

LETTER

Constraint Propagation in $(N + 1)$ -Dimensional Space-Time

Hisa-aki Shinkai^{1,3} and Gen Yoneda²

Received September 25, 2003

Higher dimensional space-time models provide us an alternative interpretation of nature, and give us different dynamical aspects than the traditional four-dimensional space-time models. Motivated by such recent interests, especially for future numerical research of higher-dimensional space-time, we study the dimensional dependence of constraint propagation behavior. The $N + 1$ Arnowitt-Deser-Misner evolution equation has matter terms which depend on N , but the constraints and constraint propagation equations remain the same. This indicates that there would be problems with accuracy and stability when we directly apply the $N + 1$ ADM formulation to numerical simulations as we have experienced in four-dimensional cases. However, we also conclude that previous efforts in re-formulating the Einstein equations can be applied if they are based on constraint propagation analysis.

KEY WORDS: Numerical relativity; formulation problem; constraint propagation.

1. INTRODUCTION

Higher dimensional space-time models have been investigated from many viewpoints in physics. Current research interests come from brane-world models

¹Computational Science Division, Institute of Physical & Chemical Research (RIKEN), Hirosawa, Wako, Saitama, 351-0198 Japan; e-mail: hshinkai@riken.jp

²Department of Mathematical Science, Waseda University, Okubo, Shinjuku, Tokyo, 169-8555, Japan; e-mail: yoneda@waseda.jp

³Present address: Inamori Foundation, Research Department, 88 Kankoboko, Shimogyo, Kyoto 600-8009, Japan; e-mail: research@inamori-f.or.jp

that try to solve the hierarchical problem in the unified theory (e.g. [1, 2]). Since these models can be probed by future Large Hadron Collider experiments, a lot of research is being undertaken. Even apart from such brane-world models, many new physical results in higher dimensional general relativity are reported. Although we do not have space to list them all, we mention the discoveries of the black-hole solutions in five-dimensional space-time (e.g. [3]) that violate the traditional black-hole no-hair conjecture, the possibility of new stable configurations of black-string models (e.g. [4, 5]), and the modified version of cosmic hoop conjecture (e.g. [6]).

In order to investigate such topics, especially their dynamical and nonlinear behavior, numerical simulations are necessary. Numerical relativity is a promising research field, but it is also true that we have not yet obtain the recipe to perform long-term stable and accurate dynamical evolution. Many trial simulations of binary compact objects have revealed that the mathematically equivalent sets of evolution equations are showing different numerical stability in the free evolution schemes. Current research target in numerical relativity is to find out better reformulation of the Einstein equation (see reviews, e.g. [7–9]).

In this *Letter*, we study the dimensional dependence of constraint propagation in the standard Arnowitt-Deser-Misner (ADM) formulation of the Einstein equation (space-time decomposition) [10, 11]. The reader might think that starting with the ADM equation is old-fashioned since recent large-scale numerical simulations are not using the ADM equation due to its stability problem. However, we still think that ADM is the starting formulation for analyzing the dynamical behavior both analytically and numerically. The plenty of re-formulations of the Einstein equations have been proposed in the last decade. Most of them are starting from the ADM variables. The practical advantages of such re-formulations are extensively under investigation by many groups now, but, in our viewpoint, the essential improvements of them can be explained in a unified way via constraint propagation equations [8]. As we have shown in [12, 13], the stability problem of ADM can be controlled by adjusting constraints appropriately to evolution equations, and that the key idea also works in other formulations [14, 15]. Therefore the analysis of the ADM equation is still essential.

The idea of constraint propagation (originally reported in [16, 17]) is a useful tool for calibrating the Einstein equations for numerical simulations. The modifications to the evolution equations change the property of the associated constraint propagation, and several particular adjustments to evolution equations using constraints are expected to diminish the constraint violating modes. We proposed to apply eigenvalue analysis to constraint propagation equations and to judge the property of the constraint violation. The proposed adjusted equations have been confirmed as showing better stability than before by numerical experiments (e.g. [18, 19]). The purpose of this letter is to show this idea is also applicable to all higher dimensional cases.

