Advantages of a modified ADM formulation: Constraint propagation analysis of the Baumgarte-Shapiro-Shibata-Nakamura system

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Several numerical relativity groups are using a modified Arnowitt-Deser-Misner (ADM) formulation for their simulations, which was developed by Nakamura and co-workers (and widely cited as the Baumgarte-Shapiro-Shibata-Nakamura system). This so-called BSSN formulation is shown to be more stable than the standard ADM formulation in many cases, and there have been many attempts to explain why this reformulation has such an advantage. We try to explain the background mechanism of the BSSN equations by using an eigenvalue analysis of constraint propagation equations. This analysis has been applied and has succeeded in explaining other systems in our series of works. We derive the full set of the constraint propagation equations, and study it in the flat background space-time. We carefully examine how the replacements and adjustments in the equations change the propagation structure of the constraints, i.e., whether violation of constraints (if it exists) will decay or propagate away. We conclude that the better stability of the BSSN system is obtained by their adjustments in the equations, and that the combination of the adjustments is in a good balance, i.e., a lack of their adjustments might fail to obtain the present stability. We further propose other adjustments to the equations, which may offer more stable features than the current BSSN equations.

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I. INTRODUCTION

One of the most important current topics in the field of numerical relativity is to find a formulation of the Einstein equations which gives us stable and accurate long-term evolution. We all know that simulating space-time and matter based on general relativity is the essential research direction to go in the future, but we do not have a definite recipe for controlling numerical blow-ups. We concentrate our discussion on the free evolution of the Einstein equations based on the 3+1 (space+time) decomposition of space-time, which requires solving the constraints only on the initial hypersurface and monitors the violation (error) of the calculation by checking constraints during the evolution.

Over the decades, the Arnowitt-Deser-Misner (ADM) [1] formulation has been treated as the default by numerical relativists. (More precisely, the version introduced by Smarr and York [2] was taken as the default, which we denote the standard ADM formulation hereafter.) Although the ADM formulation mostly works for gravitational collapse or cosmological models in numerical treatments, it does not satisfy the requirement for long-term evolution, e.g., the studies of gravitational wave sources.

As we mentioned in our previous paper [3], we think we can classify the current efforts of formulating equations for numerical relativity in the following three ways: (i) apply a modified ADM [Baumgarte-Shapiro-Shibata-Nakamura (BSSN)] formulation [4,5], (ii) apply a first-order hyperbolic formulation (see the references, e.g., in [6–8]), or (iii) apply an asymptotically constrained system [9–12].

The first refers to using a modified ADM formulation, originally proposed by Nakamura in the late 1980s, and subsequently modified by Nakamura-Oohara and Shibata-Nakamura [4]. This introduces conformal decomposition of the ADM variables, a new variable for calculating Ricci curvature, and adjusts the equations of motion using constraints. The advantage of this formulation was reintroduced by Baumgarte and Shapiro [5], and therefore this is often cited as the BSSN formulation, which we follow also. The BSSN equations are now widely used in the large-scale numerical computations, including coalescence of binary neutron stars [13] and binary black holes [14].

The second and third efforts use similar modifications such as introductions of new variables and/or adjustments of the equations, but differ in their purposes: to construct a hyperbolic formulation or to construct a formulation which constraints will decay or propagate away. The latter is intended to control numerical evolution so that the constrained manifold is its attractor. While the hyperbolic formulations have been extensively studied in this direction, we think the worrisome point in the discussion is the treatment of the nonprincipal part which is ignored in the hyperbolic formulation. As Kidder, Scheel, and Teukolsky [8] reported recently, unless we reduce the effect of the nonprincipal part of the equations we may not gain any advantages from the hyperbolic formulation for numerical results [6,15].

Through the series of studies [3,6,12,16], we propose a systematic treatment for constructing a robust evolution system against perturbative error. We call it an asymptotically constrained (or asymptotically stable) system if the error decays itself. The idea is to adjust evolution equations using

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constraints (we call this an adjusted system) and to decide the coefficients (multipliers) by analyzing constraint propagation equations. We propose to apply an eigenvalue analysis of the propagation equations of the constraints, especially in its Fourier components, so as to include the nonprincipal part in the analysis. The characters of eigenvalues will be changed according to the adjustments to the original evolution equations. We conjectured that the constraint violation that occurred during the evolution will decay (if negative real eigenvalues) or propagate away (if pure imaginary eigenvalues).

This conjecture was confirmed to explain the following numerical behaviors: wave propagation in the Maxwell equations [12], in the Ashtekar version of the Einstein equations [12], and in the ADM formulation (flat space-time background) [16]. The advantage of this construction scheme is that it can be applied to a formulation which is not a first-order hyperbolic form, such as to the ADM formulation [3,16]. We think, therefore, that our proposal is an alternative way to control or predict the violation of constraints. (We believe that the idea of the constraint propagation analysis first appeared in Frittelli [17], where she derived a hyperbolicity classification for the standard ADM formulation.)

The purpose of this paper is to apply this constraint propagation analysis to the BSSN formulation, and understand how each improvement contributes to more stable numerical evolution. Together with numerical comparisons with the standard ADM case [18,19], this topic has been studied by many groups with different approaches. Using numerical test evolutions, Alcubierre *et al.* [20] found that the essential improvement is in the process of replacing terms by constraints, and that the eigenvalues of the BSSN *evolution equations* have fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability can be caused by "zero eigenvalues" that violate the "gauge mode." Miller [21] applied von Neumann's stability analysis to the plane-wave propagation, and reported that BSSN has a wider range of parameters that give us stable evolution. These studies provide some support regarding the advantage of BSSN, while it also showed an example of an ill-posed solution in BSSN (as well as in ADM) [22]. (Inspired by BSSN's conformal decomposition, several related hyperbolic formulations have also been proposed [23–25].)

We think our analysis will offer a new vantage point on the topic, and contribute an alternative understanding of its background. Consequently, we propose a more effective improvement of the BSSN system that has not yet been tried in numerical simulations.

The construction of this paper is as follows. We review the BSSN system in Sec. II, and also we discuss where the adjustments are applied. In Sec. III, we apply our constraint propagation analysis to show how each improvement works in the BSSN equations, and in Sec. IV we extend our study to seek a better formulation which might be obtained by small steps. We only consider the vacuum space-time throughout the paper, but the inclusion of matter is straightforward.

