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Asymptotically Constrained Systems for Numerical Relativity

真貝寿明

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References

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1 動機・結論　これ一枚！

- **安定な数値計算を可能にするEinstein方程式の定式化は何か？**
  重力波発生の一般相対論的数値シミュレーションが，世界各地で精力的に進んでいるが，Einstein方程式の定式化の違いで，数値計算の安定性が変わることが報告されている．
- **現状 ＝ 混沌　試行錯誤 我々はこれらの統一的な理解を試みる**
  現在のアプローチ，主として３つ．(1) 柴田中村によるADM改良形式，(2) 双曲性を陽にもつ3+1形式の利用，(3) 漸近的に拘束面に発展するシステムの構築
- **我々の提案 ＝「拘束条件式の発展方程式を固有値解析せよ」**
  数値発展安定性に対する十分条件を固有値問題として解析的に説明．
  Fourier変換を用いて，双曲型偏微分方程式の議論で無視されていた低次項の効果も解析
- **我々の発見１ ＝「右辺に拘束条件式を加えることで安定性が変化」**
  “Adjusted System” **と命名．Maxwell方程式とAshtekar変数方程式で有効性確認．**
- **我々の発見２ ＝「ADM発展方程式でも，Adjusted Systemで拘束面がアトラクター」**
  Detweiler (1987) **による修正ADM方程式の乗数パラメータの有効範囲を解析的に説明**
  さまざまな修正方法を提案．予想される安定性を提示
- **今後 ＝ 実際の数値計算で実証，より汎用性のある方法を検討**
  来年5月，メキシコでこのトピックの研究会

2 これまでの試み 数値的安定性に適した，Einstein 方程式の 定式化？

従来のADM形式をそのまま数値計算に適用するのは，好ましくない．
constraint **項の付加により，**error**の**growing mode**が抑えられることがある．** $\Rightarrow$ Why?

作戦１ ADM変数を若干変更したShibata-Nakamura (Baumgarte-Shapiro) 形式の成功

新変数：$(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, where $\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}$, $\tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K)$, $\tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}$, and use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.
なぜ，この方法が本来のADMより優れているのか，明確な説明は未だなされていない．

作戦２ 双曲型 Einstein 運動方程式の定式化，およびその数値計算への応用

weakly hyperbolic $\ni$ strongly hyperbolic $\ni$ symmetric hyperbolic systems,

本当に役に立つのか？　どのレベルの双曲性が必要なのか？

Using Ashtekar's variables between we found that [HS-Yoneda, 2000]

(1) 3**レベルの違いは，運動方程式の右辺に**constraint**を加えたり，ゲージ条件を制限すること．**

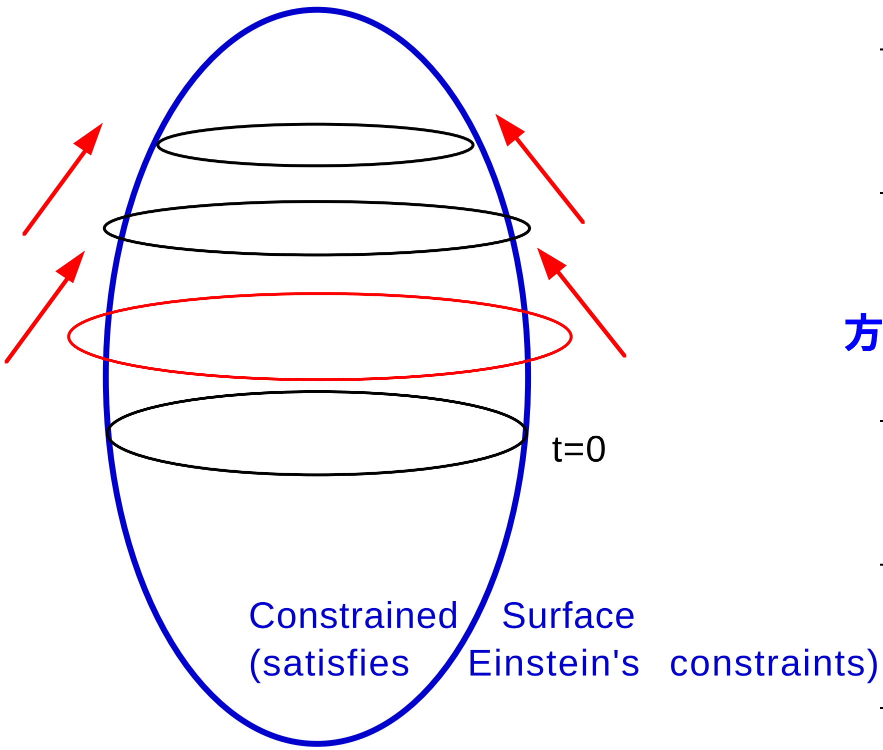
(2) **数値計算比較の結果（面対称時空での重力波伝播），**3**レベル間に顕著な差は見られない．**

