9) and (3.10), we can understand the systems of flexibly changing the initial con-

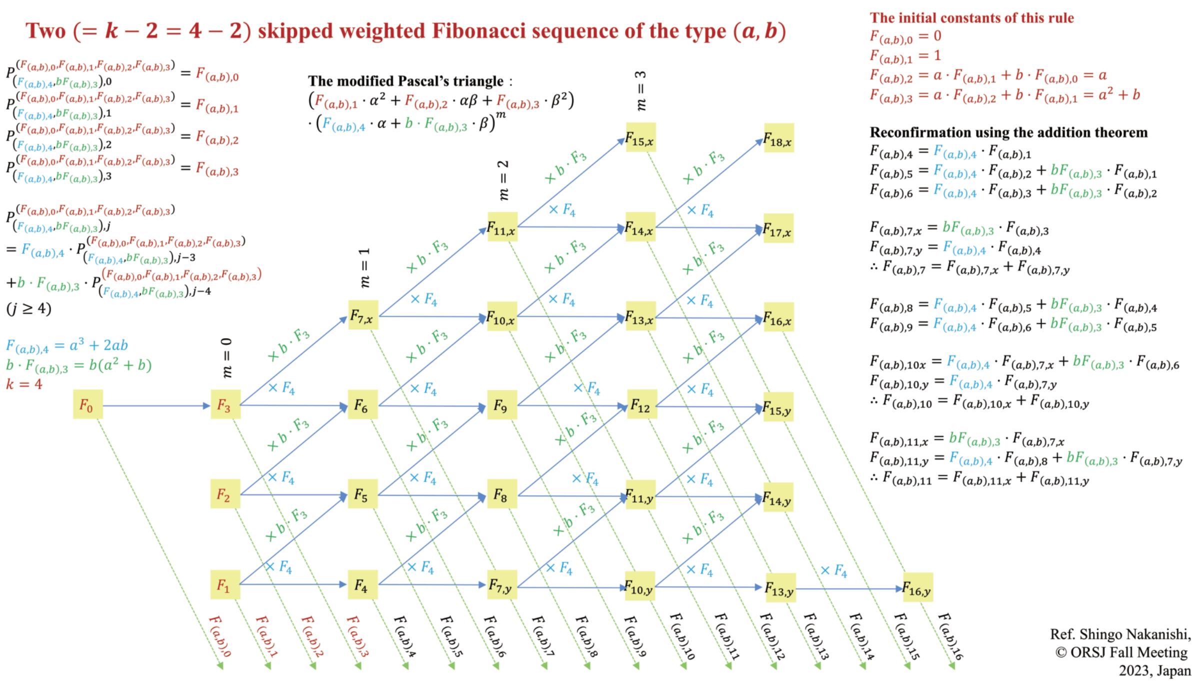


Figure 6: Concepts of the one skipped weighted Fibonacci sequence using the modified Pascal's triangle [17]

stants of the modified Pascal's triangles. It can create that with these conditions as skipped weighed Gibonacci sequences in Figure 8 instead of Figures 6 and 7. From these visualizations, we can imagine the summation for the diagonals as the extended knight moving on the modified Pascal's triangles [6] to obtain the flexibly proper numbers of skipped weighted Fibonacci or Lucas sequences specifically. Changing the initial constants according to the order of sequences enables much attractive visualizations shown in Figures 9 and 10 if we confirm to alter the constants along the flow on the conveyor belts in these Figures.

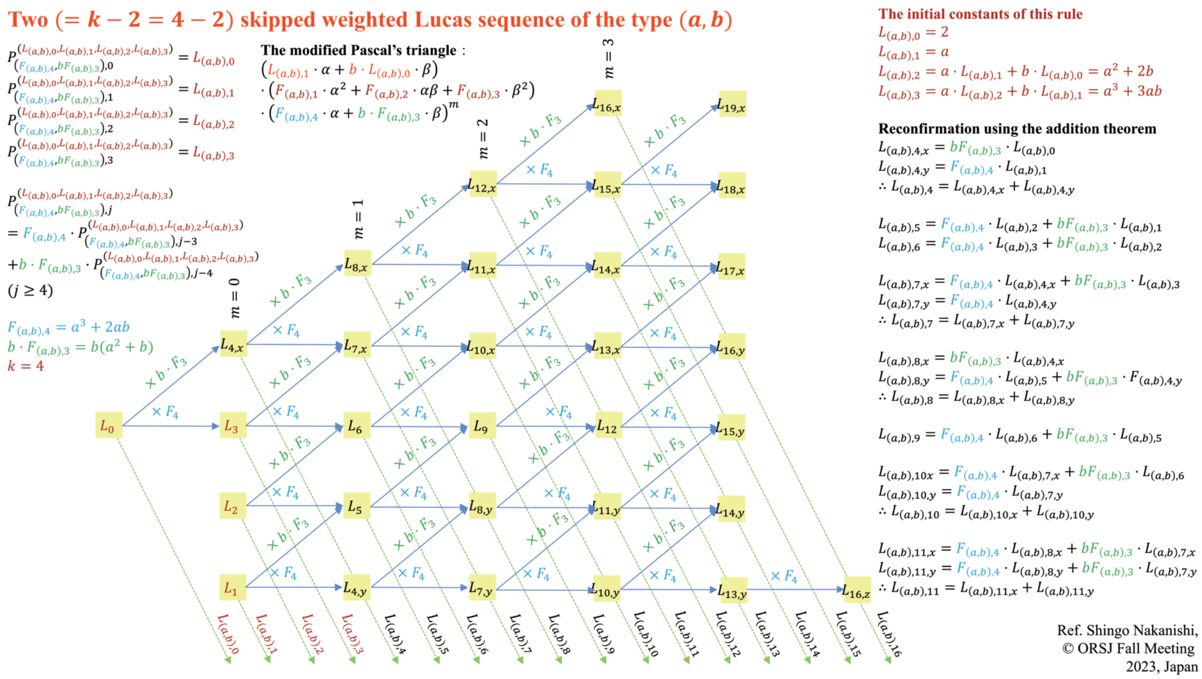


Figure 7: Concepts of the two skipped weighted Lucas sequence using the modified Pascal's triangle [17]