II. BSSN EQUATIONS AND THEIR CONSTRAINT PROPAGATION EQUATIONS

A. BSSN equations

We start by presenting the standard ADM formulation, which expresses the space-time with a pair of 3-metric γ_{ij} and extrinsic curvature K_{ij}. The evolution equations become

$$\partial_t^A \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{2.1}$$

$$\partial_t^A K_{ij} = \alpha R_{ij}^{\mathrm{ADM}} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha + (D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij}, \tag{2.2}$$

where α, β_i are the lapse and shift function and D_i is the covariant derivative on 3-space. The symbol ∂_t^A means the time derivative defined by these equations, and we distinguish them from those of the BSSN equations ∂_t^B, which will be defined in Eqs. (2.15)–(2.19). The associated constraints are the Hamiltonian constraint $\mathcal{H}$ and the momentum constraints $\mathcal{M}_i$:

$$\mathcal{H}^{\mathrm{ADM}} = R^{\mathrm{ADM}} + K^2 - K_{ij}K^{ij}, \tag{2.3}$$

$$\mathcal{M}_i^{\mathrm{ADM}} = D_j K^j{}_i - D_i K. \tag{2.4}$$

The widely used notation [4,5] is to introduce the variables $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$ instead of (γ_{ij}, K_{ij}), where

$$\varphi = (1/12)\log(\det \gamma_{ij}), \tag{2.5}$$

$$\tilde{\gamma}_{ij} = e^{-4\varphi}\gamma_{ij}, \tag{2.6}$$

$$K = \gamma^{ij}K_{ij}, \tag{2.7}$$

$$\tilde{A}_{ij} = e^{-4\varphi}[K_{ij} - (1/3)\gamma_{ij}K], \tag{2.8}$$

$$\tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}. \tag{2.9}$$

The new variable $\tilde{\Gamma}^i$ was introduced in order to calculate Ricci curvature more accurately. $\tilde{\Gamma}^i$ also contributes to making the system reproduce wave equations in its linear limit. In the BSSN formulation, Ricci curvature is not calculated as

$$R_{ij}^{\mathrm{ADM}} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li}, \tag{2.10}$$

but

$$R_{ij}^{\mathrm{BSSN}} = R_{ij}^{\varphi} + \tilde{R}_{ij}, \tag{2.11}$$

$$R_{ij}^{\varphi} = -2\tilde{D}_i\tilde{D}_j\varphi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\varphi + 4(\tilde{D}_i\varphi)(\tilde{D}_j\varphi) - 4\tilde{\gamma}_{ij}(\tilde{D}^k\varphi)(\tilde{D}_k\varphi), \tag{2.12}$$

$$\tilde{R}_{ij} = -(1/2)\tilde{\gamma}^{lk}\partial_l\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{\gamma}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}, \tag{2.13}$$

where $\tilde{D}_i$ is a covariant derivative associated with $\tilde{\gamma}_{ij}$. These are weakly equivalent, but R_{ij}^{BSSN} does have a wave operator apparently in the flat background limit, so that we can expect more natural wave propagation behavior.

Additionally, BSSN requires us to impose the conformal factor as

$$\tilde{\gamma}(:=\det \tilde{\gamma}_{ij})=1 \tag{2.14}$$

during the evolution. This is a kind of definition, but can also be thought of as a constraint. We will return to this point shortly.

BSSN's improvements are not only the introductions of new variables, but also the replacement of terms in the evolution equations using the constraints. The purpose of this paper is to understand and to identify which improvement works for the stability. Before doing that, we first show the standard set of the BSSN evolution equations:

$$\partial_t^B \varphi=-(1/6)\alpha K+(1/6)\beta^i(\partial_i\varphi)+(\partial_i\beta^i), \tag{2.15}$$

$$\partial_t^B\tilde{\gamma}_{ij}=-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{ik}(\partial_j\beta^k)+\tilde{\gamma}_{jk}(\partial_i\beta^k)-(2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k)+\beta^k(\partial_k\tilde{\gamma}_{ij}), \tag{2.16}$$

$$\partial_t^B K=-D^iD_i\alpha+\alpha\tilde{A}_{ij}\tilde{A}^{ij}+(1/3)\alpha K^2+\beta^i(\partial_i K), \tag{2.17}$$

$$\partial_t^B\tilde{A}_{ij}=-e^{-4\varphi}(D_iD_j\alpha)^{\mathrm{TF}}+e^{-4\varphi}\alpha(R_{ij}^{\mathrm{BSSN}})^{\mathrm{TF}}+\alpha K\tilde{A}_{ij}-2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j+(\partial_i\beta^k)\tilde{A}_{kj}+(\partial_j\beta^k)\tilde{A}_{ki}-(2/3)(\partial_k\beta^k)\tilde{A}_{ij}+\beta^k(\partial_k\tilde{A}_{ij}), \tag{2.18}$$

$$\begin{aligned}\partial_t^B\tilde{\Gamma}^i=&-2(\partial_j\alpha)\tilde{A}^{ij}\\&+2\alpha[\tilde{\Gamma}^i_{jk}\tilde{A}^{kj}-(2/3)\tilde{\gamma}^{ij}(\partial_jK)+6\tilde{A}^{ij}(\partial_j\varphi)]\\&-\partial_j[\beta^k(\partial_k\tilde{\gamma}^{ij})-\tilde{\gamma}^{kj}(\partial_k\beta^i)-\tilde{\gamma}^{ki}(\partial_k\beta^j)\\&+(2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)].\end{aligned} \tag{2.19}$$

We next summarize the constraints in this system. The normal Hamiltonian and momentum constraints (the "kinematic" constraints) are naturally written as

$$\mathcal{H}^{\mathrm{BSSN}}=R^{\mathrm{BSSN}}+K^2-K_{ij}K^{ij}, \tag{2.20}$$

$$\mathcal{M}_i^{\mathrm{BSSN}}=\mathcal{M}_i^{\mathrm{ADM}}, \tag{2.21}$$

where we use the Ricci scalar defined by Eq. (2.11). Additionally, we regard the following three as the constraints (the "algebraic" constraints):

$$\mathcal{G}^i=\tilde{\Gamma}^i-\tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{2.22}$$

$$\mathcal{A}=\tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{2.23}$$

$$\mathcal{S}=\tilde{\gamma}-1, \tag{2.24}$$

where the first two are from the algebraic definition of the variables (2.8) and (2.9), and Eq. (2.24) is from the requirement of Eq. (2.14). Hereafter, we write $\mathcal{H}^{\mathrm{BSSN}}$ and $\mathcal{M}^{\mathrm{BSSN}}$ simply as $\mathcal{H}$ and $\mathcal{M}$, respectively.