(3) symmetric hyp.**形式が，必ずしも数値的誤差を最小に抑える，というわけではない．**

（初期値境界値IBVP**問題として考えるときには，対称双曲型または強双曲型方程式レベルが必要だ，という報告もある．）**

作戦３ 破れたconstraintが，自己回復してゆく発展形式の定式化，およびその数値計算への応用

“Asymptotically Constrained System”　拘束面がアトラクターになるシステム



方針１： λ-system (Brodbeck et al, 2000)

- **拘束条件の破れに対して，それを修正するように，外力を加える．**
- **対称双曲型運動方程式に応用すれば，解の一意な発展が保証される**

方針２： Adjusted system (Yoneda HS, 2000, 2001)

- **運動方程式に拘束条件式を加えることで，拘束条件式の破れ方を制御可能**
- **拘束条件の発展方程式の固有値解析で，拘束条件破れの予測．**
- **対称双曲型運動方程式でなくても，このシステムは構築可能** ⇒ **ADM形式に対しても!!**

3 Adjusted system 一般論　と　我々の仮説

一般的なprocedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) + F(C^a, \partial_b C^a, \cdots)$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

どのように $F(C^a, \partial_b C^a, \cdots)$ を決定するか？

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) + G(C^a, \partial_b C^a, \cdots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = A(\hat{C}^a)\, \hat{C}^k$

仮説： 拘束条件の発展方程式をFourier分解し，その固有値を計算する．固有値の（1）「実部が負」あるいは（2）「虚数部分を持つ」ならば，元の発展方程式は，より安定である．

4 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \tag{1}$$

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{2}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \tag{3}$$

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{4}$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}, \tag{5}$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \tag{6}$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \tag{7}$$

$$\partial_t \mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \tag{8}$$

The constraint propagation equations of the original ADM equation:

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (1)

$$\begin{aligned}\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.\end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (2)

$$\begin{aligned}\partial_t \mathcal{H} &= \beta^l \partial_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l \\ &= \beta^l \nabla_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha(\nabla_l \mathcal{M}^l) - 4(\nabla_l \alpha)\mathcal{M}^l, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l \\ &= -(1/2)\alpha(\nabla_i \mathcal{H}) - (\nabla_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l,\end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (3): by using Lie derivatives along αn^μ,

$$\begin{aligned}\pounds_{\alpha n^\mu} \mathcal{H} &= +2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l, \\ \pounds_{\alpha n^\mu} \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \alpha K \mathcal{M}_i.\end{aligned}$$

- Expression using γ_{ij} and K_{ij}

$$\begin{aligned}\partial_t \mathcal{H} &= H_1^{mn}(\partial_t \gamma_{mn}) + H_2^{imn}\partial_i(\partial_t \gamma_{mn}) + H_3^{ijmn}\partial_i \partial_j(\partial_t \gamma_{mn}) + H_4^{mn}(\partial_t K_{mn}), \\ \partial_t \mathcal{M}_i &= M_{1i}{}^{mn}(\partial_t \gamma_{mn}) + M_{2i}{}^{jmn}\partial_j(\partial_t \gamma_{mn}) + M_{3i}{}^{mn}(\partial_t K_{mn}) + M_{4i}{}^{jmn}\partial_j(\partial_t K_{mn}),\end{aligned}$$

