Adjusted ADM systems and their expected stability properties [1]

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Abstract

In order to find a way to have a better formulation for numerical evolution of the Einstein equations, we study the propagation equations of the constraints based on the Arnowitt-Deser-Misner formulation. By adjusting constraint terms in the evolution equations, we try to construct an "asymptotically constrained system" which is expected to be robust against violation of the constraints, and to enable a long-term stable and accurate numerical simulation. We first provide useful expressions for analyzing constraint propagation in a general spacetime, then apply it to Schwarzschild spacetime. We search when and where the negative real or non-zero imaginary eigenvalues of the homogenized constraint propagation matrix appear, and how they depend on the choice of coordinate system and adjustments. The predictions here may help the community to make further improvements. [a]

[a] The full version of the article is available as [1]

1 Introduction

Computing the Einstein equations numerically is a necessary direction for general relativity research. This approach, so-called numerical relativity, already gives us much feedback to help understand the nature of strong gravity, such as singularity formations, critical behavior of gravitational collapses, cosmology, and so on [2].

There are several different approaches to simulate the Einstein equations, among them the most robust is to apply 3+1 (space + time) decomposition of spacetime, which was first formulated by Arnowitt, Deser and Misner (ADM) [3] (we refer to this as the "original ADM" system). A current important issue in the 3+1 approach is to control long-term stability of time integrations in simulating black hole or neutron star binary coalescences. So far, the use of fundamental variables (g_{ij}, K_{ij}), the internal 3-metric and extrinsic curvature, has been the most popular approach [4] (we refer to this as the "standard ADM" system). In recent years, however, many groups report that another re-formulation of the Einstein equations provides more stable and accurate simulations. We try here to understand these efforts through "adjusting" procedures to the evolution equations (that we explain later) using eigenvalue analysis of the constraint propagation equations.

At this moment, there may be three major directions to obtain longer time evolutions.

1. The first is to use a modification of the ADM system that was developed by Japanese group, which is now often abbreviated as BSSN (Baumgarte-Shapiro-Shibata-Nakamura) fromulation [5]. This is a combination of the introduction of new variables, conformal decompositions, rescaling the conformal factor, and the replacement of terms in the evolution equation using momentum constraints. Although there are many studies to show why this re-formulation is better than the standard ADM, as far as we know, there is no definite conclusion yet. One observation [6] pointed out that to replace terms using momentum constraints appears to change the stability properties.

2. The second is to re-formulate the Einstein equations in a first-order hyperbolic form [7]. This is motivated from the expectation that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and also that the boundary treatment can be improved if

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we know the characteristic speed of the system. We note that, in constructing hyperbolic systems, the essential procedures are to adjust equations using constraints and to introduce new variables, normally the spatially derivatived metric. Several groups report that the hyperbolic formulation actually has advantages over the direct use of ADM formulation [8, 9, 10]. However it is also reported that there are no drastic changes in the evolution properties *between* hyperbolic systems (weakly/strongly and symmetric hyperbolicity) by systematic numerical studies by Hern [11] based on Frittelli-Reula formulation [12], and by the authors [13] based on Ashtekar's formulation [16, 14, 15]. Therefore we may say that the mathematical notion of "hyperbolicity" is not always applicable for predicting the stability of numerical integration of the Einstein equations (See also §4.4.2 in [2]). It will be useful if we have an alternative procedure to predict stability including the effects of the non-principal parts of the equation, which are neglected in the discussion of hyperbolicity.

3. The third is to construct a robust system against the violation of the constraints, such that the constraint surface is the attractor. The idea was first proposed as "λ-system" by Brodbeck et al [18] in which they introduce artificial flow to the constraint surface using a new variable based on the symmetric hyperbolic system. This idea was tested and confirmed to work as expected in some test simulations by the authors[19] (based on the formulation developed by [20]). Although it is questionable whether the recovered solution is true evolution or not [21], we think that to enforce the decay of errors in its initial perturbative stage is the key to the next improvements. Actually, by studying the evolution equations of the constraints (hereafter we call them constraint propagation) and by evaluating eigenvalues (amplification factors, AFs) of constraint propagation in its homogenized form, we found that a similar "asymptotically constrained system" can be obtained by simply adjusting constraints to the evolution equations, even for the ADM equations [22].