Taking careful account of these constraints, Eqs. (2.20) and (2.21) can be expressed directly as

$$\mathcal{H}=e^{-4\varphi}\tilde{R}-8e^{-4\varphi}\tilde{D}^j\tilde{D}_j\varphi-8e^{-4\varphi}(\tilde{D}^j\varphi)(\tilde{D}_j\varphi)+(2/3)K^2-\tilde{A}_{ij}\tilde{A}^{ij}-(2/3)\mathcal{A}K, \tag{2.25}$$

$$\mathcal{M}_i=6\tilde{A}^j{}_i(\tilde{D}_j\varphi)-2\mathcal{A}(\tilde{D}_i\varphi)-(2/3)(\tilde{D}_iK)+\tilde{\gamma}^{kj}(\tilde{D}_j\tilde{A}_{ki}). \tag{2.26}$$

In summary, the fundamental dynamical variables in BSSN are $(\varphi,\tilde{\gamma}_{ij},K,\tilde{A}_{ij},\tilde{\Gamma}^i)$, a total of 17. The gauge quantities are (α,β^i), which is four, and the constraints are $(\mathcal{H},\mathcal{M}_i,\mathcal{G}^i,\mathcal{A},\mathcal{S})$, i.e., nine components. As a result, four (2 by 2) components are left which correspond to two gravitational polarization modes.

B. Adjustments in evolution equations

Next, we show the BSSN evolution equations (2.15)–(2.19) again, identifying where the terms are replaced using the constraints (2.20)–(2.24).

By a straightforward calculation, we get

$$\partial_t^B\varphi=\partial_t^A\varphi+(1/6)\alpha\mathcal{A}-(1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, \tag{2.27}$$

$$\partial_t^B\tilde{\gamma}_{ij}=\partial_t^A\tilde{\gamma}_{ij}-(2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A}+(1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{2.28}$$

$$\partial_t^BK=\partial_t^AK-(5/3)\alpha K\mathcal{A}-\alpha\mathcal{H}+\alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{2.29}$$

$$\begin{aligned}\partial_t^B\tilde{A}_{ij}=&\partial_t^A\tilde{A}_{ij}+[(1/3)\alpha\tilde{\gamma}_{ij}K-(2/3)\alpha\tilde{A}_{ij}]\mathcal{A}\\&+[(1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij})-(1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S})]\mathcal{G}^k+\alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k)-(1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\partial_t^B\tilde{\Gamma}^i=&\partial_t^A\tilde{\Gamma}^i+[-(2/3)(\partial_j\alpha)\tilde{\gamma}^{ji}-(2/3)\alpha(\partial_j\tilde{\gamma}^{ji})-(1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S})+4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi)]\mathcal{A}-(2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A})+2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j\\&-(1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S})+(1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S})+(1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S})+(5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S})\\&+(1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S})+(1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}),\end{aligned} \tag{2.31}$$

where ∂_t^A denotes the part of no replacements, i.e., the terms only use the standard ADM evolution equations in its time derivatives.

From Eqs. (2.27)–(2.31), we understand that all the BSSN evolution equations are *adjusted* using constraints. This fact will give us the importance of the scaling constraint $\mathcal{S}=0$ and the trace-free operation $\mathcal{A}=0$ during the evolution.

As we have pointed out in the case of adjusted ADM systems [16,3], certain combinations of adjustments (replacements) in the evolution equations change the eigenvalues of constraint propagation equations drastically. For example, all negative eigenvalues can be negative real by applying Detweiler's adjustment [26] or its simplified version. One common fact we found is that such a case has an adjustment which breaks time-reversal parity of the original equation. That is, with a change of time integration direction $\partial_t \to -\partial_t$, an adjusted term might become effective if it breaks time-reversal symmetry. (This time asymmetric feature was first implemented as a "lambda-system" in [9].) Unfortunately, for the case of the BSSN equations, Eqs. (2.27)–(2.31), all the above adjustments keep the time-reversal symmetry, so that we cannot expect direct decays of constraint violation in the present form. We will give the details on this point later.

III. CONSTRAINT PROPAGATION ANALYSIS IN FLAT SPACE-TIME

A. Procedures

We start this section overviewing the procedures and our goals. In our series of previous works [3,12,16], we have concluded that eigenvalue analysis of the constraint propagation equations is quite useful for explaining or predicting how the constraint violation grows.

Suppose we have a set of dynamical variables $u^a(x^i,t)$ and their evolution equations

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots), \tag{3.1}$$

and the (first class) constraints

$$C^\alpha(u^a, \partial_i u^a, \ldots) \approx 0. \tag{3.2}$$

For monitoring the violation of constraints, we propose to investigate the evolution equations of C^α (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \ldots). \tag{3.3}$$

[We do not mean to integrate Eq. (3.3) numerically, but rather to evaluate it analytically in advance.] In order to analyze the contributions of all right-hand-side terms in Eq. (3.3), we propose to reduce Eq. (3.3) in ordinary differential equations by Fourier transformation,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \tag{3.4}$$

where $C(x,t)^\rho = \int \hat{C}(k,t)^\rho \exp(ik\cdot x) d^3k$, and then to analyze the set of eigenvalues, say Λ_α, of the coefficient matrix, $M^\alpha{}_\beta$, in Eq. (3.4). We call Λ's and $M^\alpha{}_\beta$ the constraint amplification factors (CAFs) of Eq. (3.3) and constraint propagation matrix, respectively. Our guidelines to have "better stability" are that (A) if the CAFs have a *negative real part* (the constraints are forced to be diminished), then we see more stable evolution than a system which has a positive real part, and (B) if the CAFs have a *nonzero imaginary part* (the constraints are propagating away), then we see more stable evolution than a system which has zero CAFs. We found heuristically that the system becomes more stable when more Λ's satisfy the above criteria [6,12]. We note that these guidelines are confirmed numerically for wave propagation in the Maxwell system and in the Ashtekar version of the Einstein system [12], and also for error propagation in Minkowskii space-time using adjusted ADM systems [16]. Supporting theorems for guideline (A) were recently discussed [31].

The above features of the constraint propagation, Eq. (3.3), will differ when we modify the original evolution equations. If we add (adjust) the evolution equations using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots) + F(C^\alpha, \partial_i C^\alpha, \ldots), \tag{3.5}$$

then Eq. (3.3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \ldots) + G(C^\alpha, \partial_i C^\alpha, \ldots). \tag{3.6}$$

Therefore, the problem is how to adjust the evolution equations so that their constraint propagation satisfies the above criteria as much as possible.