where

$$
\begin{aligned}
H_1^{mn} &:= -2R^{(3)mn} - \Gamma^p_{kj}\Gamma^k_{pi}\gamma^{mi}\gamma^{nj} + \Gamma^m\Gamma^n \\
&\quad +\gamma^{ij}\gamma^{np}(\partial_i\gamma^{mk})(\partial_j\gamma_{kp}) - \gamma^{mp}\gamma^{ni}(\partial_i\gamma^{kj})(\partial_j\gamma_{kp}) - 2KK^{mn} + 2K^n{}_jK^{mj}, \\
H_2^{imn} &:= -2\gamma^{mi}\Gamma^n - (3/2)\gamma^{ij}(\partial_j\gamma^{mn}) + \gamma^{mj}(\partial_j\gamma^{in}) + \gamma^{mn}\Gamma^i, \\
H_3^{ijmn} &:= -\gamma^{ij}\gamma^{mn} + \gamma^{in}\gamma^{mj}, \\
H_4^{mn} &:= 2(K\gamma^{mn} - K^{mn}), \\
M_{1i}{}^{mn} &:= \gamma^{nj}(\partial_iK^m{}_j) - \gamma^{mj}(\partial_jK^n{}_i) + (1/2)(\partial_j\gamma^{mn})K^j{}_i + \Gamma^nK^m{}_i, \\
M_{2i}{}^{jmn} &:= -\gamma^{mj}K^n{}_i + (1/2)\gamma^{mn}K^j{}_i + (1/2)K^{mn}\delta^j_i, \\
M_{3i}{}^{mn} &:= -\delta^n_i\Gamma^m - (1/2)(\partial_i\gamma^{mn}), \\
M_{4i}{}^{jmn} &:= \gamma^{mj}\delta^n_i - \gamma^{mn}\delta^j_i,
\end{aligned}
$$

where we expressed $\Gamma^m = \Gamma^m_{ij}\gamma^{ij}$.

5 Constraint propagations in spherically symmetric spacetime

5.1 The procedure

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics.

$$C_\mu = \sum_{l,m} \left(A^{lm}(t,r) a_{lm}(\theta,\varphi) + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{1}$$

where we choose the basis of the vector harmonics as

$$a_{lm} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_{lm} = \begin{pmatrix} 0 \\ Y_{lm} \\ 0 \\ 0 \end{pmatrix}, c_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ \partial_\theta Y_{lm} \\ \partial_\varphi Y_{lm} \end{pmatrix}, d_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sin\theta}\partial_\varphi Y_{lm} \\ \sin\theta\, \partial_\theta Y_{lm} \end{pmatrix}.$$

The basis are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} C_\mu^* C_\rho \eta^{\nu\rho} \sin\theta d\theta,$$

where $\eta^{\nu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.}$$

We also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t)\, e^{ikr} \quad \text{etc.} \tag{2}$$

So that we will be able to obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

5.2 Constraint propagations in Schwarzschild spacetime

1. the standard Schwarzschild coordinate

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)}$$

2. the isotropic coordinate, which is given by, $r = (1 + M/2r_{iso})^2 r_{iso}$:

$$ds^2 = -(\frac{1 - M/2r_{iso}}{1 + M/2r_{iso}})^2 dt^2 + (1 + \frac{M}{2r_{iso}})^4 [dr_{iso}^2 + r_{iso}^2 d\Omega^2], \qquad \text{(the isotropic expression)}$$

3. the ingoing Eddington-Finkelstein (iEF) coordinate, by $t_{iEF} = t + 2M \log(r - 2M)$:

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r} dt_{iEF} dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)}$$

4. the Painlevé-Gullstrand (PG) coordinates,

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right) dt_{PG}^2 + 2\sqrt{\frac{2\,M}{r}} dt_{PG}\, dr + dr^2 + r^2 d\Omega^2\,, \qquad \text{(the PG expression)}$$

which is given by $t_{PG} = t + \sqrt{8Mr} - 2M \log\{(\sqrt{r/2M} + 1)/(\sqrt{r/2M} - 1)\}$