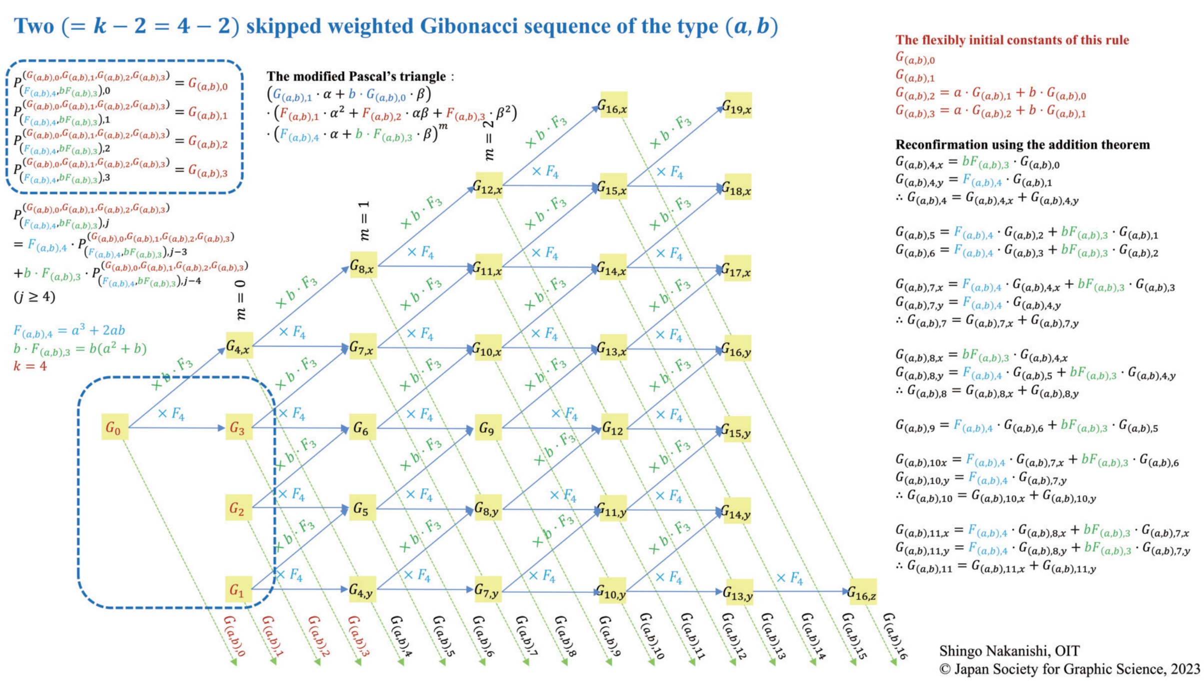


Figure 8: Concepts of the two skipped weighted Gibonacci sequence using the modified Pascal's triangle [18]

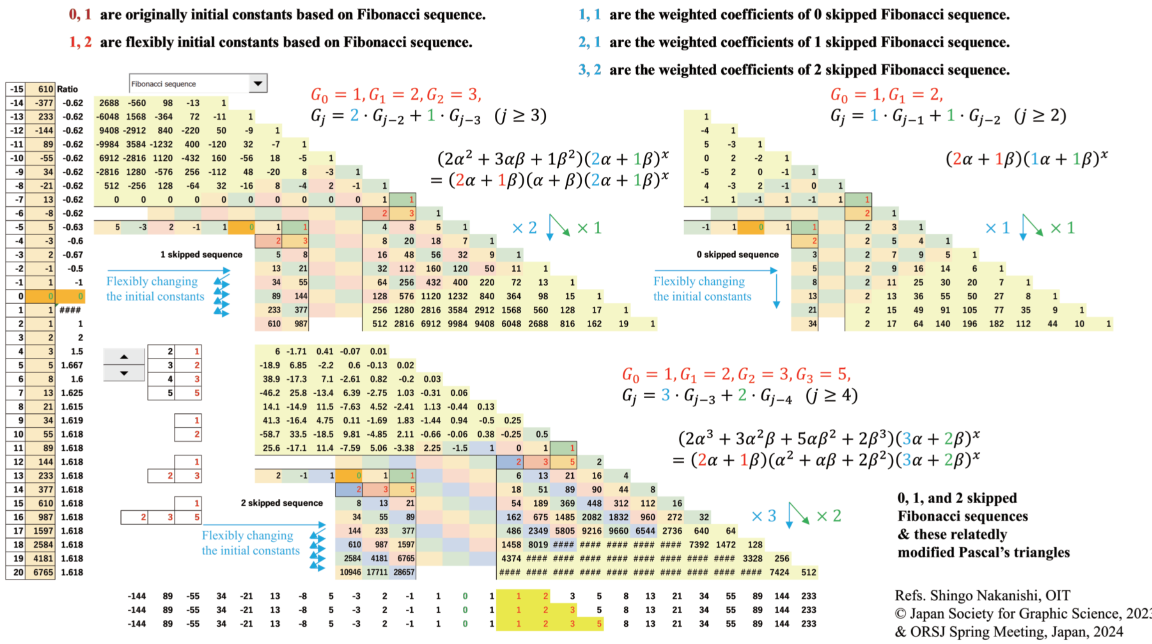


Figure 9: Visualizations of the no, one, or two skipped weighted Fibonacci sequences using the modified Pascal's triangles with flexibly changing the initial constants [18]

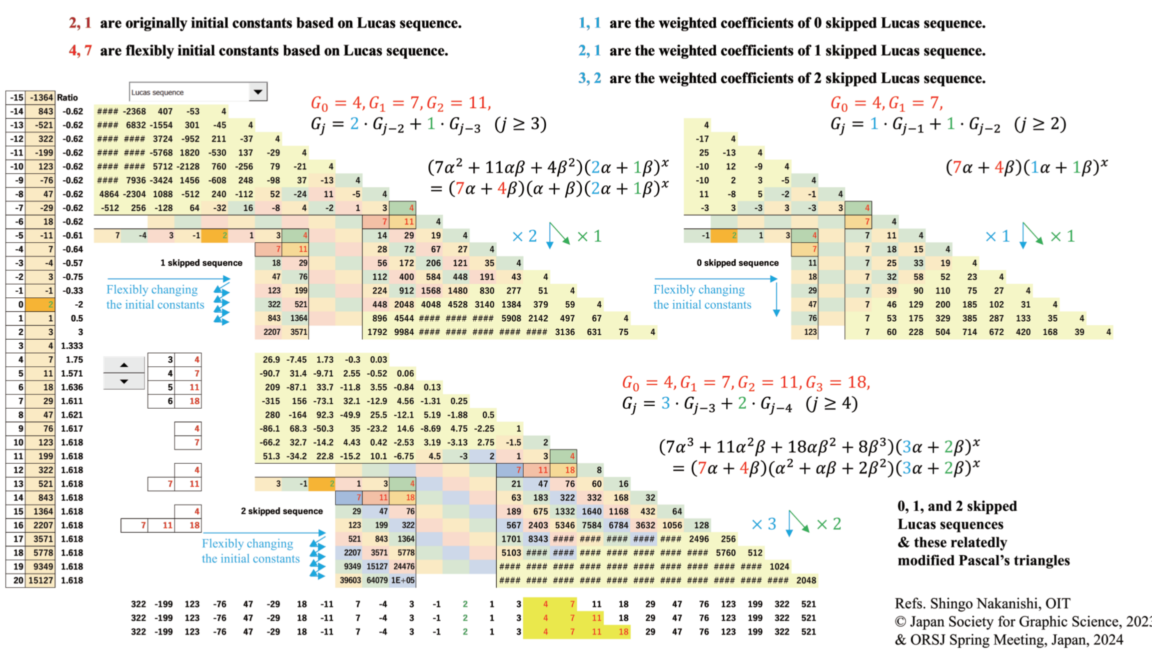


Figure 10: Visualizations of the no, one, or two skipped weighted Lucas sequences using the modified Pascal's triangles with flexibly changing the initial constants [18]

