Why & How we know there are black-holes? Introduction to Einstein's Relativity

Hisaaki Shinkai

Abstract:
Einstein's theories of relativity explains that our space and time can bend like a trampoline. The theory predicts the possibility of time travel, the existence of black holes, and the expansion of the Universe. The session picks up the following three subjects: (1) delay of clocks in the rockets, (2) how we observe black holes, and (3) how we know the Universe is expanding. The lecture introduces the basic idea of physics (equation of motion, conservation laws), including its historical backgrounds.

Messages:
Physics explains the nature using equations. Physicists believes the nature can be explained in a simple form. Enjoy thinking how and why.

0 Homework before the session

Check out the following issues.

0.1 Kepler's laws of planetary motion

Johannes Kepler discovered the following laws on the motion of planets.

Kepler's law (1609, 1618)

1st law	The orbit of a planet is an ellipse with the Sun at one of the two foci.
2nd law	A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3rd law	The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Figure 1: Johannes Kepler (1571-1630)

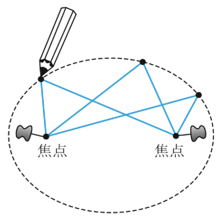

Figure 2: An ellipse and two foci.

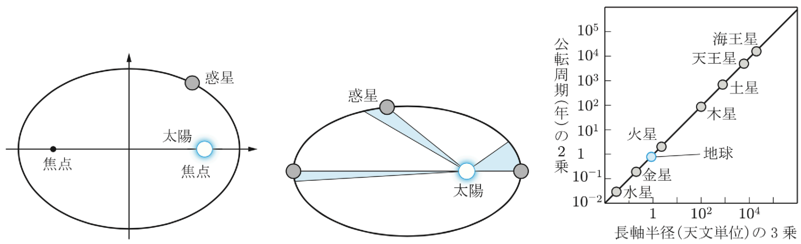

Figure 3: Kepler's first, second and third law of planetary motion.

0.2 Newton's laws of motion

Isaac Newton formulated the basic laws of the motion.

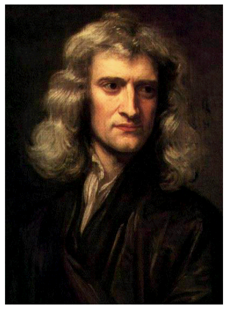

Figure 4: Isaac Newton (1642–1726)

Newton's law of motion (1687)

1st law	If there is no net force, an object either remains at rest or continues to move at a constant velocity.
2nd law	The force $\boldsymbol{F}$ produces the acceleration $\boldsymbol{a}$ to an object ([m/s^2]). The relation ("**equation of motion**") is $$\boldsymbol{F} = m\boldsymbol{a}, \qquad (0.1)$$ where m is the mass of the object ([kg]).
3rd law	When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

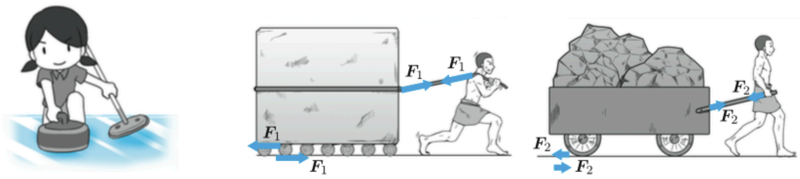

Figure 5: (Left) Newton's first law. (Right) Newton's third law (action-reaction).

These laws derive some conservation laws.

(A) conservation of linear momentum

When two objects (with mass m_A, m_B) interact each other (such as collision, merger, separations, penetration), then the total linear momentum is conserved:

$$m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B = m_A\boldsymbol{v}'_A + m_B\boldsymbol{v}'_B \qquad (0.2)$$

where $\boldsymbol{v}_A, \boldsymbol{v}_B$ are velocities of each object before interaction ([m/s] with its directional information), $\boldsymbol{v}'_A, \boldsymbol{v}'_B$ are those of after interaction ([m/s] with its directional information).

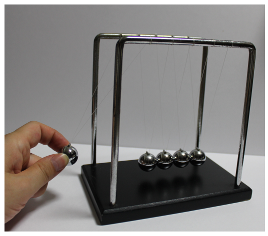

Figure 6: Newton's cradle.

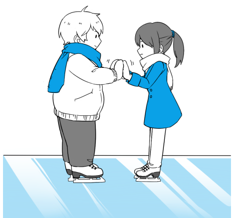

Figure 7: On the ice, two standing couple begin pushing each other. How will they move?

(B) conservation of energy

If there is no friction, the total energy of the system E (sum of the potential energy and kinetic energy) is conserved:

$$E = mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2 \tag{0.3}$$

where m, g, h, v are the mass of an object ([kg]), gravitational acceleration (=9.8 [m/s^2]), height location of the object ([m]), and the speed of the object ([m/s]), respectively.