The purpose of this report is to extend our previous study [22] in more general expressions and also to apply the systems to spacetime which has non trivial curvature, Schwarzschild black hole spacetime. The actual numerical simulations require many ingredients to be considered such as the choice of integration schemes, boundary treatments, grid structures and so on. However, we think that our approach to the stability problem through an *implementation to the equations* is definitely one of the aspects that should be improved.

Adjusting evolution equations is not a new idea. Actually the standard ADM system for numerical relativists [4] is adjusted from the original one [3] using the Hamiltonian constraint (See Frittelli's analysis on constraint propagation between the original and standard ADM formulations [23]). Detweiler [24] proposed a system using adjustments so that the L2 norm of constraints may not blow up. Several numerical relativity groups recently report the advantages of the adjusting procedure with a successful example [9]. We try here to understand the background mathematical features systematically by using AFs of constraint propagation.

2 Adjusted systems

2.1 Procedure and background – general discussion –

We begin with an overview of the adjusting procedure and the idea of background structure, which were described in our previous work [19, 22].

Suppose we have a set of dynamical variables $u^a(x^i, t)$, and their evolution equations

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \tag{1}$$

and the (first class) constraints

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \tag{2}$$

For monitoring the violation of constraints, we propose to investigate the evolution equation of C^α (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots). \tag{3}$$

(We do not mean to integrate (3) numerically, but to evaluate them analytically in advance.)

We propose to homogenize (3) by a Fourier transformation, e.g.

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \quad \text{where } C(x,t)^\rho = \int \hat{C}(k,t)^\rho \exp(ik\cdot x) d^3k, \tag{4}$$

then to analyze the set of eigenvalues, say Λs, of the coefficient matrix, $M^\alpha{}_\beta$, in (4). We call Λs the amplification factors (AFs) of (3). As we have proposed and confirmed in [19]:

Conjecture:

(a) If the amplification factors have a *negative real-part* (the constraints are forced to be diminished), then we see more stable evolutions than a system which has positive amplification factors.

(b) If the amplification factors have a *non-zero imaginary-part* (the constraints are propagating away), then we see more stable evolutions than a system which has zero amplification factors.

We found heuristically that the system becomes more stable when more Λs satisfy the above criteria [13, 19]. We remark that this eigenvalue analysis requires the fixing of a particular background spacetime, since the AFs depend on the dynamical variables, u^a.

The above features of the constraint propagation, (3), will differ when we modify the original evolution equations. Suppose we add (adjust) the evolution equations using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots), \tag{5}$$

then (3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots). \tag{6}$$

Therefore, the problem is how to adjust the evolution equations so that their constraint propagations satisfy the above criteria as much as possible.

2.2 Standard ADM system and its constraint propagation

We start by analyzing the standard ADM system. By "standard ADM" we mean here the most widely adopted system, due to York [4], with the evolution equations,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \tag{7}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \tag{8}$$

and the constraint equations,

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij} K^{ij}, \tag{9}$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K, \tag{10}$$

where (γ_{ij}, K_{ij}) are the induced three-metric and the extrinsic curvature, (α, β_i) are the lapse function and the shift covector, ∇_i is the covariant derivative adapted to γ_{ij}, and $R^{(3)}_{ij}$ is the three-Ricci tensor.

The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (9) and the momentum constraints (10), can be written as

$$\partial_t \mathcal{H} = \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha \gamma^{ij}(\partial_i \mathcal{M}_j) + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \tag{11}$$

$$\partial_t \mathcal{M}_i = -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \tag{12}$$

Further expressions of these constraint propagations are introduced in Appendix A of [1].