## 4. Description and Definition of Modified Pascal's Triangles Using Matrices

### 4.1. Description of modified Pascal's triangles and these matrices for no skipped Fibonacci or Lucas sequences

In subsection 3.1, we can describe the skipped Fibonacci or Lucas sequences. These findings can be also shown by using the Pascal's matrices in the following expansion. First, we can define the original Pascal's triangle [5] such as

$$\mathbf{L}_{S(0),F(a,b),l} = \mathbf{L}_{F(a,b),l} = \begin{pmatrix} \binom{0}{0}\cdot 1 & 0 & 0 & \cdots & 0 & 0 \\ \binom{1}{0}\cdot a & \binom{1}{1}\cdot b & 0 & \cdots & 0 & 0 \\ \binom{2}{0}\cdot a^2 & \binom{2}{1}\cdot ab & \binom{2}{2}\cdot b^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \binom{l-2}{0}\cdot a^{(l-2)} & \binom{l-2}{1}\cdot a^{(l-3)}\cdot b & \binom{l-2}{2}\cdot a^{(l-4)}b^2 & \cdots & \binom{l-2}{l-2}\cdot b^{(l-2)} & 0 \\ \binom{l-1}{0}\cdot a^{(l-1)} & \binom{l-1}{1}\cdot a^{(l-2)}\cdot b & \binom{l-1}{2}\cdot a^{(l-3)}b^2 & \cdots & \binom{l-1}{l-2}\cdot ab^{(l-2)} & \binom{l-1}{l-1}\cdot b^{(l-1)} \end{pmatrix}. \tag{4.1}$$

When we focus on the sum of each reverse diagonal of the (4.1), it is known to obtain the original or weighted Fibonacci sequence generally. If we suppose the following condition is shown

$$\mathbf{L}_{S(0),F(n,1),l} = \mathbf{L}_{F(n,1),l} = \begin{pmatrix} \binom{0}{0}\cdot 1 & 0 & 0 & \cdots & 0 & 0 \\ \binom{1}{0}\cdot n & \binom{1}{1}\cdot 1 & 0 & \cdots & 0 & 0 \\ \binom{2}{0}\cdot n^2 & \binom{2}{1}\cdot n & \binom{2}{2}\cdot 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \binom{l-2}{0}\cdot n^{(l-2)} & \binom{l-2}{1}\cdot n^{(l-3)} & \binom{l-2}{2}\cdot n^{(l-4)} & \cdots & \binom{l-2}{l-2}\cdot 1 & 0 \\ \binom{l-1}{0}\cdot n^{(l-1)} & \binom{l-1}{1}\cdot n^{(l-2)} & \binom{l-1}{2}\cdot n^{(l-3)} & \cdots & \binom{l-1}{l-2}\cdot n & \binom{l-1}{l-1}\cdot 1 \end{pmatrix}, \tag{4.2}$$

we can estimate

$$\begin{aligned} \mathbf{L}_{F(n+1,1),l} &= \mathbf{L}_{F(n,1),l}\mathbf{L}_{F(1,1),l} \\ &= {\mathbf{L}_{F(1,1),l}}^{(n+1)} \end{aligned} \tag{4.3}$$

precisely [5]. On the other hand, if we use the diagonal matrix

$$\mathrm{diag}[b^h] = \begin{pmatrix} b^0 & 0 & 0 & \cdots & 0 \\ 0 & b^1 & 0 & \cdots & 0 \\ 0 & 0 & b^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & b^{(l-1)} \end{pmatrix}, \quad (h = 0, 1, 2, \cdots, l-1), \tag{4.4}$$

we can also confirm the following calculation results shown in Figure 11. These are

$$\mathbf{L}_{S(0),F(n,1),l} = \mathbf{L}_{F(n,1),l} = {\mathbf{L}_{F(1,1),l}}^{n}, \tag{4.5}$$

$$\mathbf{L}_{S(0),F(1,n),l} = \mathbf{L}_{F(1,n),l} = \mathbf{L}_{F(1,1),l)}\mathrm{diag}[n^h], \tag{4.6}$$

$$\mathbf{L}_{F(a,b),l} = {\mathbf{L}_{F(1,1),l)}}^{a}\mathrm{diag}[b^h]. \tag{4.7}$$

From (4.5), we can imply the primary metallic ratios [24] based on k-Pell sequences such as

$$\lambda_{(n,1)} = \lim_{n\to+\infty} \frac{F_{(n,1),j+1}}{F_{(n,1),j}} \quad \text{or}$$
$$\lambda_{(n,1)} = \frac{n}{2} + \frac{\sqrt{n^2+4}}{2}, \quad \because \lambda_{(n,1)}^2 - n\lambda_{(n,1)} - 1 = 0. \tag{4.8}$$

Similarly, from (4.6), we can also verify the secondary metallic ratios [24] based on k-Jacobsthal sequences such as

$$\lambda_{(1,n)} = \lim_{n\to+\infty} \frac{F_{(1,n),j+1}}{F_{(1,n),j}} \quad \text{or}$$
$$\lambda_{(1,n)} = \frac{1}{2} + \frac{\sqrt{1+4n}}{2}, \quad \because \lambda_{(1,n)}^2 - \lambda_{(1,n)} - n = 0. \tag{4.9}$$

If we think of the 2 types of the initial numbers [5] as follows

$$\mathbf{G}_{(L((a,b),i),b\cdot L((a,b),i-1)),l)} = \begin{pmatrix} L_{(a,b),i} & b\cdot L_{(a,b),i-1} & 0 & 0 & \cdots & 0 \\ 0 & L_{(a,b),i} & b\cdot L_{(a,b),i-1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & L_{(a,b),i} & b\cdot L_{(a,b),i-1} & 0 \\ 0 & 0 & \cdots & 0 & L_{(a,b),i} & b\cdot L_{(a,b),i-1} \\ 0 & 0 & \cdots & 0 & 0 & L_{(a,b),i} \end{pmatrix}, \tag{4.10}$$

$$\mathbf{G}_{(F((a,b),i),b\cdot F((a,b),i-1)),l)} = \begin{pmatrix} F_{(a,b),i} & b\cdot F_{(a,b),i-1} & 0 & 0 & \cdots & 0 \\ 0 & F_{(a,b),i} & b\cdot F_{(a,b),i-1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & F_{(a,b),i} & b\cdot F_{(a,b),i-1} & 0 \\ 0 & 0 & \cdots & 0 & F_{(a,b),i} & b\cdot F_{(a,b),i-1} \\ 0 & 0 & \cdots & 0 & 0 & F_{(a,b),i} \end{pmatrix}, \tag{4.11}$$

we can indicate the modified Pascal's matrices for the flexibly no skipped weighted Lucas sequence. These are

$$\mathbf{L}_{S(0),L(n,1),l} = \mathbf{L}_{F(n,1),l}\mathbf{G}_{(L(n,1),0,L(n,1),-1),l}, \tag{4.12}$$
$$\mathbf{L}_{S(0),L(1,n),l} = \mathbf{L}_{F(1,n),l}\mathbf{G}_{(L(1,n),0,L(1,n),-1),l}. \tag{4.13}$$