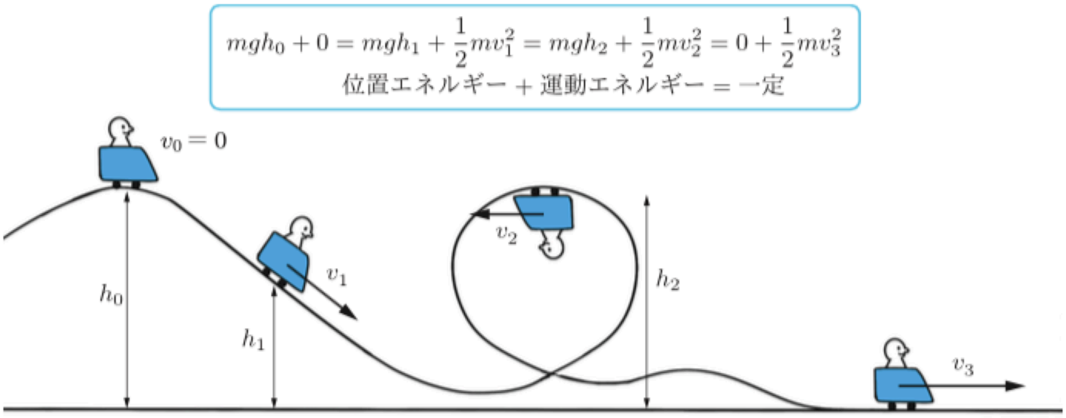

Figure 8: Jet coaster does not have its engine. The initial potential energy decides its later velocity.

(C) conservation of angular momentum

If an object begins rotation, the angular momemtum,

$$\boldsymbol{r} \times m\boldsymbol{v} \tag{0.4}$$

is conserved, where $\boldsymbol{r}, m, \boldsymbol{v}$ are directional vector from rotating axis ([m]), mass of an object ([kg]), and the velocity of an object ([m/s] with directional information), respectively.

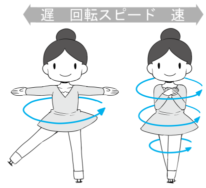

Figure 9: Figure skater rotates faster by shrinking her arms.

0.3 Least approaching time of life saver

The next problem is independent from the previous issues.

(Prob.) Least approaching time of life saver

A beautiful girl has fallen out of a boat, and she is screaming for help in the water (at C in the figure). We are at point A on land, and we see the accident, and we can run and can also swim. We can run 3 times faster than swim. If we plan to reach C as quick as we can, we should run a little greater distance on land. Where, then, is the point B on the shoreline at which we start to swim? Let the coordinate $A(0, -50)$ and $C(100, 50)$, and find $B(x, 0)$ using a software.

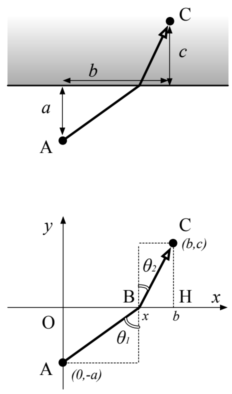

Figure 10

Hint. Procedures: Prepare a spreadsheet application like "Excel".

(1) Prepare columns A, B, $\cdots$, F, putting coordinate a, b, c, and the speed v_1, v_2, together with trial location x.

(2) At the cell G2, input "`=SQRT(A2*A2+F2*F2)/D2`", which means $= \sqrt{(\mathtt{A2})^2 + (\mathtt{F2})^2}/\mathtt{D2}$.
Copy the cell G2 and past it to G3, then it automatically inputs the value of the next line, "`=SQRT(A3*A3+F3*F3)/D3`".

(3) Prepare the cell H2 with "`=SQRT((B2-F2)*(B2-F2)+C2*C2)/E2`", and the column H similarly.

(4) Prepare the cell I2 with "`=G2+H2`", and the column I similarly.

(5) From $x = 50$ to $x = 100$, find out the one which minimizes $t_1 + t_2$.

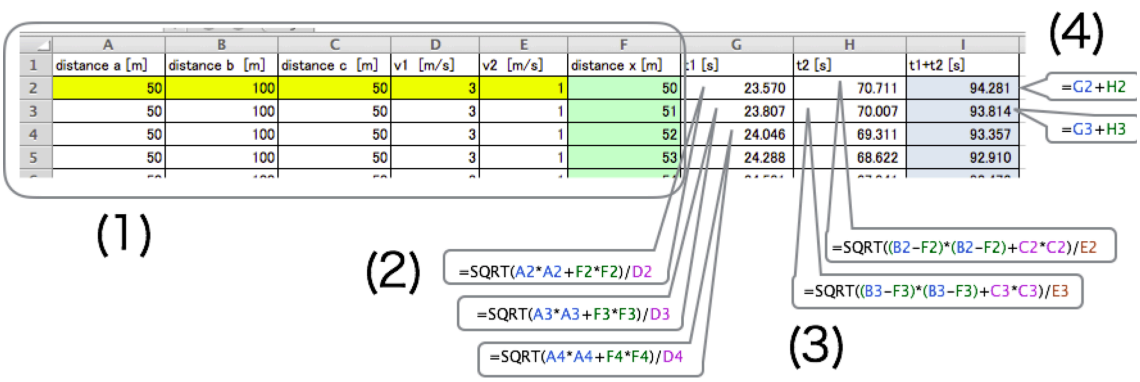

Figure 11: Sample sheet of Excel.