2.3 Adjustment to ADM evolution equations and its effects on constraint propagations

Generally, we can write the adjustment terms to (7) and (8) using (9) and (10) by the following combinations (using up to the first derivatives of constraints for simplicity in order to include Detweiler's case, see the next subsection),

$$\begin{aligned}
&\text{adjustment term of} \quad \partial_t \gamma_{ij}: \quad && +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (13)\\
&\text{adjustment term of} \quad \partial_t K_{ij}: \quad && +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (14)
\end{aligned}$$

where P, Q, R, S and p, q, r, s are multipliers.

According to this adjustment, the constraint propagation equations are also modified as

$$\begin{aligned}
\partial_t \mathcal{H} &= (11) + H_1^{mn}(13) + H_2^{imn}\partial_i(13) + H_3^{ijmn}\partial_i\partial_j(13) + H_4^{mn}(14), && (15)\\
\partial_t \mathcal{M}_i &= (12) + M_{1i}{}^{mn}(13) + M_{2i}{}^{jmn}\partial_j(13) + M_{3i}{}^{mn}(14) + M_{4i}{}^{jmn}\partial_j(14). && (16)
\end{aligned}$$

with appropriate changes in indices. (See detail in Appendix A in [1]. The definitions of $H_1, \cdots, M_1, \cdots$ are also there.)

3 Eigenvalue analysis of constraint propagation equations

We proposed several examples of adjustments (ways to fix above multipliers), and showed their resultant AFs in graphs [1, 22]. (We have to omit the details due to the space limitation. Please refer our articles.) In conclusion, we found that it is possible to obtain an asymptotically constrained system in ADM formulation by adjusting a particular combination of the multipliers. In our studies, Detweiler's proposal (1987) is still the best one according to our conjecture but has a growing mode of error near the horizon if we apply the system to the Schwarzschild spacetime.

The reader might ask why we can break the time-reversal invariant feature of the evolution equations by a particular choice of adjusting multipliers against the fact that the "Einstein equations" are time-reversal invariant. This question can be answered by the following. If we take a time-reversal transformation ($\partial_t \to -\partial_t$), the Hamiltonian constraint and the evolution equations of K_{ij} keep their signatures, while the momentum constraints and the evolution equations of γ_{ij} change their signatures. Therefore if we adjust γ_{ij}-equations using Hamiltonian constraint and/or K_{ij}-equations using momentum constraints (supposing the multiplier has +-parity), then we can break the time-reversal invariant feature of the "ADM equations". In fact, the examples we obtained all obey this rule.

4 Concluding Remarks

Motivated by performing a long-term stable and accurate numerical simulation of the Einstein equation, we proposed to adjust evolution equations by adding constraint terms by analyzing the constraint propagation equations in advance. The idea is to construct an asymptotically constrained evolution system, which is robust against violation of the constraint. This method works even for the ADM formulation (which is not a hyperbolic system) against flat background spacetime [22], and here we applied the analyses to a curved spacetime, a spherically symmetric black hole spacetime.

Recently, several numerical relativity groups report the effects of adjustments. They are mostly searching for a suitable combination of multipliers through trial and error. We hope our discussion here helps to understand the background mathematics systematically, though it may not be the perfect explanation. The main difference between our analysis and actual numerical studies is that this is a local analysis only on the evolution equations. The actual numerical results are obtained under a certain treatment of the boundary conditions, the choice of numerical integration schemes and grid structures, and also depend on the accuracy of the initial data. We think, however, that our proposal is an alternative to the hyperbolicity classification in that it includes the non-principal part of the evolution equations, and we expect that the discussion here will provide fundamental information on the stable formulation of the Einstein equations to the community.

We observed that the effects of adjustments depend on the choice of coordinate, gauge conditions, and also on its time evolution. Therefore our basic assumption of the constancy of the multipliers may be better to be replaced with more general procedures in our future treatment.

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