Notably, we can confirm the k-Pell-Lucas sequences based on $\mathbf{L}_{S(0),L(n,1),l}$ and k-Jacobsthal-Lucas sequences from $\mathbf{L}_{S(0),L(1,n),l}$ shown in Figure 12. Moreover, if we use the transposed matrix $\mathbf{L}^T_{F(b,a),l}$, we can get the other types of modified Pascal's matrices as follows

$$\mathbf{A}_{F(a,b),l} = \mathbf{A}^T_{F(b,a),l} = \mathbf{L}_{F(b,a),l}\mathbf{L}^T_{F(a,b),l} \tag{4.14}$$
$$\mathbf{A}_{F(b,a),l} = \mathbf{L}_{F(a,b),l}\mathbf{L}^T_{F(b,a),l} \tag{4.15}$$
$$\mathbf{A}_{L(1,n),l} = \mathbf{A}_{F(a,b),l}\mathbf{G}_{(L(n,1),0,L(n,1),-1),l}. \tag{4.16}$$

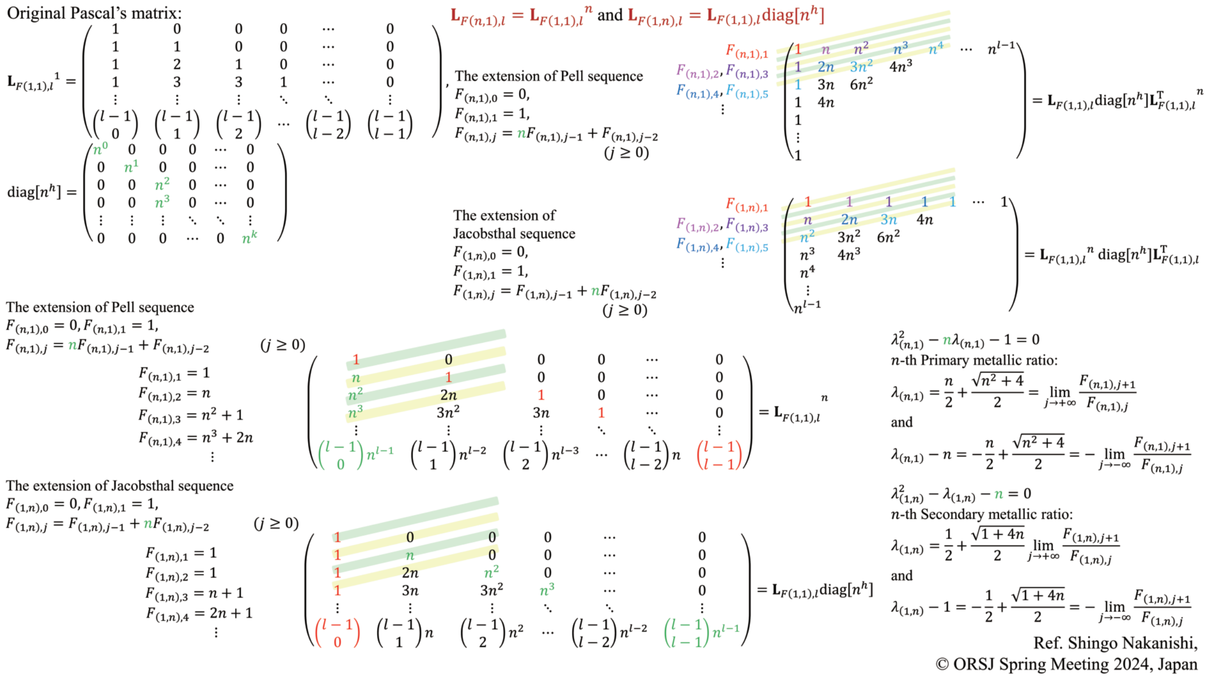

Figure 11: Visualization of weighted Fibonacci sequences using the metallic ratios [19]

These findings are illustrated in Figure 11. At the same time, we can imply the primary and secondary metallic ratios (4.8) and (4.9) in Figure 11. Notably, we call $\lambda_{(1,1)}$, $\lambda_{(2,1)}$, $\lambda_{(3,1)}$, $\lambda_{(1,2)}$, and $\lambda_{(1,3)}$ as the golden ratio, silver ratio, bronze ratio, nickel ratio, and kappa ratio that are proposed by de Spinadel in 1990s [24] respectively. However, there seem to be two types of major naming customs after searching for that throughout the internet. This study attempts to define the naming of that as the first ratio, second ratio, third ratio of the primary or secondary metallic ratios and so on simply. From Figure 12, we can also understand the no skipped Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, or Jacobsthal-Lucas sequences be displayed by the proper modified Pascal's matrices specifically.

### 4.2. Description of modified Pascal's triangles and these matrices for $(k-2)$ skipped Fibonacci or Lucas sequences

Bacially, the modified Pascal's matrices of this subsection requires to be considered as the same thinking of 4.1 as much as possible to understand that easily. First, we suppose flexibly changing the initial constants from $F_{((a,b),1)}$ to $F_{((a,b),k-1)}$ for the band matrices $\mathbf{B}_{(F((a,b),1),\cdots,F((a,b),k-1)),l}$. Then, we aim to define the modified Pascal's matrices for the $(k-2)$ skipped weighted Fibonacci sequences $\mathbf{L}_{S(k-2),F(F((a,b),k),F((a,b),k-1)),l,i}$ or that of Lucas sequences $\mathbf{L}_{S(k-2),L(F((a,b),k),F((a,b),k-1)),l,i}$ as follows. These modified Pascal's matrices can be computed as follows

$$\begin{aligned}&\mathbf{L}_{S(k-2),F(F((a,b),k),F((a,b),k-1)),l,i}\\&=\mathbf{L}_{(F((a,b),k),F((a,b),k-1)),l}\mathrm{diag}[b^h]\mathbf{B}_{(F((a,b),1),\cdots,F((a,b),k-1)),l}\mathbf{G}_{(F((a,b),i),b\cdot F((a,b),i-1)),l},\end{aligned}\tag{4.17}$$

$$\begin{aligned}&\mathbf{L}_{S(k-2),L(F((a,b),k),F((a,b),k-1)),l,i}\\&=\mathbf{L}_{(F((a,b),k),F((a,b),k-1)),l}\mathrm{diag}[b^h]\mathbf{B}_{(F((a,b),1),\cdots,F((a,b),k-1)),l}\mathbf{G}_{(L((a,b),i),b\cdot L((a,b),i-1)),l}.\end{aligned}\tag{4.18}$$