1 Special Relativity: Theory of time

1.1 Confusion from the Maxwell equations

Equations of electro-magnetic wave

The theory of electro-magnetism, which was summarized by Maxwell in 1864, predicted the existence of electro-magnetic wave (radio wave, including light itself) which propagates *at the speed of light* even *in the vacuum space.* These arise two questions.

Figure 12: James C. Maxwell (1831-1879)

- First one is why the wave propagates in the vacuum. We need some medium, like molecules of air for sound wave. Physicists, therefore, named the medium, *Ether* [1], and began experiments to detect it.
- Second question is the appearance of the speed of light in the equations. The measure of speed certainly depends on the observers, so physicists had to define some special observer.

Does *Ether* exist?

Albert Michelson began his experiments to find *Ether* in 1881. He developed a new idea for measuring tiny distance, which are now called "interferometer". When two waves (like sound wave or light) overlap, those are superposed. Depending on the location, superposed wave can be stronger or weaker, sometimes two waves compensate each other and disappear. This phenomena is called *interference* of wave. Michelson applied this phenomenon for measuring a tiny distance up to the half wavelength of light.

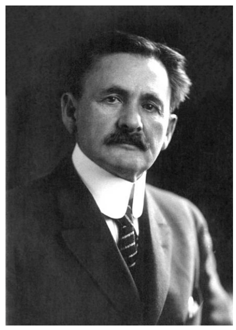

Figure 13: Albert Michelson (1852-1931)

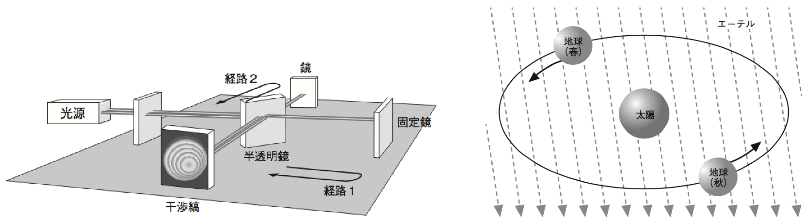

Figure 14: (Left) Michelson's interferometer. (Right) Idea for finding *Ether*.

He invented the following device. Split one light beam in two and each light ray travels in two different perpendicular directions. Two beams reflect at the end-mirror and merge again together at the splitting

[1] *Ether* means the matter which fills everywhere in ancient Greek.

mirror. The interference of light will make pattern depending on the difference of each path.

Suppose that the Universe is filled with *Ether*. Since the Earth circulates the Sun with the speed 30 [km/s], two light rays of the interferometer will feel the different directional velocity of *Ether*. The difference should cause the change of interference pattern through a year, because the Earth will move uploading and downloading in the river of *Ether*, which will prove the existence of *Ether*. These were the idea of Michelson.

Edward W. Morley (1838–1923)

Michelson and his assistant Morley continued their experiments for 6 years, and finally concluded that *Ether* had not been detected. They expressed themselves that they did not succeed the experiment.

Experiment **Interference of waves**

H.W. Check out how noise-canceling head-set works?

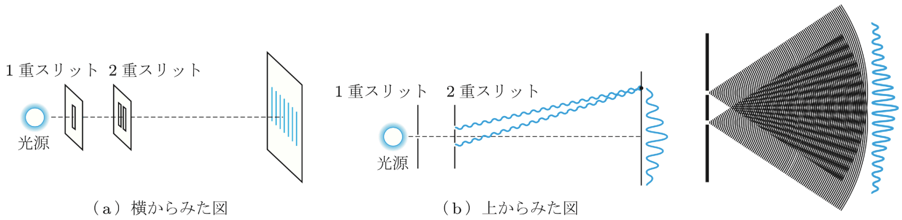

Figure 15: Double-slit experiment of light.

Lorentz contraction

The result of Michelson and Morley made physicists in confusion. The result indicated that the wind of *Ether* is not observable. Many physicists proposed explanations for this fact which were 'consistent' with the existence of *Ether*. For example, if *Ether* will circulate together with the Earth, the experiment had no contradiction. This explanation predicts that *Ether* moves due to the gravity of the Earth. Michelson tried his experiment again with the heavy gravity source at one arm of interferometer, but there was no differences in the interference patterns.

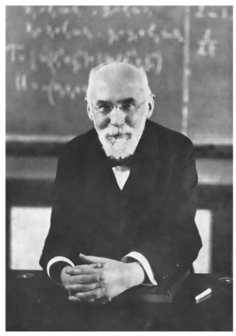

Figure 16: Hendrik A. Lorentz (1853–1928)

The theory of *Ether* became difficult. FitzGerald and Lorentz began changing the Newton's physics. They proposed that all the matter will shrink its length if they move at large speed, which is called the Lorentz-FitzGerald contraction. If we suppose that the matter of length L at the speed $v = 0$ will change its length if it moves with the speed v in its moving direction as

$$L' = \sqrt{1 - (v/c)^2} L \tag{1.5}$$

George F. FitzGerald (1851–1901)

where c is the speed of light, then there is no contradiction with the experiments. This relation can be interpreted also as the clock timing of the observer who moves at the speed v will be longer with the factor $1/\sqrt{1-(v/c)^2}$.