B. BSSN constraint propagation equations

Our purpose in this section is to apply the above procedure to the BSSN system. The set of the constraint propagation equations, $\partial_t(\mathcal{H}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$, turns to be quite long and not elegant (it is not a first-order hyperbolic and includes many nonlinear terms), and we put them in the Appendix. In order to understand the fundamental structure, we hereby show an analysis on the flat space-time background.

For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first-order perturbation equations of Eqs. (2.27)–(2.31) can be written as

$$\partial_t {}^{(1)}\phi = -(1/6)\,{}^{(1)}K + (1/6)(\kappa_\phi - 1)\,{}^{(1)}\mathcal{A}, \tag{3.7}$$

$$\partial_t {}^{(1)}\tilde{\gamma}_{ij} = -2\,{}^{(1)}\tilde{A}_{ij} - (2/3)(\kappa_{\tilde{\gamma}} - 1)\,\delta_{ij}\,{}^{(1)}\mathcal{A}, \tag{3.8}$$

$$\partial_t {}^{(1)}K = -(\partial_j \partial_j {}^{(1)}\alpha) + (\kappa_{K1} - 1)\,\partial_j {}^{(1)}\mathcal{G}^j - (\kappa_{K2} - 1)\,{}^{(1)}\mathcal{H}, \tag{3.9}$$

$$\begin{aligned}\partial_t {}^{(1)}\tilde{A}_{ij} = {}& {}^{(1)}(R^{\rm BSSN}_{ij})^{\rm TF} - {}^{(1)}(\tilde{D}_i \tilde{D}_j \alpha)^{\rm TF} \\ &+ (\kappa_{A1} - 1)\,\delta_{k(i}(\partial_{j)}\,{}^{(1)}\mathcal{G}^k) \\ &- (1/3)(\kappa_{A2} - 1)\,\delta_{ij}(\partial_k {}^{(1)}\mathcal{G}^k),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\partial_t {}^{(1)}\tilde{\Gamma}^i = {}& -(4/3)(\partial_i {}^{(1)}K) - (2/3)(\kappa_{\tilde{\Gamma}1} - 1)(\partial_i {}^{(1)}\mathcal{A}) \\ &+ 2(\kappa_{\tilde{\Gamma}2} - 1)\,{}^{(1)}\mathcal{M}_i,\end{aligned} \tag{3.11}$$

TABLE I. Summary of Sec. III C: contributions of adjustment terms and effects of introductions of new constraints in the BSSN system. The center column indicates whether each constraint is taken as a component of constraints in each constraint propagation analysis (''use''), and whether each adjustment is on (''adj''). The column ''diag?'' indicates diagonalizability of the constraint propagation matrix. The right column shows CAFs, where Im and Re mean pure imaginary and real eigenvalue, respectively. Case (0) (standard ADM) is shown in [16].

No. in text		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	diag?	CAFs in Minkowskii background
(0)	standard ADM	use	use				yes	(0,0,Im,Im)
(i)	BSSN no adjustment	use	use	use	use	use	yes	(0,0,0,0,0,0,0,Im,Im)
(ii)	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	no	(0,0,0,Im,Im,Im,Im,Im,Im)
(iii)	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no	no difference in flat background
(iv)	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	no	(0,0,0,Im,Im,Im,Im,Im,Im)
(v)	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	no	(0,0,0,0,0,0,0,Im,Im)
(vi)	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	no	(0,0,0,0,0,0,0,Re,Re)
(vii)	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	no	(0,0,0,Im,Im,Im,Im,Im,Im)

where we introduced parameters κ's, all $\kappa=0$ reproduce the no-adjustment case from the standard ADM equations, and all $\kappa=1$ correspond to the BSSN equations. We express them as

$$\kappa_{\rm adj}:=(\kappa_\varphi,\kappa_{\tilde\gamma},\kappa_{K1},\kappa_{K2},\kappa_{A1},\kappa_{A2},\kappa_{\tilde\Gamma 1},\kappa_{\tilde\Gamma 2}). \quad (3.12)$$

Constraint propagation equations at the first order in the flat space-time, then, become

$$\partial_t{}^{(1)}\mathcal{H}=[\kappa_{\tilde\gamma}-(2/3)\kappa_{\tilde\Gamma 1}-(4/3)\kappa_\varphi+2]\partial_j\partial_j{}^{(1)}\mathcal{A} +2(\kappa_{\tilde\Gamma 2}-1)(\partial_j{}^{(1)}\mathcal{M}_j), \quad (3.13)$$

$$\partial_t{}^{(1)}\mathcal{M}_i=[-(2/3)\kappa_{K1}+(1/2)\kappa_{A1} -(1/3)\kappa_{A2}+(1/2)]\partial_i\partial_j{}^{(1)}\mathcal{G}^j+(1/2)\kappa_{A1}\partial_j\partial_j{}^{(1)}\mathcal{G}^i +[(2/3)\kappa_{K2}-(1/2)]\partial_i{}^{(1)}\mathcal{H}, \quad (3.14)$$

$$\partial_t{}^{(1)}\mathcal{G}^i=2\kappa_{\tilde\Gamma 2}{}^{(1)}\mathcal{M}_i+[-(2/3)\kappa_{\tilde\Gamma 1}-(1/3)\kappa_{\tilde\gamma}](\partial_i{}^{(1)}\mathcal{A}), \quad (3.15)$$

$$\partial_t{}^{(1)}\mathcal{S}=-2\kappa_{\tilde\gamma}{}^{(1)}\mathcal{A}, \quad (3.16)$$

$$\partial_t{}^{(1)}\mathcal{A}=(\kappa_{A1}-\kappa_{A2})(\partial_j{}^{(1)}\mathcal{G}^j). \quad (3.17)$$

We will discuss the CAFs of Eqs. (3.13)–(3.17).

C. Effect of adjustments

We check the CAFs of the BSSN equations in detail. The list of examples is shown also in Table I. Hereafter, we let $k^2=k_x^2+k_y^2+k_z^2$ for Fourier wave numbers.

(i) No-adjustment case, $\kappa_{\rm adj}=$(all zeros). This is the starting point of the discussion. In this case,

$$X_{\rm CAFs}=(0(\times 7),\pm\sqrt{-k^2}),$$

i.e., $(0(\times 7),\pm$ pure imaginary(one pair)). In the standard ADM formulation, which uses (γ_{ij},K_{ij}), CAFs are (0,0, $\pm$ pure imaginary) [16]. Therefore, if we do not apply adjustments in the BSSN equations, the constraint propagation structure is quite similar to that of the standard ADM formalism.