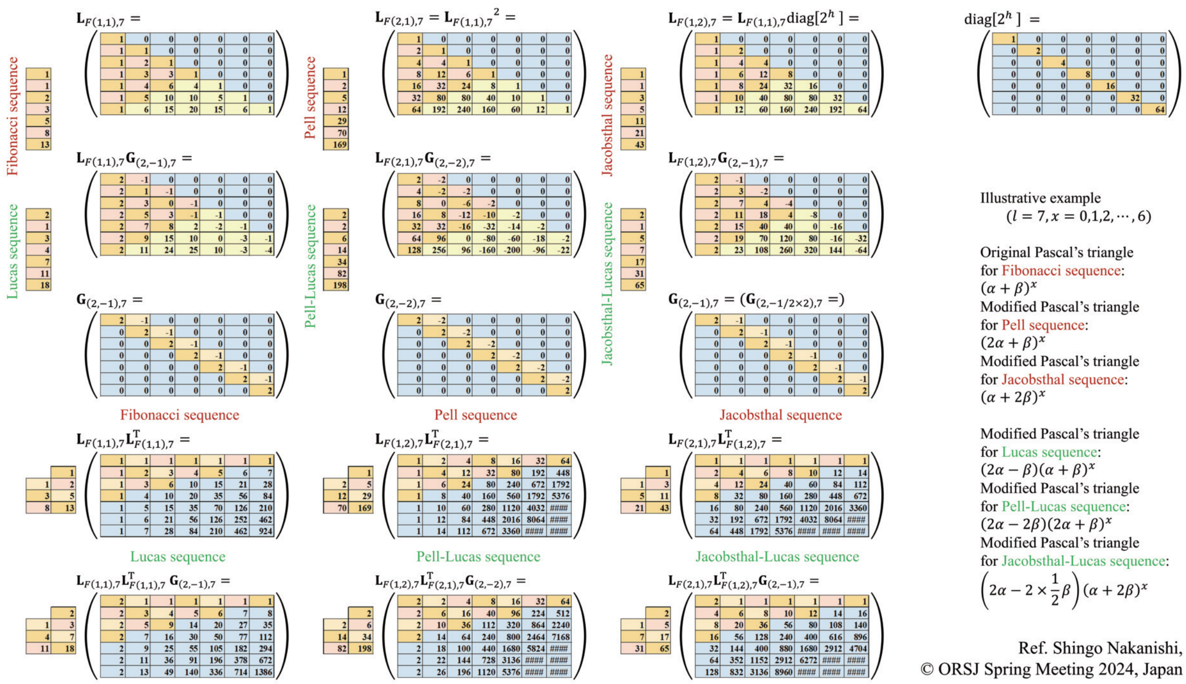


Figure 12: Visualization of weighted Fibonacci or Lucas sequences using the modified Pascal's matrices [19]

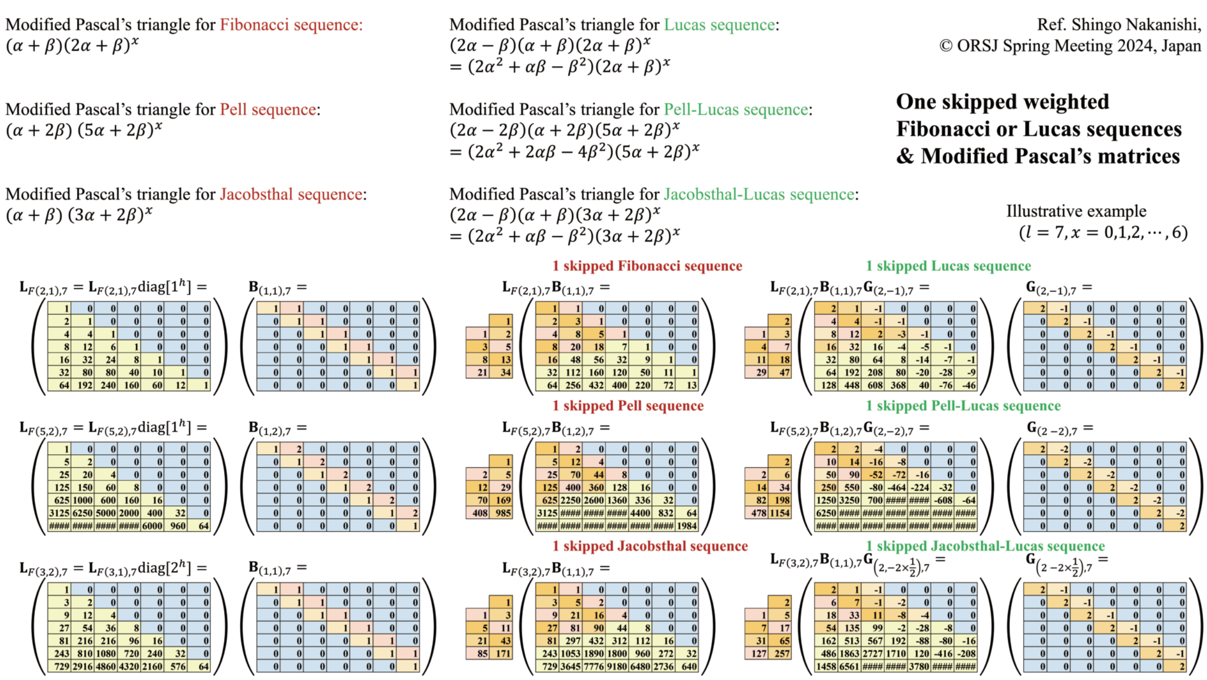


Figure 13: Visualization of weighted one skipped Fibonacci or Lucas sequences using the modified Pascal's matrices [19]

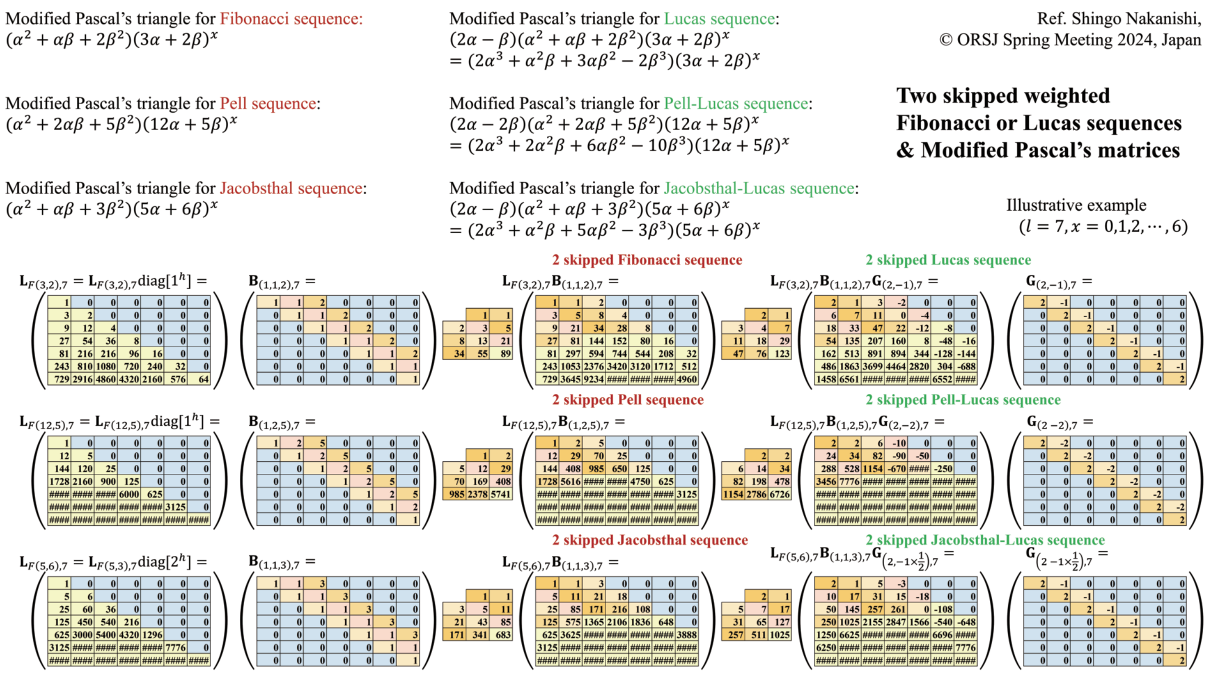


Figure 14: Visualization of weighted two skipped Fibonacci or Lucas sequences using the modified Pascal's matrices [19]

where the band matrix using from 1 to $(k-1)$-th initial constants of the weighted Fibonacci numbers. That is

$$\mathbf{B}_{(F((a,b),1),\cdots,F((a,b),k-1)),l} = \begin{pmatrix} F_{(a,b),1} & \cdots & F_{(a,b),k-1} & 0 & \cdots & 0 \\ 0 & F_{(a,b),1} & \cdots & F_{(a,b),k-1} & \ddots & \vdots \\ 0 & 0 & \ddots & \cdots & \ddots & 0 \\ \vdots & 0 & 0 & F_{(a,b),1} & \cdots & F_{(a,b),k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & F_{(a,b),1} \end{pmatrix}. \tag{4.19}$$