2. $N+1$ -DIMENSIONAL ADM EQUATIONS

We start from the $(N + 1)$ -dimensional Einstein equation,

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + g_{\mu\nu}\Lambda = 8\pi T_{\mu\nu}, \tag{1}$$

and decompose it into N -dimensional space plus time, using the projection operator \perp_ν^μ ,

$$\gamma_\nu^\mu = \delta_\nu^\mu + n^\mu n_\nu \equiv \perp_\nu^\mu. \tag{2}$$

where n^μ is a unit normal vector of the spacelike hypersurface Σ , and we write the metric components,

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{3}$$

where γ_{ij} expresses N -dimensional intrinsic metric, and α and β^i the lapse and shift function, respectively. (Greek indices proceed $0, 1, \dots, N$, while Latin indices proceed $1, \dots, N$).

The projections of the Einstein equation are the following three:

$$G_{\mu\nu} n^\mu n^\nu = 8\pi T_{\mu\nu} n^\mu n^\nu \equiv 8\pi\rho_H, \tag{4}$$

$$G_{\mu\nu} n^\mu \perp_\rho^\nu = 8\pi T_{\mu\nu} n^\mu \perp_\rho^\nu \equiv -8\pi J_\rho, \tag{5}$$

$$G_{\mu\nu} \perp_\rho^\mu \perp_\sigma^\nu = 8\pi T_{\mu\nu} \perp_\rho^\mu \perp_\sigma^\nu \equiv 8\pi S_{\rho\sigma}, \tag{6}$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \tag{7}$$

which gives $T = -\rho_H + S^\ell_\ell$.

To express the decomposition, we introduce the extrinsic curvature K_{ij} as

$$\begin{aligned} K_{ij} &\equiv -\perp_i^\mu \perp_j^\nu \nabla_\nu n_\mu \\ &= \frac{1}{2\alpha}(-\partial_i \gamma_{ij} + D_j \beta_i + D_i \beta_j), \end{aligned} \tag{8}$$

where ∇ and D_i is the covariant differentiation with respect to $g_{\mu\nu}$ and γ_{ij} , respectively.

The projection of the Einstein equation onto Σ is given using the Gauss equation,

$${}^{(N)}R_{jkl}^i = \mathcal{R}_{\nu\rho\sigma}^\mu \perp_\mu^i \perp_j^\nu \perp_k^\rho \perp_l^\sigma - K_k^i K_{jl} + K_l^i K_{jk}, \tag{9}$$

and the Codazzi equation,

$$D_j K_i^j - D_i K = -\mathcal{R}_{\rho\sigma} n^\sigma \perp_i^\rho, \tag{10}$$

where $K = K_i^i$. For later convenience, we contract (9) to

$${}^{(N)}R_{ij} = \mathcal{R}_{ij}^\mu (\delta_\mu^\rho + n_\mu n^\rho) - KK_{ij} + K_j^\ell K_{i\ell}, \tag{11}$$

$${}^{(N)}R = \mathcal{R} + 2\mathcal{R}_{\mu\nu} n^\mu n^\nu - K^2 + K^{ij} K_{ij}. \tag{12}$$

Eqs. (4) and (12) give the Hamiltonian constraint, $\mathcal{C}_H \approx 0$, where

$$\begin{aligned} \mathcal{C}_H &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu \\ &= \frac{1}{2}({}^{(N)}R + K^2 - K^{ij} K_{ij}) - 8\pi\rho_H - \Lambda, \end{aligned} \tag{13}$$

while (5) and (10) give the momentum constraint, $\mathcal{C}_{Mi} \approx 0$, where

$$\begin{aligned} \mathcal{C}_{Mi} &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp_i^\nu \\ &= D_j K_i^j - D_i K - 8\pi J_i. \end{aligned} \tag{14}$$

Both (13) and (14) have the same expression as those of the four-dimensional version.