【**Detail explanation**】(you can skip this part)
Lorentz explained this proposal with coordinate transformation (1904). That is, if we change our coordinate (t, x, y, z) into the new coordinate (t', x', y', z') which moves in x-direction with the speed v, then we obtain

Lorentz transformation

$$\begin{aligned} t' &= \frac{t-(v/c^2)x}{\sqrt{1-(v/c)^2}} \\ x' &= \frac{x-vt}{\sqrt{1-(v/c)^2}}, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{1.6}$$

These relation can be written in a matrix form as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{-v/c^2}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{1.7}$$

This proposal sounds strange, and did not have any principle. However, Lorentz transformation is consistent with the Maxwell equations. If this is true, then we can not observe *Ether* by Michelson's interferometer because our measure scale will change along our movements which compensates *Ether* effect. This 'first-aid' was accepted by most physicists by 1905.

1.2 Einstein's Special Relativity

Two principles by Einstein

Albert Einstein proposed a new interpretation to the problem. He starts from proposing two principles, which *derive* the Lorentz transformation (1.7), and also conclude that we do not need *Ether* for explaining the Maxwell equations. Two principles are the followings:

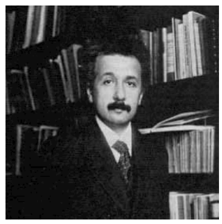

Figure 17: Albert Einstein (1879-1955)

(a) **Principle of Relativity**: All the physics laws should be the same equation, independent from the observers' coordinate.

(b) **Principle of the constant speed of light**: The speed of light in vacuum is the constant at any coordinate in the Universe.

The first principle came from the beauty of physics which Einstein believed. The second one, however, is totally new and brave idea which was never thought by anybody except Einstein. Einstein's strategy was to believe the simple or minimal assumptions which can derive the laws of real nature.

Einstein named this theory *Principle of Relativity* in 1905. This theory explained Lorentz contraction as one of the interpretations, and we do not need *Ether* at all. The truth is that light moves constant speed as the Maxwell equations says.

$c + c = c$
At first sight, it is hard to believe Einstein's conclusion that light propagates at the constant speed from any observer. The speed of light is c = 300,000 [km/s][2]. Einstein says even if we observed that light from a high-tech rocket moving at 100,000 [km/s] [3], light is still the speed c.

In our daily life, if we throw a ball with speed v_1 in the train with speed v_2, then the ball moves at the speed $v_1 + v_2$ (measured from the ground observer). Einstein's theory, however, says that this is not true. The true additive calculation should be

$$v_1 + v_2 \implies \frac{v_1 + v_2}{1 + (v_1 v_2 / c^2)} \tag{1.8}$$

which is derived from Lorentz transformation, (1.7). This rule says if the speed is small compared with c, then $v_1 + v_2$ is *approximately* true. Actually, this approximation is valid unless speed is close to c.
Exercise 1 Using the additive rule (1.8), fill in the blanks. Ex.1 $c + c = c$.

v_1	v_2	$v_1 + v_2$
0.1% of light	0.1% of light	0.1999998% of light
$0.10c$	$0.10c$	
$0.50c$	$0.50c$	
$0.90c$	$0.90c$	
$0.99c$	$0.99c$	
c	c	

1.3 Time is relative

Thought experiment of a light-clock
Try an experiment in our brain, which Einstein called "thought experiment". If we make a clock using light, then we can measure the time precisely. For example, suppose we set two mirrors apart 50 [cm], and make light back and forth between these mirrors. One return is 1 [m], so that after 299,792,458 returns of light, we can declare one second (see Fig. 18).

Suppose that we put this light-clock in a high-speed rocket. Light, then, has to move back and forth between mirrors with additional distance. Therefore one second in the rocket is apparently longer than one second on the Earth ground. However, Einstein's principle says the light speed is constant. This means that for the people in the rocket, this light-clock shows the right one second.

How can we solve this paradox?

The only solution is to admit that one second is different between two observers. That is, **time proceeds differently** depending on observer's speed. The 'one second' in the rocket is longer than 'one second' on the Earth, and the difference becomes larger when the rocket is moving fast. However, each observer believes their own clock is right, and that is right actually.

Before Einstein proposed the principle of constant light speed, people had believed that time proceeds homogeneously in all the Universe. Such

[2] The real (and defined) speed is c =299,792,458 [m/s].
[3] The maximum speed of rocket in our technology is 70 [km/s].