For making the weighted sequences, we can calculate the following matrices

$$\begin{aligned}&\mathbf{L}_{(F((a,b),k),b\cdot F((a,b),k-1)),l} = \\ &\mathbf{L}_{(F((a,b),k),F_{(a,b),k-1}),l}\mathrm{diag}[b^h] = \\ &\mathbf{L}_{(F((a,b),k),1),l}\mathrm{diag}[(b\cdot F_{(a,b),k-1})^h] = \\ &\begin{pmatrix} \binom{0}{0}\cdot 1 & 0 & \cdots & 0 \\ \binom{1}{0}\cdot F_{(a,b),k} & \binom{1}{1}\cdot b\cdot F_{(a,b),k-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \binom{l-1}{0}\cdot (F_{(a,b),k})^{(l-1)} & \binom{l-1}{1}\cdot (F_{(a,b),k})^{(l-2)}\cdot b\cdot F_{(a,b),k-1} & \cdots & \binom{l-1}{l-1}\cdot (b\cdot F_{(a,b),k-1})^{(l-1)} \end{pmatrix}\end{aligned} \tag{4.20}$$

where

$$\mathbf{L}_{(F((a,b),k),F((a,b),k-1)),l} = \begin{pmatrix} \binom{0}{0}\cdot 1 & 0 & \cdots & 0 \\ \binom{1}{0}\cdot F_{(a,b),k} & \binom{1}{1}\cdot F_{(a,b),k-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \binom{l-1}{0}\cdot (F_{(a,b),k})^{(l-1)} & \binom{l-1}{1}\cdot (F_{(a,b),k})^{(l-2)}\cdot F_{(a,b),k-1} & \cdots & \binom{l-1}{l-1}\cdot (F_{(a,b),k-1})^{(l-1)} \end{pmatrix} \tag{4.21}$$

and

$$\mathbf{L}_{(F((a,b),k),1)} = \begin{pmatrix} \binom{0}{0}\cdot 1 & 0 & \cdots & 0 \\ \binom{1}{0}\cdot F_{(a,b),k} & \binom{1}{1}\cdot 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \binom{l-1}{0}\cdot (F_{(a,b),k})^{(l-1)} & \binom{l-1}{1}\cdot (F_{(a,b),k})^{(l-2)}\cdot 1 & \cdots & \binom{l-1}{l-1}\cdot 1^{(l-1)} \end{pmatrix} \tag{4.22}$$

properly. From Figures 13 and 14, we can also understand the one or two skipped Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, or Jacobsthal-Lucas sequences be displayed by the proper modified Pascal's matrices effectively. However, we must admit the errors about the numerical calculation results of the lower parts of the modified Pascal's matrices because $(l \times l)$ of matrices means the numbers of the rows and columns of that. These calculation results are restricted within $(l \times l)$ of the matrices.

### 4.3. Computing techniques of modified Pascal's matrices for the negative numbers of the $(k-2)$ skipped Fibonacci or Lucas sequences

In case of the no skipped sequences, we can estimate the negative numbers of the sequences from the inverse matrices [29] of (4.7) easily. However, we should consider that the inverse matrices of several modified Pascal's matrices for $(k-2)$ skipped sequences are not available effectively about our proposals. It remains unclear because of new findings about modified Pascal's triangles. Thus, it necessitates to search for the proper calculations by using several computing techniques as follows. If we consider that the order is fewer than 0, there seems to be two types of the arrays. One is the zigzag line using the pairs of positive or negative values in case that the number of columns for the sequence is an odd number shown in Figure 15. At this time, we can set the number $f$ to distinguish the types of sequences. If types of the sequences belong to the weighted Fibonacci sequence, we can dfine the distinguished signal as $f = 1$. And if types of sequences belong to the weighted Lucas sequence, we can also put that as $f = 0$. Under these conditions, we can calculate the modified Pascal's matrices as follows

$$\begin{aligned} &\mathbf{L}_{SN(k-2),F(F((a,b),k),F((a,b),k-1)),l,i} \\ &\quad = (-1)^{(i+f)} \cdot (\mathbf{S}_O \mathbf{L}_{S(k-2),F(F((a,b),k),F((a,b),k-1)),l,i} \mathbf{S}_O) \odot \mathbf{B}_I, \end{aligned} \tag{4.23}$$

$$\begin{aligned} &\mathbf{L}_{SN(k-2),L(F((a,b),k),F((a,b),k-1)),l,i} \\ &\quad = (-1)^{(i+f)} \cdot (\mathbf{S}_O \mathbf{L}_{S(k-2),L(F((a,b),k),F((a,b),k-1)),l,i} \mathbf{S}_O) \odot \mathbf{B}_I. \end{aligned} \tag{4.24}$$

If the number of columns of the conveyor belts for the sequence is an even number in