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

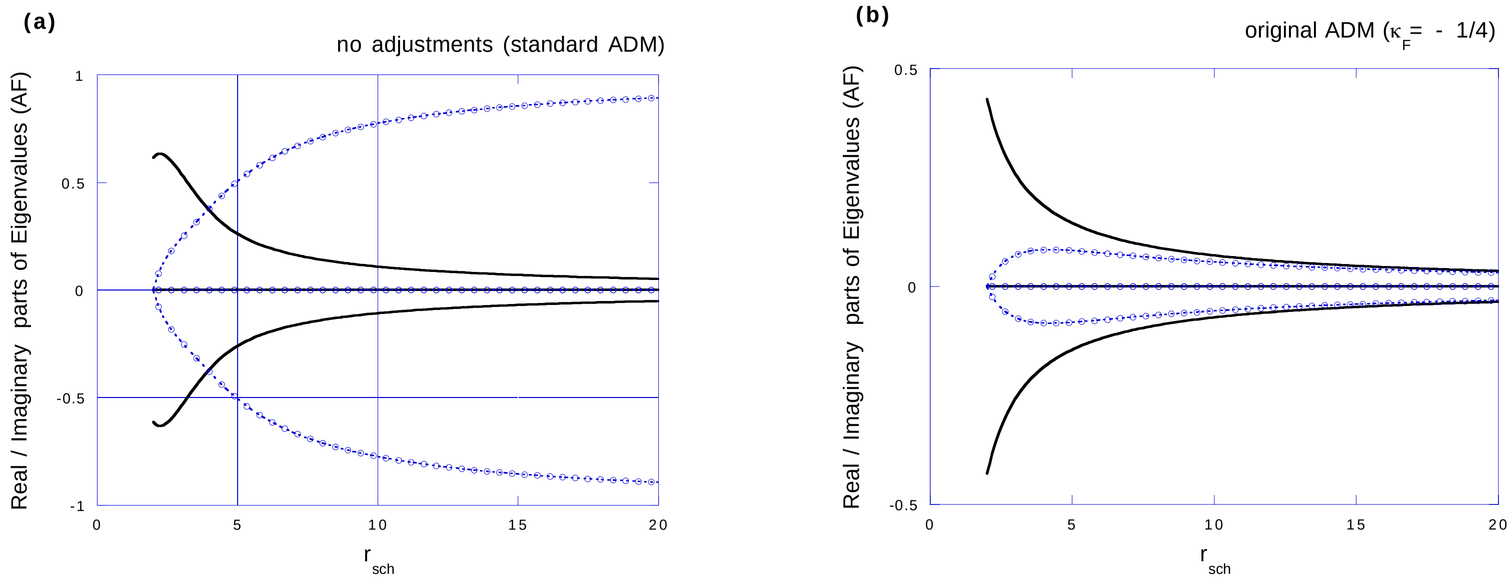


図 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

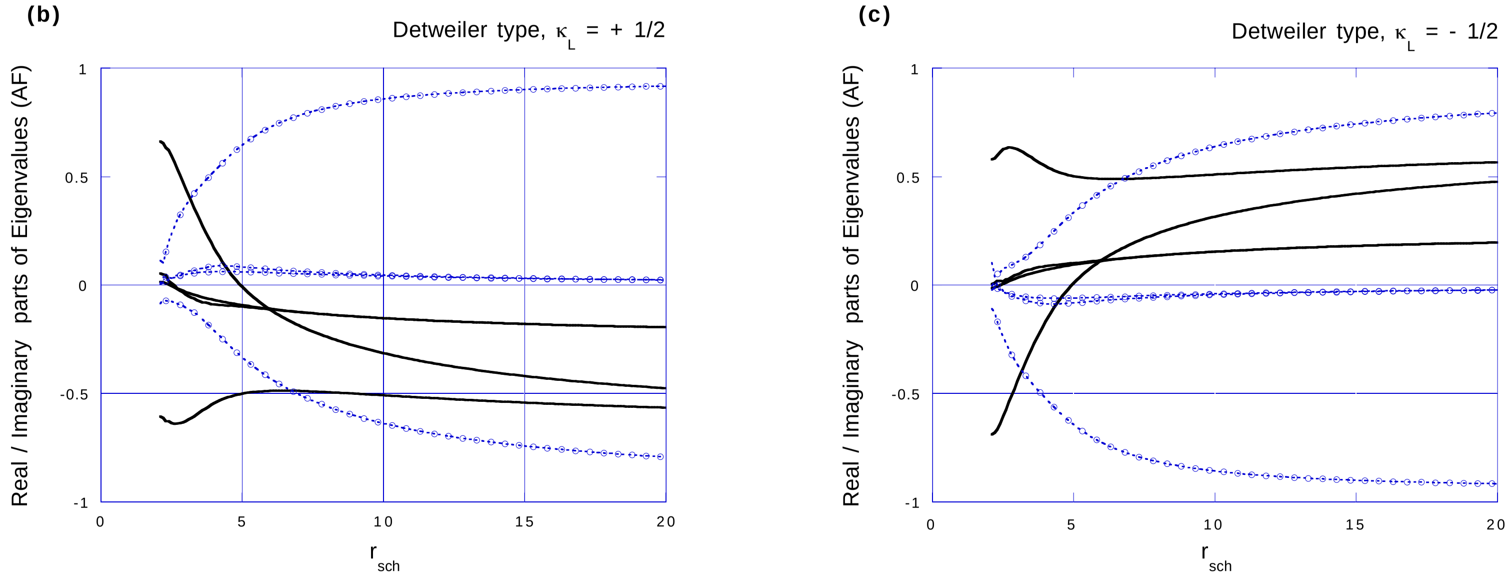


図 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\partial_t \gamma_{ij} = \text{(original terms)} + P_{ij}\mathcal{H},$$