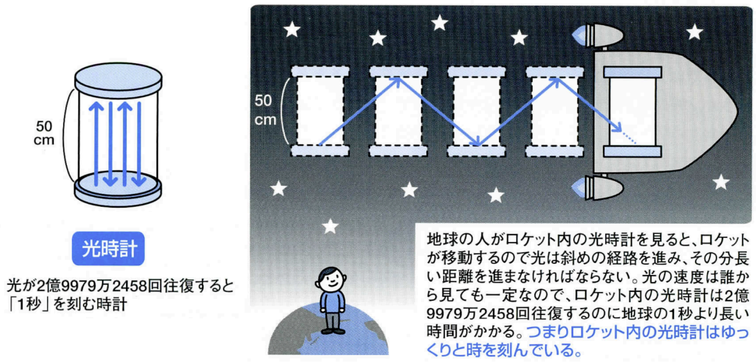

Figure 18: (Left) A light-clock. (Right) When the clock is in a rocket at large speed, a light-ray has to move longer distance for traveling between two mirrors, which means one second in a rocket is longer than that on the Earth.

a notion of 'absolute time' turned out wrong. What Einstein found is that we have to consider space and time as a set of coordinate, and time is relative to observers.

Time travel to the future

Actual difference of time can be expressed by the equation

$$T' = \sqrt{1-(v/c)^2}\,T \tag{1.9}$$

where T is the clock time (say, one second) in a rocket moving at speed v and T is the time (say, one second) at rest observer.

The right table is examples of the value of $\sqrt{1-(v/c)^2}$. This table shows that the difference of time will crucially large when the speed v is close to c.

v	$\sqrt{1-(v/c)^2}$
0.1 c	0.99499
0.5 c	0.86603
0.9 c	0.43589
0.99c	0.14107

Exercise 2a The ISS (International Space Station) circulates around the Earth with a speed 7.8 [km/s], which is 0.000026c. How much is the difference of time for a staff who stayed a year in ISS comparing to a person on the Earth?

Ex.2 **Time travel to the future.**

Exercise 2b According to a story of *Mr. Taro Urashima*, a famous fairy-tale in Japan, Mr. Urashima spended three-years travel to Ryugu castle and traveled back to his town, but he found that he suddenly got old and all the people he knew turned out to have died. If we interpreted this story as a time travel to the future (say 300 years) using a rocket, what is the speed of the rocket? Use the approximation,

$$\sqrt{1-(v/c)^2} \approx 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 \tag{1.10}$$

if your calculator is not smart.

2 General Relativity: Theory of gravity

2.1 Starting point of general relativity

The theory of Einstein in 1905 concluded that time is relative to observers, which was one of the big revolutions in physics. However, he knew the weak point of the theory as well. The relativity theory in 1905 could not treat observers who are moving with acceleration (changing his/her speed). Einstein began constructing his new generalized theory of relativity.

What he seek was the equation of motion which were common to all the observers in the Universe. He again started from two principles:

(a) **Principle of General Relativity** : All the physics laws should be the same equation, independent from the observers movements.

(b) **Principle of the constant speed of light** : The speed of light in vacuum is the constant at any coordinate in the Universe.

The main problem was how to treat the equation of motion in the coordinate with acceleration. For example, in the accelerating train, we feel the additional backward force. On the contrary, in the stopping train, we feel the additional forward force. These are due to the moving effects called *inertial force*, and this deos not exist for observers who stay at rest. Such an inertial force, therefore, is not a physical one.

His question focused to the origin of acceleration. The most common accelerating phenomenon is gravity such as falling apples. Einstein began thinking the origin of the gravity. It took another 10 years for him.

2.2 Gravity occurs from the bending of space and time

Einstein's most exciting finding in his life

When Einstein was considering what the origin of gravity, he got an excellent idea which he wrote himself as the most exciting finding in his life: a thought experiment of a free-falling elevator.

> If an elevator lost its wire, then it falls down with the acceleration with 9.8 [m/s^2]. A person in the elevator also falls down with the same acceleration, so that the inside space of elevator becomes the space with no gravity. That is, the gravity can be cancelled out in a local space.

The main message is that we can not distinguish the gravity from an inertial force, which is called the principle of equivalence. Using this principle, we can treat the speciality of accelerated frame. We can recognize gravity is not a kind of force, but something else which cause the effects to a coordinate.

Figure 19: Einstein in 1912.

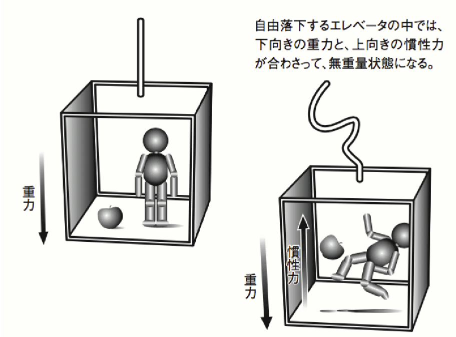

Figure 20: Inside of a free-falling elevator became a space with no-gravity.

In a free-falling elevator, we can cancel out the effect of the gravity. However, we cannot erase the gravity overall. Suppose that we prepare an Earth-size huge elevator (Fig. 21). Gravity force directs to the center of the Earth, so that even in the free-falling frame, we cannot erase the gravity in such a huge scale.