(ii) For the BSSN equations, $\kappa_{\rm adj}=$(all 1s),

$$X_{\rm CAFs}=(0(\times 3),\pm\sqrt{-k^2}(\text{three pairs})),$$

i.e., $(0(\times 3),\pm$ pure imaginary(three pairs)). The number of pure imaginary CAFs is increased over that of case (i), and we conclude this is the advantage of adjustments used in the BSSN equations.

(iii) No $\mathcal{S}$-adjustment case. All the numerical experiments so far apply the scaling condition $\mathcal{S}$ for the conformal factor φ. The $\mathcal{S}$-originated terms appear many places in the BSSN equations (2.15)–(2.19), so that we suspect nonzero $\mathcal{S}$ is a kind of source of the constraint violation. However, since all $\mathcal{S}$-originated terms do not appear in the flat space-time background analysis [no adjusted terms in Eqs. (3.7)–(3.11)], our analysis is independent of the $\mathcal{S}$ constraint. (Note that we do not deny the effect of $\mathcal{S}$ adjustment in other situations.)

(iv) No $\mathcal{A}$-adjustment case. The trace (or traceout) condition for the variables is also considered necessary (e.g., [27]). This can be checked with $\kappa_{\rm adj}=(\kappa,\kappa,1,1,1,1,\kappa,1)$, and we get

$$X_{\rm CAFs}=(0(\times 3),\pm\sqrt{-k^2}(\text{three pairs})),$$

independent of κ. Therefore, the effect of $\mathcal{A}$ adjustment is unimportant according to this analysis, i.e., on flat space-time background. (Note that we do not deny the effect of $\mathcal{A}$ adjustment in other situations.)

(v) No $\mathcal{G}^i$-adjustment case. The introduction of Γ^i is the key in the BSSN system. If we do not apply adjustments by $\mathcal{G}^i$ [$\kappa_{\rm adj}=(1,1,0,1,0,0,1,1)$], then we get

$$X_{\rm CAFs}=(0(\times 7),\pm\sqrt{-k^2}),$$

which is the same as case (i). That is, adjustments due to $\mathcal{G}^i$ terms are effective to make progress from the ADM method.

(vi) No $\mathcal{M}_i$-adjustment case. This can be checked with $\kappa_{\rm adj}=(1,1,1,1,1,1,1,\kappa)$, and we get

$$X_{\rm CAFs}=(0,\pm\sqrt{-\kappa k^2}(\text{two pairs}),\ \pm\sqrt{-k^2(-1+4\kappa+|1-4\kappa|)/6},\ \pm\sqrt{-k^2(-1+4\kappa-|1-4\kappa|)/6}).$$

If $\kappa=0$, then $(0(\times 7),\pm\sqrt{k^2/3})$, which is $(0(\times 7)$, $\pm$real value). Interestingly, these real values indicate the existence of the error-growing mode together with the decaying mode. Alcubierre *et al.* [20] found that the adjustment due to the momentum constraint is crucial for obtaining stability. We think that they picked up this error-growing mode. Fortunately at the BSSN limit ($\kappa=1$), this error-growing mode disappears and turns into a propagation mode.

(vii) No $\mathcal{H}$-adjustment case. The set $\kappa_{\rm adj}=(1,1,1,\kappa,1,1,1,1)$ gives

$$X_{\rm CAFs}=(0(\times 3),\pm\sqrt{-k^2}(3\ \text{ pairs})),$$

independently of κ. Therefore the effect of $\mathcal{H}$ adjustment is unimportant according to this analysis, i.e., on a flat spacetime background. (Note again that we do not deny the effect of $\mathcal{H}$ adjustment in other situations.)

These tests are on the effects of adjustments. We will consider whether much better adjustments are possible in the next section.

We list the above results in Table I. (Table I includes a column of diagonalizability of constraint propagation matrix M, the importance of which was pointed out in [31].) The most characteristic points of the above are (v) and (vi), which denote the contribution of the momentum constraint adjustment and the importance of the new variable $\tilde{\Gamma}^i$. It is quite interesting that the unadjusted BSSN equations [case (i)] does not have apparent advantages from the ADM system. As we showed in (v) and (vi), if we missed a particular adjustment, then the expected stability behavior occasionally gets worse than the starting ADM system. Therefore, we conclude that the better stability of the BSSN formulations is obtained by their adjustments in the equations, and the combination of the adjustments is in a good balance. That is, a lack of their adjustments might fail to bring about the stability of their system.

IV. PROPOSALS OF IMPROVED BSSN SYSTEMS

In this section, we consider the possibility of whether we can obtain a system which has much better properties, whether more pure imaginary CAFs or negative real CAFs.

A. Heuristic examples

1. A system which has eight pure imaginary CAFs

One direction is to seek a set of equations which make fewer zero CAFs than the standard BSSN case [point (ii) in the previous section]. Using the same set of adjustments in Eqs. (3.7)–(3.11), CAFs are written in general as

$$X_{\rm CAFs}=(0,\pm\sqrt{-k^2\kappa_{A1}\kappa_{\tilde{\Gamma}2}}(2\ \text{ pairs}),\ \pm\text{complicated expression},\ \pm\text{complicated expression}).$$

The terms in the first line certainly give four pure imaginary CAFs (two positive and negative real pairs) if $\kappa_{A1}\kappa_{\tilde{\Gamma}2}>0$ (<0). Keeping this in mind, by choosing $\kappa_{\rm adj}=(1,1,1,1,1,\kappa,1,1)$, we find

$$X_{\rm CAFs}=(0,\pm\sqrt{-k^2}(2\ \text{ pairs}),\ \pm\sqrt{-k^2(2+\kappa+|\kappa-4|)/6},\ \pm\sqrt{-k^2(2+\kappa-|\kappa-4|)/6}).$$

Therefore, the adjustment $\kappa_{\rm adj}=(1,1,1,1,1,4,1,1)$ gives

$$X_{\rm CAFs}=(0,\pm\sqrt{-k^2}(4\ \text{ pairs})),$$

which is one step advanced from BSSN's according our guidelines.

We note that such a system can be obtained in many ways, e.g., $\kappa_{\rm adj}=(0,0,1,0,2,1,0,1/2)$ also gives four pairs of pure imaginary CAFs.