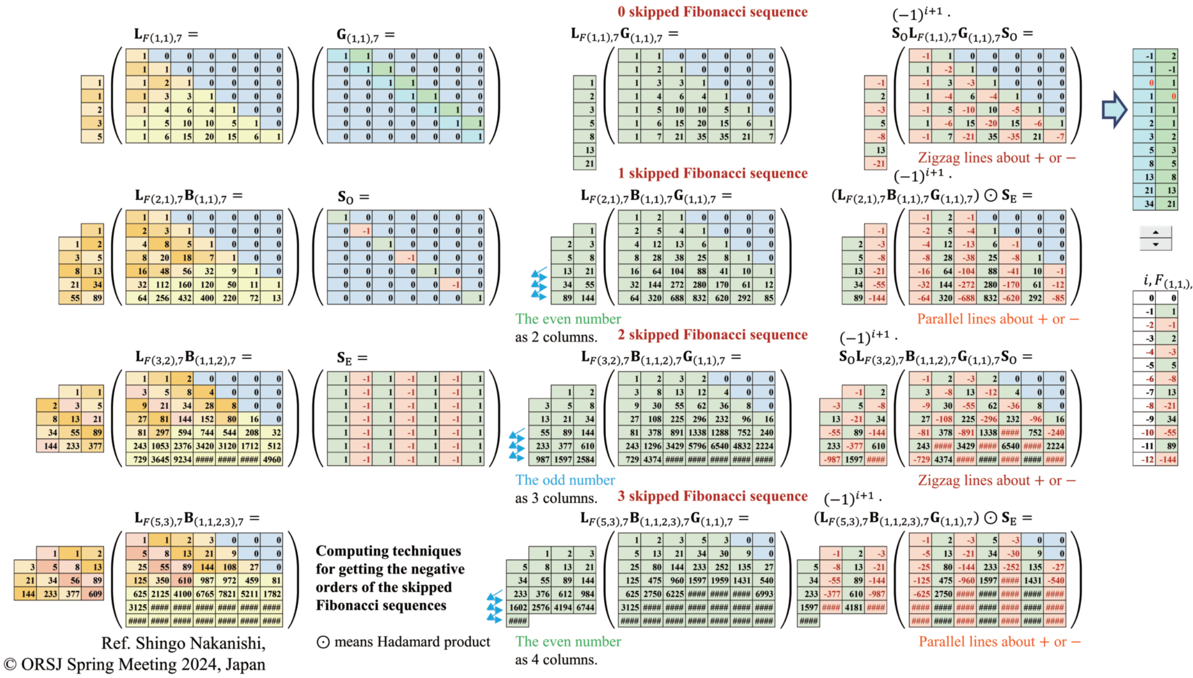


Figure 15: Visualization of negative numbers of weighted skipped Fibonacci sequence using the modified Pascal's matrices [19]

Figure 15, these are the parallel lines using that as follows

$$\begin{aligned}
&\mathbf{L}_{SN(k-2),F(F((a,b),k),F((a,b),k-1)),l,i} \\
&\quad = (-1)^{(i+f)} \cdot \mathbf{L}_{S(k-2),F(F((a,b),k),F((a,b),k-1)),l,i} \odot \mathbf{S}_E \odot \mathbf{B}_I, \qquad (4.25)\\
&\mathbf{L}_{SN(k-2),L(F((a,b),k),F((a,b),k-1)),l,i} \\
&\quad = (-1)^{(i+f)} \cdot \mathbf{L}_{S(k-2),L(F((a,b),k),F((a,b),k-1)),l,i} \odot \mathbf{S}_E \odot \mathbf{B}_I. \qquad (4.26)
\end{aligned}$$

where $\odot$ means Hadamard product, $\mathrm{sgn}(\cdot)$ means the sign function of $(\cdot)$, and the matrices $\mathbf{S}_O$ and $\mathbf{S}_E$ are used as the computing techniques for the estimation of the zigzag or parallel lines such as

$$\mathbf{S}_E = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & (-1)^{(l-1)} \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{(l-1)} \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{(l-1)} \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{(l-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{(l-1)} \end{pmatrix}, \qquad (4.27)$$

$$\mathbf{S}_O = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{(l-1)} \end{pmatrix} = \mathrm{diag}[(-1)^h] \quad (h = 0, 1, 2, \cdots, l-1) \qquad (4.28)$$

and

$$
\mathbf{B}_I = \begin{pmatrix}
b^{-i} & b^{-i-1} & b^{-i-2} & b^{-i-3} & \cdots & b^{-i-(l-1)} \\
b^{-i-(k-1)} & b^{-i-(k-1)-1} & b^{-i-(k-1)-2} & b^{-i-(k-1)-3} & \cdots & b^{-i-(k-1)-(l-1)} \\
b^{-i-2(k-1)} & b^{-i-2(k-1)-1} & b^{-i-2(k-1)-2} & b^{-i-2(k-1)-3} & \cdots & b^{-i-2(k-1)-(l-1)} \\
b^{-i-3(k-1)} & b^{-i-3(k-1)-1} & b^{-i-3(k-1)-2} & b^{-i-3(k-1)-3} & \cdots & b^{-i-3(k-1)-(l-1)} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
b^{-i-(l-1)(k-1)} & b^{-i-(l-1)(k-1)-1} & b^{-i-(l-1)(k-1)-2} & b^{-i-(l-1)(k-1)-3} & \cdots & b^{-i-(l-1)(k-1)-(l-1)}
\end{pmatrix}. \tag{4.29}
$$

In case of $b = 1$, we need not to use $\mathbf{B}_I$ to calculate (4.23) or (4.25) for the negative numbers of flexibly skipped sequences. From Figure 15, we can understand to rely on the processes of the calculations throughout the numerical examples specifically. However, for the proposals, we must inform the readers that these techniques might give some numerical calculation errors about the upper or lower parts of the modified Pascal's matrices because the calculation bounds are within $(l \times l)$ even if the practical orders of the sequences are widely spreading for positive or negative directions infinitely.

## 5. Other Flexibly Original One Skipped Sequences

### 5.1. Padovan or Perrin sequences as original one skipped sequences based on plastic ratio

As the readers know, there are two types of original one skipped sequence as follows. One is Padovan sequence in the following equation

$$G_0 = 1, \quad G_1 = 1, \quad G_2 = 1, \quad G_j = G_{j-2} + G_{j-3} \quad (j \geq 3), \tag{5.1}$$

or the following description

$$
\begin{aligned}
&P^{(1,1,1)}_{(1,1),0} = 1, \quad P^{(1,1,1)}_{(1,1),1} = 1, \quad P^{(1,1,1)}_{(1,1),2} = 1, \\
&P^{(1,1,1)}_{(1,1),j} = P^{(1,1,1)}_{(1,1),j-2} + P^{(1,1,1)}_{(1,1),j-3} \qquad (j \geq 3).
\end{aligned} \tag{5.2}
$$

The other is Perrin sequence in the following equation

$$G_0 = 3, \quad G_1 = 0, \quad G_2 = 2, \quad G_j = G_{j-2} + G_{j-3} \quad (j \geq 3), \tag{5.3}$$

or the following description

$$
\begin{aligned}
&P^{(3,0,2)}_{(1,1),0} = 3, \quad P^{(3,0,2)}_{(1,1),1} = 0, \quad P^{(3,0,2)}_{(1,1),2} = 2, \\
&P^{(3,0,2)}_{(1,1),j} = P^{(3,0,2)}_{(1,1),j-2} + P^{(3,0,2)}_{(1,1),j-3} \qquad (j \geq 3).
\end{aligned} \tag{5.4}
$$

From (5.1) and (5.2), we can estimate the following relation shown in Figure 16 if we consider the plastic ratio as the symbol $\rho$. That is

$$
\begin{aligned}
\rho^j &= \rho^{j-2} + \rho^{j-3} \\
&= P^{(1,1,1)}_{(1,1),j-3} \cdot \rho + P^{(1,1,1)}_{(1,1),j-2} + P^{(1,1,1)}_{(1,1),j-4} \cdot \rho^{-1}.
\end{aligned} \tag{5.5}
$$

Similarly, from (5.3) and (5.4), we can clarify the following relation using the constant, $C = 1.3848029\cdots$, in Figure 16. That is also

$$\begin{aligned} C\cdot\rho^{j} &= C\cdot\rho^{j-2} + C\cdot\rho^{j-3} \\ &= P^{(3,0,2)}_{(1,1),j-3}\cdot\rho + P^{(3,0,2)}_{(1,1),j-2} + P^{(3,0,2)}_{(1,1),j-4}\cdot\rho^{-1}, \qquad C = 1.3848029\cdots. \end{aligned} \tag{5.6}$$

These findings obtain several new visualizations shown in Figure 16. One is that the negative numbers of Padovan sequence are illustrated. The other is that the modified Pascal's triangle about Perrin sequence is displayed. In the next subsection, we plan to demonstrate flexibly changing the initial constants about these sequences using (5.5) for Padovan sequence or (5.6) for Perrin sequence throughout the above described and shown in Figure 16.