That is, the gravity can be cancelled out in a small region, but that cancellation is not possible in large scale. Einstein began thinking that gravity can be expressed by a sort of "geometry".

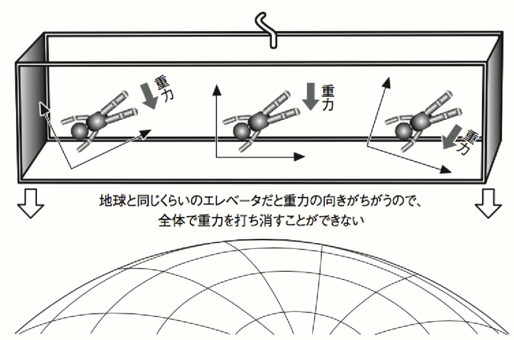

Figure 21: Gravity can be cancelled out in a local space, but not in larger space. Suppose an Earth-size elevator. Gravity force has direction, so that we cannot erase gravity in one coordinate.

Einstein, at his age 33, asked to his mathematician friend Marcel Grossmann how to treat such a problem. Grossmann introduced Riemann's geometry which is the geometry for curved space. He also suggested to Einstein that this mathematics is speculative and not appropriate for physicists. However, Einstein got into the deep inside of Riemannian geometry, and devoted his next three years for looking for the relation of curved space-time and gravity.

Marcel Grossmann (1878–1936)

Georg F.B. Riemann (1826–66)

Curved space

We learn the geometry in our elementary school, such as the total angles of any triangle is 180°. This is the result in the flat space, and this geometry is called Euclidian geometry.

Euclid (B.C.365?–B.C.275?)

Take a look at the meaning of *curved space*. Actual space is 3-

dimensional, but it would be better start imagining a globe, as an example of 2-dimensional curved space.

Airplane route from Japan to United State takes Kuril Islands and Alaska. If we map this route on a globe, this is almost the minimum distance between two points, which is called a *great circle*, an intersection of the sphere (globe) and a plane that passes through the center point of the sphere (globe). A great circle is relevant to a straight line on the sphere.

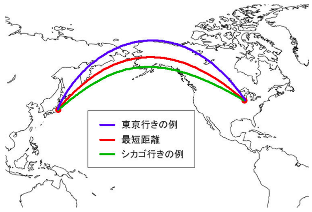

Figure 22: Flight route from Japan to US.

If we do not have a view from outside, then we naturally regard a great circle as a straight line. Is it, then, possible to get them know this straight line is bending or they live on a sphere?

The answer is yes. Let them write a triangle and calculate the total inner angles of it. If they live on a flat space, then it is 180°, while on a sphere it is larger than 180°. That is, measuring the total inner angles of a triangle tells us how our space is curved.

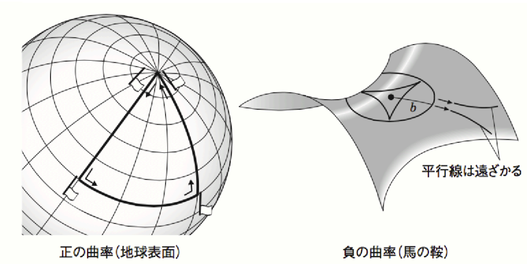

Figure 23: Riemannian geometry treats such a curved space. When we draw a triangle on a globe, the total inner angles of a triangle is greater than 180°. If we draw a triangle on a horse back, then it will be less than 180°. If you hold a flag and walk on this triangle, then you will find the direction of the flag is different when you come back to the starting point. These facts suggests the space is curved.

This analysis can be also expressed as a person who holds a flag and make a circle turn around a point (Fig.23). When he/she returns to the starting point and finds that the direction of the flag is different, then we can judge the space is curved.

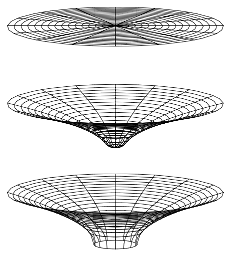

Figure 24: Imagine trampoline for curved space.

Curvature in Riemannian geometry is the generalization of this idea. If we hold a flag (a vector in precisely) and go around a point, and if we find the direction of a flag (vector) is the same with the initial one then we judge the space is flat. Otherwise we say the space is curved. Einstein tried to express the effect of gravity by an idea of "massive object make surrounding space-time curved".

Imagine a trampoline. If we put nothing, then a trampoline keeps flat surface. If we put an object, then the surface of trampoline begins bending until its tension compensates against the mass of the object. If we put a drop of water on the bended trampoline, it will move to the object along the curved surface. Therefore we can regard the effect of gravity as the curved surface of trampoline.

General relativity is the generalization of this trampoline (2-dimensional surface) to 4-dimensional space and time. Einstein found out that the origin of gravity is the curvature of space-time.

2.3 Riemannian Geometry

Let's get some feeling of Riemannian Geometry from equations (you can skip this subsection).

You can skip this subsection, §2.3.