2. A system which has negative real CAF

One criterion to obtain a decaying constraint mode (i.e., an asymptotically constrained system) is to adjust an evolution equation as it breaks time-reversal symmetry [16,3]. For example, we consider an additional adjustment to the BSSN equation as

$$\partial_t\tilde{\gamma}_{ij}=\partial_t^B\tilde{\gamma}_{ij}+\kappa_{\rm SD}\alpha\tilde{\gamma}_{ij}\mathcal{H}, \tag{4.1}$$

which is a similar adjustment of the simplified Detweiler type (SD) [26] that was discussed in [3]. The first-order constraint propagation equations on the flat background spacetime become

$$\partial_t{}^{(1)}\mathcal{H}=\partial_j\partial_j{}^{(1)}\mathcal{A}-(3/2)\kappa_{\rm SD}\partial_j\partial_j{}^{(1)}\mathcal{H},$$

$$\partial_t{}^{(1)}\mathcal{M}_i=(1/6)\partial_i{}^{(1)}\mathcal{H}+(1/2)\partial_j\partial_j{}^{(1)}\mathcal{G}^i,$$

$$\partial_t{}^{(1)}\mathcal{G}^i=-\partial_i{}^{(1)}\mathcal{A}+(1/2)\kappa_{\rm SD}\partial_i{}^{(1)}\mathcal{H}+2\,{}^{(1)}\mathcal{M}_i,$$

$$\partial_t{}^{(1)}\mathcal{A}=-(\partial_j\partial_j{}^{(1)}\alpha)^{\rm TF}+({}^{(1)}R_{jj}^{\rm BSSN})^{\rm TF},$$

$$\partial_t{}^{(1)}\mathcal{S}=-2\,{}^{(1)}\mathcal{A}+3\kappa_{\rm SD}{}^{(1)}\mathcal{H},$$

where we wrote only additional terms to Eqs. (3.13)–(3.17). The CAFs become

$$X_{\rm CAFs}=(0(\times 2),\pm\sqrt{-k^2}(\text{three pairs}),(3/2)k^2\kappa_{\rm SD}),$$

in which the last one becomes negative real if $\kappa_{\rm SD}<0$.

3. Combination of Secs. IV A 1 and IV A 2

Naturally we next consider both adjustments,

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{\rm SD} \alpha \tilde{\gamma}_{ij} \mathcal{H}, \tag{4.2}$$

$$\partial_t \tilde{A}_{ij} = \partial_t^B \tilde{A}_{ij} - \kappa_8 \alpha e^{-4\varphi} \tilde{\gamma}_{ij} \partial_k \mathcal{G}^k, \tag{4.3}$$

where the second one produces the eight pure imaginary CAFs. The additional terms in the constraint propagation equations (3.13)–(3.17) are

$$\partial_t {}^{(1)}\mathcal{H} = \partial_j \partial_j {}^{(1)}\mathcal{A} - (3/2)\kappa_{\rm SD} \partial_j \partial_j {}^{(1)}\mathcal{H},$$

$$\partial_t {}^{(1)}\mathcal{M}_i = (1/6)\partial_i {}^{(1)}\mathcal{H} + (1/2)\partial_j \partial_j {}^{(1)}\mathcal{G}^i - \kappa_8 \partial_i \partial_k {}^{(1)}\mathcal{G}^k,$$

$$\partial_t {}^{(1)}\mathcal{G}^i = -\partial_i {}^{(1)}\mathcal{A} + (1/2)\kappa_{\rm SD} \partial_i {}^{(1)}\mathcal{H} + 2\, {}^{(1)}\mathcal{M}_i,$$

$$\partial_t {}^{(1)}\mathcal{A} = -3\kappa_8 \partial_k {}^{(1)}\mathcal{G}^k,$$

$$\partial_t {}^{(1)}\mathcal{S} = -2\, {}^{(1)}\mathcal{A} + 3\kappa_{\rm SD} {}^{(1)}\mathcal{H}.$$

We then obtain

$$X_{\rm CAFs} = (0, \pm\sqrt{-k^2}(3 \text{ pairs}),$$
$$(3/4)k^2\kappa_{\rm SD} \pm \sqrt{k^2[-\kappa_8 + (9/16)k^2\kappa_{\rm SD}^2]}),$$

which reproduces Sec. IV A 1 when $\kappa_{\rm SD}=0$, $\kappa_8=1$, and Sec. IV A 2 when $\kappa_8=0$. These CAFs can become (0, pure imaginary (three pairs), complex numbers with a negative real part (one pair)), with an appropriate combination of κ_8 and $\kappa_{\rm SD}$.

B. Possible adjustments

In order to break time-reversal symmetry of the evolution equations [3,9,16], the possible simple adjustments are (i) to add $\mathcal{H}$, $\mathcal{S}$, or $\mathcal{G}^i$ terms to the equations of $\partial_t\phi$, $\partial_t\tilde{\gamma}_{ij}$, or $\partial_t\tilde{\Gamma}^i$, or (ii) to add $\mathcal{M}_i$ or $\mathcal{A}$ terms to $\partial_t K$ or $\partial_t \tilde{A}_{ij}$. We write them generally, including the proposal of Sec. IV A 2, as

$$\partial_t \phi = \partial_t^B \phi + \kappa_{\phi\mathcal{H}} \alpha \mathcal{H} + \kappa_{\phi\mathcal{G}} \alpha \tilde{D}_k \mathcal{G}^k, \tag{4.4}$$

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{\rm SD}\alpha\tilde{\gamma}_{ij}\mathcal{H} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}, \tag{4.5}$$

$$\partial_t K = \partial_t^B K + \kappa_{K\mathcal{M}} \alpha \tilde{\gamma}^{jk} (\tilde{D}_j \mathcal{M}_k), \tag{4.6}$$

$$\partial_t \tilde{A}_{ij} = \partial_t^B \tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\mathcal{A}1}\alpha\tilde{\gamma}_{ij}\mathcal{A} + \kappa_{A\mathcal{A}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}, \tag{4.7}$$

$$\partial_t \tilde{\Gamma}^i = \partial_t^B \tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j, \tag{4.8}$$

TABLE II. Possible adjustments which make a real-part CAF negative (Sec. IV B). The column of adjustments is nonzero multipliers in terms of Eqs. (4.4)–(4.8), which all violate time-reversal symmetry of the equation. The column "diag?" indicates diagonalizability of the constraint propagation matrix. Neg./Pos. means negative/positive, respectively.