The evolution equation for γ_{ij} is obtained from (8), which is again the same expression as the four-dimensional version.

The evolution equation of K_{ij} is obtained also from (5). The contraction of (1) gives

$$\mathcal{R}_{ij} = 8\pi \left(S_{ij} - \frac{1}{N-1} \gamma_{ij} T \right) - \frac{2}{1-N} \gamma_{ij} \Lambda, \tag{15}$$

where we used $g_{\mu\nu} g^{\mu\nu} = N + 1$. A straightforward calculation of $\mathcal{R}^\mu_{\ ipj} = \partial_\rho \Gamma_{ij}^\mu - \partial_j \Gamma_{\rho i}^\mu + \Gamma_{\rho\sigma}^\mu \Gamma_{ij}^\sigma - \Gamma_{\sigma j}^\mu \Gamma_{i\rho}^\sigma$, where $\Gamma_{\nu\rho}^\mu$ is the Christoffel symbol, gives

$$\begin{aligned} \alpha \mathcal{R}_{\mu i\rho j} n^\mu n^\rho &= (\partial_t K_{ij}) + (D_j D_i \alpha) - \beta^k (D_k K_{ij}) \\ &\quad - (D_j \beta^k) K_{ik} - (D_i \beta^k) K_{kj} + \alpha K_{kj} K_i^k. \end{aligned} \tag{16}$$

Substituting (15) and (16) into (11), we obtain

$$\begin{aligned} \partial_t K_{ij} &= \alpha {}^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K_j^\ell K_{i\ell} - D_i D_j \alpha \\ &\quad + \beta^k (D_k K_{ij}) + (D_j \beta^k) K_{ik} + (D_i \beta^k) K_{kj} \\ &\quad - 8\pi \alpha \left(S_{ij} - \frac{1}{N-1} \gamma_{ij} T \right) - \frac{2\alpha}{N-1} \gamma_{ij} \Lambda, \end{aligned} \tag{17}$$

that is, only the matter and cosmological constant related terms depend on the dimension N .

If we have matter, we need to evolve the matter terms together with the metric. The evolution equations for matter terms can be derived from the conservation equation, $\nabla^\mu T_{\mu\nu} = 0$. In the next section, we will discuss them together with the constraint propagation equations.

3. $N+1$ -DIMENSIONAL CONSTRAINT PROPAGATION

The constraint propagation equation, $\partial_t(\mathcal{C}_H, \mathcal{C}_{Mi})^T$, can be derived in many ways, and among them the derivation via the Bianchi identity [17] may be the easiest.

In general, we write a $N + 1$ -dimensional symmetric tensor $\mathcal{S}_{\mu\nu}$ which obeys the Bianchi identity, $\nabla^\nu \mathcal{S}_{\mu\nu} = 0$. Let us express $\mathcal{S}_{\mu\nu}$ by decomposing as

$$\mathcal{S}_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}. \quad (18)$$

The normal and spatial projections of $\nabla^\nu \mathcal{S}_{\mu\nu}$ become

$$\begin{aligned} n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\mu Y_\mu + Y_\nu n^\mu \nabla_\mu n^\nu \\ &\quad - 2Y_\mu n_\nu (\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu (\nabla^\mu X), \end{aligned} \quad (19)$$

$$\begin{aligned} h_i^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= \nabla^\mu Z_{i\mu} + Y_i (\nabla^\mu n_\mu) + Y_\mu (\nabla^\mu n_i) \\ &\quad + X(\nabla^\mu n_i) n_\mu + n_\mu (\nabla^\mu Y_i), \end{aligned} \quad (20)$$

where we used ∇ , while [17] uses a different operator. For convenience, we rewrite them

$$\begin{aligned} \alpha n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -(\partial_t X) + \alpha K X + \beta^j (\partial_j X) - \alpha \gamma^{ji} (\partial_i Y_j) + \alpha (\partial_l \gamma_{mn}) \\ &\quad \times (\gamma^{ml} \gamma^{nj} - (1/2) \gamma^{mn} \gamma^{lj}) Y_j - 2\gamma^{im} (\partial_m \alpha) Y_i + \alpha K^{ij} Z_{ij}, \end{aligned} \quad (21)$$