We now write 3-dimensional space and time coordinate unitedly as $x^\mu = (ct, x^1, x^2, x^3)$. The index μ moves 0, 1, 2, 3 and indicates each component of space and time from now on. The time t [s] times c [m/s] (the speed of light) become the dimension of space [m], so that ct is useful combination for treating time with spacial coordinate.

First, we express the flat space-time (Minkowskii space-time). In the flat space-time, the distance ds between two points in space and time with each coordinate differences $c\,dt, dx, dy, dz$ are written as

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2. \tag{2.11}$$

In order to write (2.11) shorter, we express this equation as

$\eta_{\mu\nu}$: metric tensor of the flat space-time

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{where } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

We also write $\sum_{\mu=0}^{3}$ as $\sum_{\mu}$, hereafter.

When space-time is curved, we generalize metric tensor as a function of coordinate as $g_{\mu\nu}(x)$. Therefore the distance between two points can be written as

$$ds^2 = \sum_{\mu}\sum_{\nu} g_{\mu\nu}(x) dx^\mu dx^\nu, \qquad g_{\mu\nu}(x) = \begin{pmatrix} g_{00}(x) & g_{01}(x) & g_{02}(x) & g_{03}(x) \\ g_{01}(x) & g_{11}(x) & g_{12}(x) & g_{13}(x) \\ g_{02}(x) & g_{12}(x) & g_{22}(x) & g_{23}(x) \\ g_{03}(x) & g_{13}(x) & g_{23}(x) & g_{33}(x) \end{pmatrix}. \tag{2.13}$$

The metric tensor $g_{\mu\nu}(x)$ is symmetric ($g_{\mu\nu} = g_{\nu\mu}$), so that we have 10 independent components in 4-dimensional space-time.

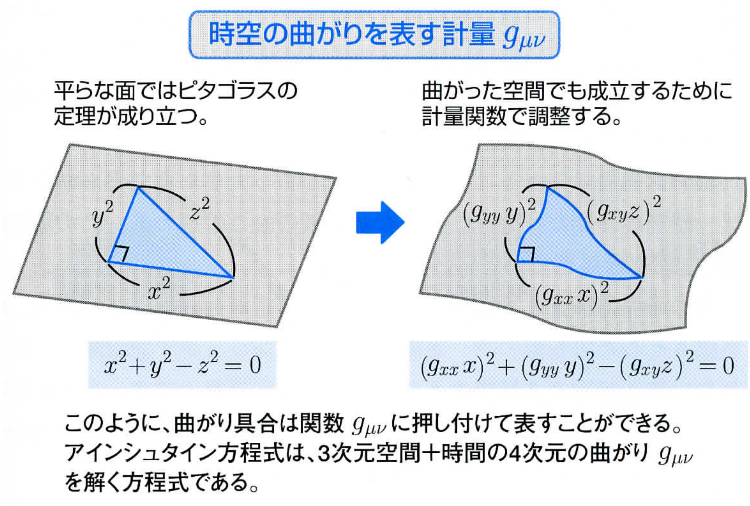

Figure 25: The metric function $g_{\mu\nu}$ expresses how the space-time is curved.

Metric $g_{\mu\nu}$ expresses how space-time curves at each point. From metric, we can calculate curvature at each point.

Elwin B. Christoffel (1829–1900)

We omit the details, but the procedure of calculating curvature is following: Define Christoffel symbol $\Gamma^\alpha_{\mu\nu}$ as

$$\Gamma^\alpha_{\mu\nu} = \sum_\beta \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right), \tag{2.14}$$

and define Riemann's curvature tensor $R^\mu{}_{\nu\alpha\beta}$

Riemann curvature tensor

$$R^\mu{}_{\nu\alpha\beta} = \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} - \frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} + \sum_\sigma \Gamma^\mu{}_{\sigma\alpha} \Gamma^\sigma{}_{\nu\beta} - \sum_\sigma \Gamma^\mu{}_{\sigma\beta} \Gamma^\sigma{}_{\nu\alpha}. \tag{2.15}$$

We also prepare Ricci tensor and Ricci scalar by contracting (summing up the internal indice) as

Gregorio Ricci-Curbastro (1853–1925)

$$\text{Ricci tensor} \quad R_{\mu\nu} = \sum_\alpha R^\alpha{}_{\mu\alpha\nu}, \tag{2.16}$$

$$\text{and Ricci scalar} \quad R = \sum_\mu \sum_\nu g^{\mu\nu} R_{\mu\nu}. \tag{2.17}$$

2.4 The Einstein equation

After many trials and errors, on November 25, 1915, Einstein reached his conclusion. He found the fundamental equation of the gravity, which we call the field equation or the Einstein equation. The massive object will bend the space and time, which produces gravity. He named this theory as the general theory of relativity (general relativity), and he called his theory of 1905 as the special theory of relativity (special relativity) since the previous one was limited to no acceleration cases.