Adjustment		CAFs	diag?	Effect of the adjustment	
$\partial_t\phi$	$\kappa_{\phi\mathcal{H}}\alpha\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),8\kappa_{\phi\mathcal{H}}k^2)$	no	$\kappa_{\phi\mathcal{H}}<0$ makes 1 Neg.	
$\partial_t\phi$	$\kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}}<0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\rm SD}\alpha\tilde{\gamma}_{ij}\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),(3/2)\kappa_{\rm SD}k^2)$	yes	$\kappa_{\rm SD}<0$ makes 1 Neg.	Sec. IV A 2
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1}>0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$	$(0,0,(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2}\pm\sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}$ $\times(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2}<0$ makes 6 Neg. 1 Pos.	Sec. IV B 2 *a*
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1}<0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2}>0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2),\ (1/3)\kappa_{K\mathcal{M}}k^2$ $\pm(1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}}<0$ makes 2 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{M}1}k^2)$	yes	$\kappa_{A\mathcal{M}1}>0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$	$(0,0,-k^2\kappa_{A\mathcal{M}2}/4\pm\sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}$ $\times(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2}>0$ makes 7 Neg	Sec. IV B 1
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\alpha\tilde{\gamma}_{ij}\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1}<0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	yes	$\kappa_{A\mathcal{A}2}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i$	$(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long exp.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1}<0$ makes 6 Neg. 1 Pos.	Sec. IV B 2 *b*
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long exp.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2}>0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long exp.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3}>0$ makes 2 Neg. 1 Pos.	

where κ's are possible multipliers (all $\kappa=0$ reduce the system of the standard BSSN evolution equations).

We show the effects of each term in Table II. The CAFs in the table are on the flat space background. We see that several terms produce negative real part in CAFs, which might improve the stability more than the previous system. (Table II includes again a column of diagonalizability of constraint propagation matrix M. Diagonalizable ones are expected to reflect the predictions from eigenvalue analysis. That is, the eigenvalue analysis with diagonalizable ones definitely avoids the diverging possibility in constraint propagation when it includes degenerated CAFs. See [31].) For the readers' convenience, we list several of the best candidates here.

1. A system which has seven negative CAFs

Simply adding the $\tilde{D}_{(i}\mathcal{M}_{j)}$ term to the $\partial_t\tilde{A}_{ij}$ equation, say

$$\partial_t\tilde{A}_{ij}=\partial_t^{\mathrm{BSSN}}\tilde{A}_{ij}+\kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) \tag{4.9}$$

with $\kappa_{A\mathcal{M}2}>0$, the CAFs on the flat background are seven negative real CAFs.

2. A system which has six negative CAFs

The below two adjustments will make six negative real CAFs, while they also produce one positive real CAF (a constraint-violating mode). The effectiveness is not clear at this moment, but we think they are worth testing in numerical experiments.

a. With $\kappa_{\tilde{\gamma}\mathcal{G}2}<0$,

$$\partial_t\tilde{\gamma}_{ij}=\partial_t^{\mathrm{BSSN}}\tilde{\gamma}_{ij}+\kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k. \tag{4.10}$$

b. With $\kappa_{\tilde{\Gamma}\mathcal{G}2}<0$,

$$\partial_t\tilde{\Gamma}^i=\partial_t^{\mathrm{BSSN}}\tilde{\Gamma}^i+\kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i. \tag{4.11}$$

V. CONCLUDING REMARKS

Applying the constraint propagation analysis, we tried to understand why and how the so-called BSSN (Baumgarte-Shapiro-Shibata-Nakamura) reformulation works better than the standard ADM equations in general relativistic numerical simulations. Our strategy was to evaluate eigenvalues of the constraint propagation equations in their Fourier modes, which succeeded to explain the stability properties in many other systems in our series of works.

We have studied step-by-step where the replacements in the equations affect and/or newly added constraints work, by checking whether the error of constraints (if it exists) will decay or propagate away. Alcubierre *et al.* [20] pointed out the importance of the replacement (adjustment) to the evolution equation using the momentum constraint, and our analysis clearly explains why they concluded that this is the key. Not only did we find this adjustment, but we also discovered that other adjustments and other introductions of new constraints also contribute to making the evolution system more stable. We found that if we missed a particular adjustment, then the expected stability behavior occasionally got worse than the ADM system. We further propose other adjustments of the set of equations which may have better features for numerical treatments.

The discussion in this paper was only in the flat background space-time, and may not be applicable directly to the general numerical simulations. However, we rather believe that the general fundamental aspects of constraint propagation analysis are already revealed in this paper. This is because, for the ADM system and its adjusted cases, we found that the better formulations in the flat background are also better in the Schwarzschild space-time, while there are differences in the effective adjusting multipliers or the effective coordinate ranges [3,16].

We have not shown any numerical tests here. However, recently, proposal (B) in Sec. IV was examined numerically using linear wave initial data and confirmed to be effective for controlling the violation of the Hamiltonian constraint with our predicted multiplier signature [28]. Systematic numerical comparisons between different formulations are underway [29], and we expect to have a chance to report them in the near future. We are also trying to explain the stability of Laguna-Shoemaker's implemented BSSN system [30] using the constraint propagation analysis.

There may not be the ultimate formulation for any models in numerical relativity, but we believe our guidelines to find a better formulation in a systematic way will contribute to progress in this field. We hope the predictions in this paper will help the community to make further improvements.

Note added in proof. Recently, Yo *et al.* [32] reported that the adjustment of Sec. IV A 2 in this paper is quite effective for long term stable numerical demonstration of Kerr-Schild space-time.

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APPENDIX: FULL SET OF BSSN CONSTRAINT PROPAGATION EQUATIONS

The constraint propagation equations of the BSSN system can be written as follows:

$$\begin{aligned}\partial_t\mathcal{H}=&[(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k]\mathcal{H}+[-4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde{\gamma}^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde{\gamma}^{jk}]\mathcal{M}_j+[-2\alpha e^{-4\varphi}\tilde{A}^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde{A}_{kl})\tilde{\gamma}^{kl}\\&-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\partial_k+(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j]\mathcal{G}^j\\&+[2\alpha e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\mathcal{A})\tilde{\gamma}^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\alpha)\tilde{\gamma}^{lk}\mathcal{A}\partial_k\end{aligned}$$