$$\partial_t K_{ij} = \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l),$$

where $P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij}),$

$$S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],$$

Example 3: standard ADM (in isotropic/iEF coord.)

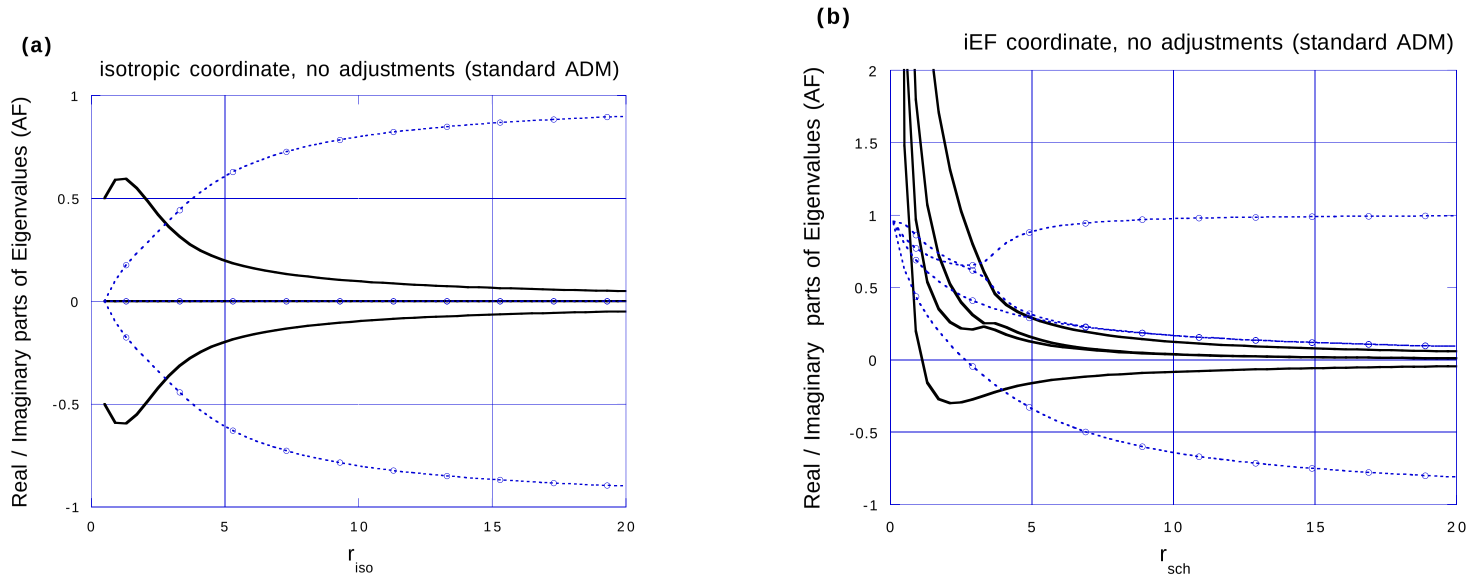
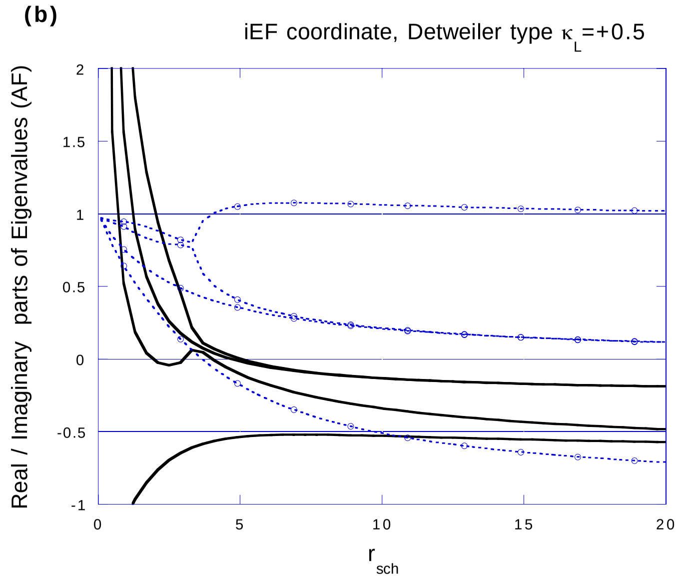