The field equation (The Einstein equation)

The field equation (The Einstein equation)

The origin of the gravity is the curvature of space-time. The relation is expressed as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.18}$$

- The left-hand side expresses how space-time is curved using Riemannian geometry.
- The right-hand side expresses how the matter distributes in space-time. ($T_{\mu\nu}$ is called the energy-momentum tensor, and $T_{\mu\nu} = 0$ if vacuum.)
- The indice μ, ν indicate the coordinate (t, x, y, z), so that (2.18) consists of 10 equations.

This equation tells us that if the matter exists then the surrounding space-time curves. If space-time curves, then the matter moves along the curve (geodesics).

General relativity describes strong gravitational field, such as compact objects (neutron stars, black holes) and/or the whole Universe.

The Einstein equation (2.18) is the equation for the metric $g_{\mu\nu}$. By imposing the matter distribution, we can solve how space-time curves. The equation (2.18) predicted the existence of black-holes, and the Universe is expanding, both of which interestingly Einstein did not believe at their first appearances.

2.5 How and Why we know there is a black hole

Schwarzschild solution (black-hole solution)

The solution of the Einstein equation (2.18) of spherically symmetric, static, and vacuum space-time.

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{2.19}$$

where G and c are gravitational constant and speed of light, M is the mass at the center, r is radial coordinate.

Figure 26: Karl Schwarzschild (1873–1916)

This solution was derived by Schwarzschild in 1916, and after 50 years it is finally understood that this metric describes a black-hole.

A black hole is a region of space-time with strong gravitational effects that nothing, including the fastest light, can escape from its inside. The theory of general relativity predicts that a sufficiently compact mass can deform space-time to form a black hole.

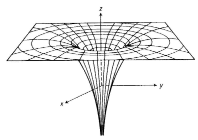

Figure 27: Black-hole is the bended trampoline to infinity.

Discussion Black holes, therefore, do not show themselves. Then, how can we know there is a black hole? And why we believe there are black holes? Let's think on these issues at the session.

Exercise 3 The radius of black-hole, R_{BH} can be given by

Ex.3 Size of black-hole.

$$R_{BH} = \frac{2GM}{c^2}, \tag{2.20}$$

where M is the mass of black-hole, $G = 6.67 \times 10^{-11}[\mathrm{m^3/kg/s^2}]$ is gravitational constant, and $c = 3.0 \times 10^8[\mathrm{m/s}]$ is the speed of light. Calculate the size of

- an Earth-mass black hole ($M = 6.0 \times 10^{24}$[kg])
- a Sun-mass black hole ($M_\odot = 2.0 \times 10^{30}$[kg])
- a black-hole of the center of our galaxy ($M = 4.2 \times 10^6 M_\odot$)

2.6 How we know the Universe is expanding

If we apply general relativity to describe the whole Universe, we obtain the following solution:

Alexander A. Friedman (1888–1925)
Georges-Henri Lemaitre (1894–1966)
Howard P. Robertson (1903–1961)
Arthur G. Walker (1909–2001)

The Expanding Universe (FLRW solution)

If we suppose the Universe is filled with normal single matter homogeneously and isotropically, then the Universe can be expressed as

$$ds^2 = -c^2dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right\} \quad (2.21)$$

where $a(t)$ is the scale factor (the radius of the Universe), and k is the total curvature of space-time [space-time is open ($k > 0$), flat ($k = 0$), or closed ($k < 0$)].

This solution was independently obtained by Friedman, Lemaitre, Robertson and Walker. The scale factor $a(t)$ indicates the dynamical behavior of the Universe is inevitable: the Universe is expanding or shrinking depending on the total energy.

Figure 28: Edwin P. Hubble(1889–1953)

Einstein did not admit the Universe is dynamical. He believed that the Universe stays itself from infinite past to infinite future. In order to realize such a solution, Einstein modified his field equation by adding a constant term, which he called the cosmological constant. However, in 1929, Edwin Hubble reported that the Universe is expanding by observing galaxies in far region.

Homework How did Hubble find that the Universe is expanding? Use the word Doppler effect for explanation.

HW The Universe is expanding.

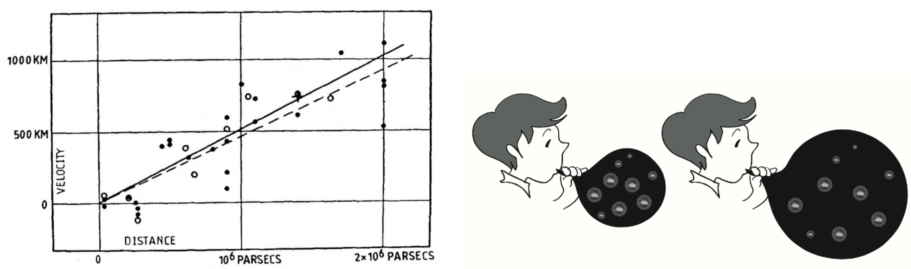

Figure 29: *Left:* Observation result by Hubble (1929). The backward moving velocity of galaxies as a function of the distance is plotted. *Right:* Far galaxies move faster, which does not mean that our galaxy is the center of the Universe.