$$+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m\tilde{\gamma}^{lk}\partial_m\partial_l\partial_k-(5/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^m\tilde{\gamma}^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m(\partial_m\tilde{\gamma}^{lk})\partial_l\partial_k$$
$$+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^i(\partial_j\partial_i\tilde{\gamma}^{jk})\partial_k+(3/4)e^{-4\varphi}\tilde{\gamma}^{-3}\beta^i\tilde{\gamma}^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^i(\partial_i\tilde{\gamma}^{jk})(\partial_j\mathcal{S})\partial_k$$
$$+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{pj}(\partial_j\beta^k)\partial_p\partial_k-(5/12)e^{-4\varphi}\tilde{\gamma}^{-2}\tilde{\gamma}^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\beta^k)\partial_i$$
$$-(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{mk}(\partial_k\partial_l\beta^l)\partial_m]\mathcal{S}+[(4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde{\gamma}^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde{\gamma}^{lk})$$
$$+\alpha e^{-4\varphi}(\partial_j\tilde{\gamma}^{jk})\partial_k+8\alpha e^{-4\varphi}\tilde{\gamma}^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde{\gamma}^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde{\gamma}^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde{\gamma}^{lk})$$
$$+2e^{-4\varphi}(\partial_l\alpha)\tilde{\gamma}^{lk}\partial_k+e^{-4\varphi}\tilde{\gamma}^{lk}(\partial_l\partial_k\alpha)]\mathcal{A}, \tag{A1}$$

$$\partial_t\mathcal{M}_i=[-(1/3)(\partial_i\alpha)+(1/6)\partial_i]\mathcal{H}+\alpha K\mathcal{M}_i+[\alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)(\partial_j\tilde{\gamma}_{mi})-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}(\partial_j\tilde{\gamma}_{mi})$$
$$+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_k\partial_j\tilde{\gamma}_{mi})+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde{\gamma}_{kl})(\partial_j\tilde{\gamma}^{kl})+\alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)\tilde{\gamma}_{ji}\partial_m$$
$$+\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}\tilde{\gamma}_{ji}\partial_m+\alpha e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\Gamma}_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{lk}\tilde{\gamma}_{ji}\partial_k\partial_l+(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_j\tilde{\gamma}_{im})$$
$$\times(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\gamma}_{ji}(\partial_k\alpha)\partial_m]\mathcal{G}^j+[-\tilde{A}^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_i K)$$
$$+(1/9)\alpha K\partial_i-\alpha\tilde{A}^k{}_i\partial_k]\mathcal{A}, \tag{A2}$$

$$\partial_t\mathcal{G}^i=2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j+[-(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)\beta^k\tilde{\gamma}^{in}(\partial_k\tilde{\gamma}_{mn})\tilde{\gamma}^{ml}\tilde{\gamma}^{-1}\partial_l+(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-1}\partial_l\partial_k$$
$$-(1/2)(\partial_m\beta^i)\tilde{\gamma}^{mk}\tilde{\gamma}^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde{\gamma}^{ik}\tilde{\gamma}^{-1}\partial_k]\mathcal{S}+[+4\alpha\tilde{\gamma}^{ij}(\tilde{D}_j\varphi)-\alpha\tilde{\gamma}^{ij}\partial_j-(\partial_k\alpha)\tilde{\gamma}^{ik}]\mathcal{A}, \tag{A3}$$

$$\partial_t\mathcal{S}=+\beta^k(\partial_k\mathcal{S})-2\alpha\tilde{\gamma}\mathcal{A}, \tag{A4}$$

$$\partial_t\mathcal{A}=(\alpha K+\beta^k\partial_k)\mathcal{A}. \tag{A5}$$

The flat background linear order equations, Eqs. (3.13)–(3.17), were obtained from these expressions.

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[1] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[2] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979); L. Smarr and J. W. York, Jr., Phys. Rev. D **17**, 2529 (1978).

[3] H. Shinkai and G. Yoneda, Class. Quantum Grav. **19**, 1027 (2002).

[4] T. Nakamura and K. Oohara, in *Frontiers in Numerical Relativity*, edited by C. R. Evans, L. S. Finn, and D. W. Hobill (Cambridge University Press, Cambridge, England, 1989); M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995).

[5] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. D **59**, 024007 (1999).

[6] H. Shinkai and G. Yoneda, Class. Quantum Grav. **17**, 4799 (2000).

[7] L. Lehner, Class. Quantum Grav. **18**, R25 (2001).

[8] L. E. Kidder, M. A. Scheel, and S. A. Teukolsky, Phys. Rev. D **64**, 064017 (2001).

[9] O. Brodbeck, S. Frittelli, P. Hübner, and O. A. Reula, J. Math. Phys. **40**, 909 (1999).

[10] F. Siebel and P. Hübner, Phys. Rev. D **64**, 024021 (2001).

[11] H. Shinkai and G. Yoneda, Phys. Rev. D **60**, 101502 (1999).

[12] G. Yoneda and H. Shinkai, Class. Quantum Grav. **18**, 441 (2001).

[13] See, e.g., M. Shibata and K. Uryu, Prog. Theor. Phys. **107**, 265 (2002).

[14] See, e.g., M. Alcubierre, W. Benger, B. Brügmann, G. Lanfermann, L. Nerger, E. Seidel, and R. Takahashi, Phys. Rev. Lett. **87**, 271103 (2001).

[15] S. D. Hern, Ph.D. thesis, Cambridge University, gr-qc/0004036.

[16] G. Yoneda and H. Shinkai, Phys. Rev. D **63**, 124019 (2001).

[17] S. Frittelli, Phys. Rev. D **55**, 5992 (1997).

[18] M. Alcubierre, B. Brügmann, T. Dramlitsch, J. A. Font, P. Papadopoulos, E. Seidel, N. Stergioulas, and R. Takahashi, Phys. Rev. D **62**, 044034 (2000).

[19] L. Lehner, M. Huq, and D. Garrison, Phys. Rev. D **62**, 084016 (2000).

[20] M. Alcubierre, G. Allen, B. Brügmann, E. Seidel, and W.-M. Suen, Phys. Rev. D **62**, 124011 (2000).

[21] M. Miller, gr-qc/0008017.

[22] S. Frittelli and R. Gomez, J. Math. Phys. **41**, 5535 (2000).

[23] A. Arbona, C. Bona, J. Massó, and J. Stela, Phys. Rev. D **60**, 104014 (1999).

[24] S. Frittelli and O. A. Reula, J. Math. Phys. **40**, 5143 (1999).

[25] M. Alcubierre, B. Brügmann, M. Miller, and W.-M. Suen, Phys. Rev. D **60**, 064017 (1999).

[26] S. Detweiler, Phys. Rev. D **35**, 1095 (1987).

[27] M. Alcubierre and B. Brügmann, Phys. Rev. D **63**, 104006

(2001).
[28] Masaru Shibata (private communication).
[29] Mexico Numerical Relativity Workshop, 2002.
[30] P. Laguna and D. Shoemaker, Class. Quantum Grav. **19**, 3679 (2002).
[31] G. Yoneda and H. Shinkai, gr-qc/0209106.
[32] H.-J. Yo, T. W. Baumgarte, and S. L. Shapiro, Phys. Rev. D **66**, 084026 (2002).