図 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \le r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

Example 4: Detweiler-type adjusted (in iEF/PG coord.)



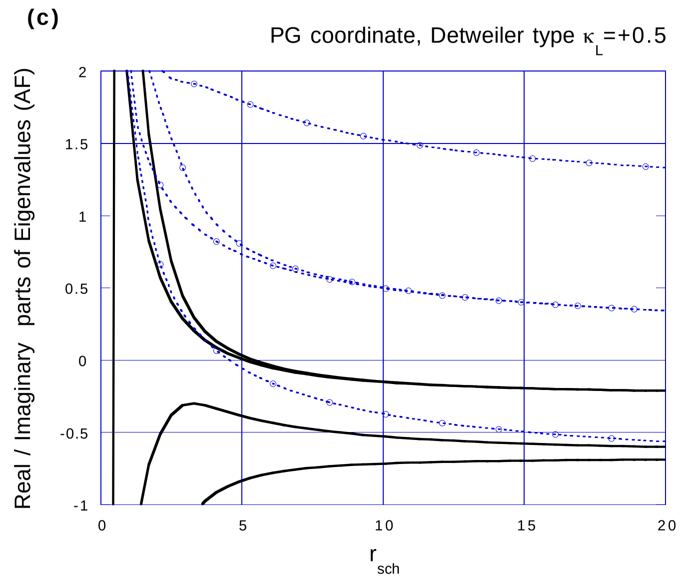


図 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

Example 5:
On Maximally-sliced hypersurfaces (standard ADM in Sch. coord.)