$$\begin{aligned} -\alpha h_i^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -(\partial_t Y_i) - (\partial_i \alpha) X + \alpha K Y_i + \beta^j (\partial_j Y_i) + \gamma^{km} (\partial_i \beta_m) Y_k \\ &\quad - \beta^j \gamma^{kp} (\partial_i \gamma_{pj}) Y_k - \alpha \gamma^{jk} (\partial_k Z_{ij}) - (\partial_j \alpha) Z_i^j + (1/2) \alpha (\partial_i \gamma_{jk}) Z^{kj} \\ &\quad + \alpha \gamma^{mk} (\partial_m \gamma_{kj}) Z_i^j - (1/2) \alpha \gamma^{mk} (\partial_j \gamma_{mk}) Z_i^j, \end{aligned} \quad (22)$$

respectively.

If we substitute $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ into (21) and (22) and assume $\nabla^\mu T_{\mu\nu} = 0$, then we obtain the matter evolution equations, $\partial_t \rho_H$ and $\partial_t J_i$. If we substitute $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, \mathcal{C}_H, \mathcal{C}_{Mi}, \kappa \gamma_{ij} \mathcal{C}_H)$ with $\kappa = \text{const.}$ and assume $\nabla^\mu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0$, then we obtain the constraint propagation equations, $\partial_t \mathcal{C}_H$ and $\partial_t \mathcal{C}_{Mi}$. [The parameter κ corresponds to adding a term to (17), $+(\kappa - 1) \mathcal{C}_H$.]

This derivation does not depend on the dimension N at all. Therefore the evolution equations both for the matter and constraints remain the same with those in the traditional four dimensional version.

The constraints include the extrinsic curvature terms, and the evolution equation of K_{ij} changes due to N as we saw in (17). Interestingly, however, such changes will be cancelled out and the resultant constraint propagation equations remain the same.

This means that a series of constraint propagation analyses can be directly applied to higher dimensional space-time. That is, the standard ADM evolution equations are likely to fail for long-term stable simulations. However, previously proposed adjustment techniques (e.g. [12, 13]) are also effective.

For example, constraint amplification factors (i.e. the eigenvalues of constraint propagation matrix) in Schwarzschild space-time [eq. (47) in [13]] are $(0, 0, \pm f(r))$ for four-dimensional standard ADM evolution equations, where $f(r)$ is a complex-valued function. In the five-dimensional Schwarzschild or black-string case, they become simply $(0, 0, 0, \pm f(r))$.

4. REMARKS

Motivated by the recent interests in higher dimensional space-time, we checked the constraint propagation equations based on the $N + 1$ ADM scheme. The evolution equation has matter terms which depend on N , but we show the constraint propagations remain the same as those in the four-dimensional ones. This indicates that there would be problems with accuracy and stability when we directly apply the $N + 1$ ADM formulation to numerical simulations as we have experienced in four-dimensional cases. However, we also conclude that previous efforts in re-formulating the Einstein equations can be applied if they are based on constraint propagation analysis. The generality holds for other systems when their constraints are written in the form of (18).

Since we only used the Bianchi identity in the core discussion, the assertion is also applicable to brane-world models. In the context of the Randall-Sundrum brane-world models [2], people study the modified four-dimensional Einstein equations [20], which are derived from five-dimensional Einstein equations with a thin-shell (3-brane) approximation. The terms there additional to the standard ADM (see eq.(17) in [20]) include extrinsic curvature (due to shell-normal vector), cosmological constant(s), and five-dimensional Weyl curvature. These terms, however, can be interpreted as a single stress-energy tensor which obeys the Bianchi identity. Therefore the properties of the constraint propagation equations are the same as the above (from the five-dimensional space-time viewpoint). Our proposals for the adjustments [12, 13] are also valid in brane-world models.