### 5.2. Padovan or Perrin sequences as one skipped sequences based on plastic ratio with flexibly changing the initial constants

If we think that the modified Pascal's triangle using the binomial theorem is described as $(G_1\cdot\alpha^2+G_2\cdot\alpha\beta+G_0\cdot\beta^2)\cdot(1\cdot\alpha+1\cdot\beta)^x$ based on the initial condition $G_1 = P^{(1,1,1)}_{(1,1),i+1},\ \ G_2 = P^{(1,1,1)}_{(1,1),i+2},\ \ G_0 = P^{(1,1,1)}_{(1,1),i}$, using the indicators $\alpha$ and $\beta$, we can propose the Padovan sequences with flexibly changing the initial constants according to (5.5) for Padovan sequence or (5.6) for Perrin sequence in Figure 16. Therefore, we can calculate the modified Pascal's triangles according to the initial condition $G_1 = P^{(1,1,1)}_{(1,1),i+1}, G_2 = P^{(1,1,1)}_{(1,1),i+2}, G_0 = P^{(1,1,1)}_{(1,1),i}$ for flexible Padovan sequence and $G_1 = P^{(3,0,2)}_{(1,1),i+1}, G_2 = P^{(3,0,2)}_{(1,1),i+2}, G_0 = P^{(3,0,2)}_{(1,1),i}$ for flexible Perrin sequence. First, we can prepare the band matrix for the initial condition such as

$$\mathbf{G}_{(G(i+1),G(i+2),G(i))),l} = \begin{pmatrix} G_{i+1} & G_{i+2} & G_i & 0 & \cdots & 0 \\ 0 & G_{i+1} & G_{i+2} & G_i & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & G_{i+1} & G_{i+2} & G_i \\ 0 & 0 & \cdots & 0 & G_{i+1} & G_{i+2} \\ 0 & 0 & \cdots & 0 & 0 & G_{i+1} \end{pmatrix}. \tag{5.7}$$

Second, if we use the original Pascal's matrix (4.1) in the case that $a = 1$ and $b = 1$, we can define the modified Pascal's matrices for the flexibly changing Padovan sequence as follows. That is

$$\mathbf{L}_{S(1),G(1,1),l} = \mathbf{L}_{F(1,1),l}\mathbf{G}_{(G(i+1),G(i+2),G(i)),l}. \tag{5.8}$$

Simply, if we use the initial condition $G_1 = P^{(3,0,2)}_{(1,1),i+1}, G_2 = P^{(3,0,2)}_{(1,1),i+2}, G_0 = P^{(3,0,2)}_{(1,1),i}$ instead of $G_1 = P^{(1,1,1)}_{(1,1),i+1}, G_2 = P^{(1,1,1)}_{(1,1),i+2}, G_0 = P^{(1,1,1)}_{(1,1),i}$, we can apply the modified Pascal's triangle using the binomial theorem as $(G_1\cdot\alpha^2 + G_2\cdot\alpha\beta + G_0\cdot\beta^2)\cdot(1\cdot\alpha + 1\cdot\beta)^x$ to that for the flexibly changing Perrin sequences. This is how we can calculate the modified Pascal's triangles for that using (5.7) and (5.8).

## 6. Conclusions

We conclude the following highlighted things. We can understand that the modified Pascal's triangles and these related matrices are proposed to create the weighted skipped Fibonacci or Lucas sequences effectively. These findings can be expanded by introducing the flexibly changing initial constants according to the weighted skipped sequences concretely and

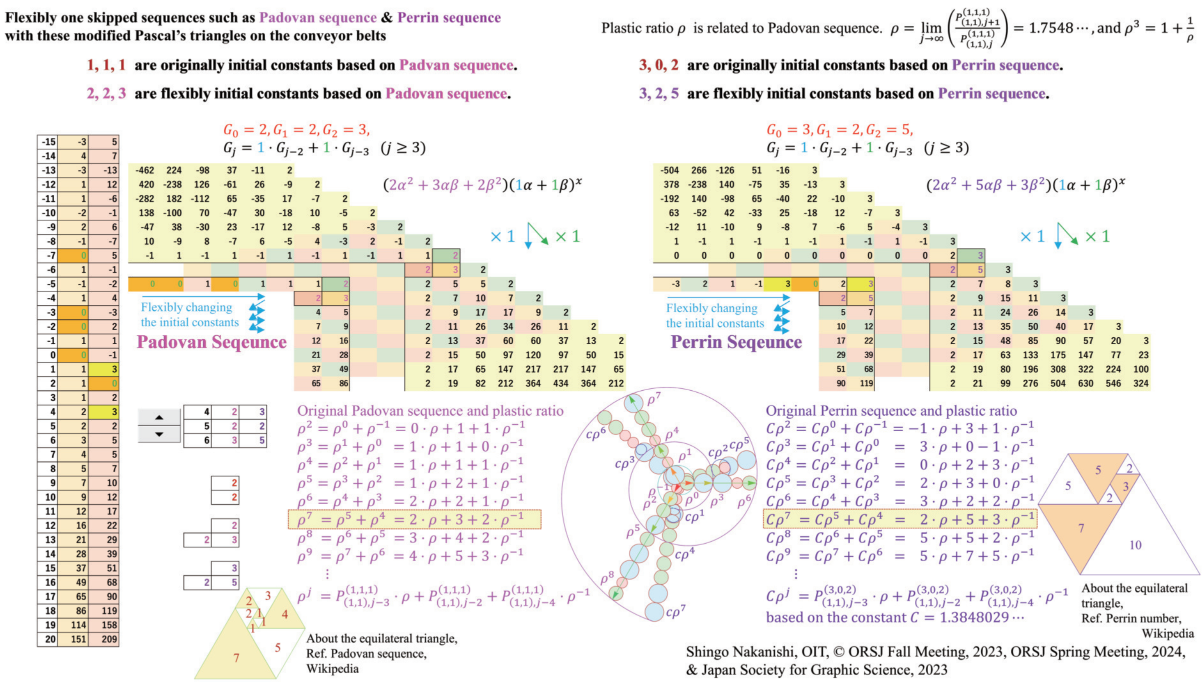

Figure 16: Visualization of Flexible Padovan or Perrin sequences using the modified Pascal's matrices [18]

systematically. To perform that more effectively, we can suggest the addition theorem of Fibonacci sequence should be indispensable for visualizing skipped sequences by using the above descriptions in this paper. Thus, we imagine the extended knight moving summations obtain the various sequences and these visualizations based on the proper conditions. Similarly, we can apply these findings to that of the original Padovan or Perrin sequences with flexibly changing the initial constants. These ideas can also provide some visual illustrative figures systematically and effectively about creating the modified Pascal's triangles and these related matrices for the various weighted skipped sequences with flexibly changing the initial constants. Finally, we need further investigation about negative numbers to create the modified Pascal's triangles more effectively even if we propose the computing techniques in this paper.

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