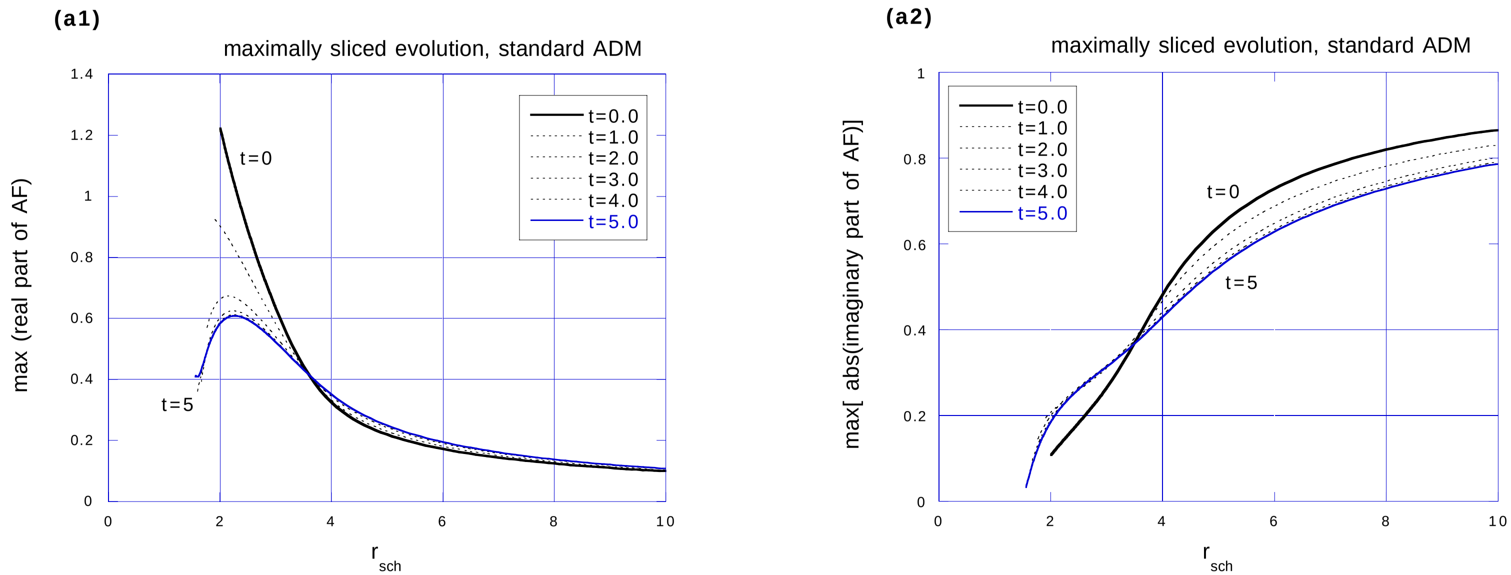


図 5: Amplification factors of snapshots of maximally-sliced evolving Schwarzschild spacetime. Fig (a1) and (a2) are of the standard ADM formulation (real and imaginary parts, respectively), Lines in (a1) are the largest (positive) AF on each time slice, while lines in (a2) are the maximum imaginary part of AF on each time slice. The lines start at $r_{min} = 2$ ($\bar{t} = 0$) and $r_{min} = 1.55$ ($\bar{t} = 5$).

No.	No. in Table.??	adjustment		1st?	Sch/iso coords.			iEF/PG coords.	
					TRS	real.	imag.	real.	imag.
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	-	P_{ij}	$P_{rr}=-\kappa$ or $P_{rr}=-\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	-	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	-	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	-	${Q^k}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa\sim 1.35$ min. vals.	not apparent
Q-2	-	${Q^k}_{ij}$	${Q^r}_{rr}=\kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	-	${Q^k}_{ij}$	${Q^r}_{ij}=\kappa\gamma_{ij}$ or ${Q^r}_{ij}=\kappa\alpha\gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	-	${Q^k}_{ij}$	${Q^r}_{rr}=\kappa\gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F=-1/4$ min. abs vals.		$\kappa_F=-1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr}=-\kappa_\mu\alpha$ or $R_{rr}=-\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	-	R_{ij}	$R_{rr}=-\kappa\gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	${S^k}_{ij}$	$\kappa_L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)}-(\partial_l\alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	-	${S^k}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	-	${p^k}_{ij}$	${p^r}_{ij}=-\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	-	${p^k}_{ij}$	${p^r}_{rr}=\kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	-	${p^k}_{ij}$	${p^r}_{rr}=\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	-	${q^{kl}}_{ij}$	${q^{rr}}_{ij}=\kappa\alpha\gamma_{ij}$	no	no	$\kappa=1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	-	${q^{kl}}_{ij}$	${q^{rr}}_{rr}=-\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	-	${r^k}_{ij}$	${r^r}_{ij}=\kappa\alpha\gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	-	${r^k}_{ij}$	${r^r}_{rr}=-\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	-	${r^k}_{ij}$	${r^r}_{rr}=-\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	${s^{kl}}_{ij}$	$\kappa_L\alpha^3[\delta^k_{(i}\delta^l_{j)}-(1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	-	${s^{kl}}_{ij}$	${s^{rr}}_{ij}=-\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	-	${s^{kl}}_{ij}$	${s^{rr}}_{rr}=-\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

表 1: List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The ‘N/A’ means that there is no effect due to the coordinate properties; ‘not apparent’ means the adjustment does not change the AFs effectively according to our conjecture; ‘enl./red./min.’ means enlarge/reduce/minimize, and ‘Pos./Neg.’ means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.