We hope this short report helps numerical relativists for developing their future simulations.

ACKNOWLEDGMENTS

HS is supported by the special postdoctoral researchers' program at RIKEN. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Japan Society of the Promotion of Science, No. 14740179 and by Waseda University Grant for Special Research Projects Individual Research 2003A-870.

Note added in proof: After we submitted the article, we found that Anderson and Tavakol posted a preprint (E. Anderson and R. Tavakol, gr-qc/0309063) on the ADM formulation in the large extra dimensions. The detail PDE analysis can be seen in their article, but the constraint propagation analysis is not available there.

REFERENCES

- [1] Arkani-Hamed, N., Dimopoulos, S., and Dvali, G. (1998). *Phys. Lett. B* **429**, 263; **436**, 257.
- [2] Randall, L., and Sundrum, R. (1999). *Phys. Rev. Lett.* **83**, 3370, 4690.
- [3] Horowitz, G. T., and Strominger, A. (1991). *Nucl. Phys. B.* **360**, 197; Emparan, R., and Reall, H. S. (2002). *Phys. Rev. Lett.* **88**, 101101.
- [4] Gregory, R., and Laflamme, R. (1993). *Phys. Rev. Lett.* **70**, 2837; Horowitz, G. T., and Maeda, K. (2001). *Phys. Rev. Lett.* **87**, 131301.
- [5] Wiseman, T. (2003). *Class. Quant. Grav.* **20**, 1177; Choptuik, M., Lehner, L., Olabarieta, I., Petryk, R., Pretorius, F., and Villegas, H. (2003). *Phys. Rev. D* **68**, 044001.
- [6] Ida, D., and Nakao, K. (2002). *Phys. Rev. D* **66**, 064026.
- [7] Lehner, L. (2001). *Class. Quant. Grav.* **18**, R25.
- [8] Shinkai, H., and Yoneda, G. (in press). In *Progress in Astronomy, and Astrophysics*, Nova Science New York. The manuscript is available as gr-qc/0209111.
- [9] Baumgarte, T. W., and Shapiro, S. L. (2003). *Phys. Rep.* **376**, 41.
- [10] Arnowitt, R., Deser, S., and Misner, C. W. (1962). In *Gravitation: An Introduction to Current Research*, L. Witten (Ed.), Wiley, New York.
- [11] York, J. W. Jr. (1979). In *Sources of Gravitational Radiation*, L. Smarr (Ed.), Cambridge; Smarr Cambridge University Press, L., and York, J. W. Jr. (1978). *Phys. Rev. D* **17**, 2529.
- [12] Yoneda, G., and Shinkai, H. (2001). *Phys. Rev. D* **63**, 124019.
- [13] Shinkai, H., and Yoneda, G. (2002). *Class. Quant. Grav.* **19**, 1027.
- [14] Shinkai, H., and Yoneda, G. (2000). *Class. Quant. Grav.* **17**, 4799; Yoneda, G., and Shinkai, H. (2001). *Class. Quant. Grav.* **18**, 441.
- [15] Yoneda, G., and Shinkai, H. (2002). *Phys. Rev. D* **66**, 124003; Yoneda, G., and Shinkai H. (2003). *Class. Quant. Grav.* **20**, L31.
- [16] Detweiler, S. (1987). *Phys. Rev. D* **35**, 1095.
- [17] Frittelli, S. (1997). *Phys. Rev. D* **55**, 5992.
- [18] Kelly, B., Laguna, P., Lockitch, K., Pullin, J., Schnetter, E. Shoemaker, D., and Tiglio, M. (2001). *Phys. Rev. D* **64**, 084013.
- [19] Yo, H.-J., Baumgarte, T. W., and Shapiro, S. L. (2002). *Phys. Rev. D* **66**, 084026.
- [20] Shiromizu, T., Maeda, K., and Sasaki, M. (2000). *Phys. Rev. D* **62**, 024012.