Reformulating the Einstein eqs for Stable Numerical Simulations
– Formulation Problem in Numerical Relativity –

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Outline

Why mathematically equivalent eqs produce different numerical stability?

- **Three approaches: ADM/BSSN, hyperbolic formulation, attractor systems**
- **Proposals : A unified treatment as Adjusted Systems**

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A series of works with **Gen Yoneda** Math. Sci. Dept., Waseda Univ., Japan

@ 12th JGRG, Tokyo, November 27, 2002

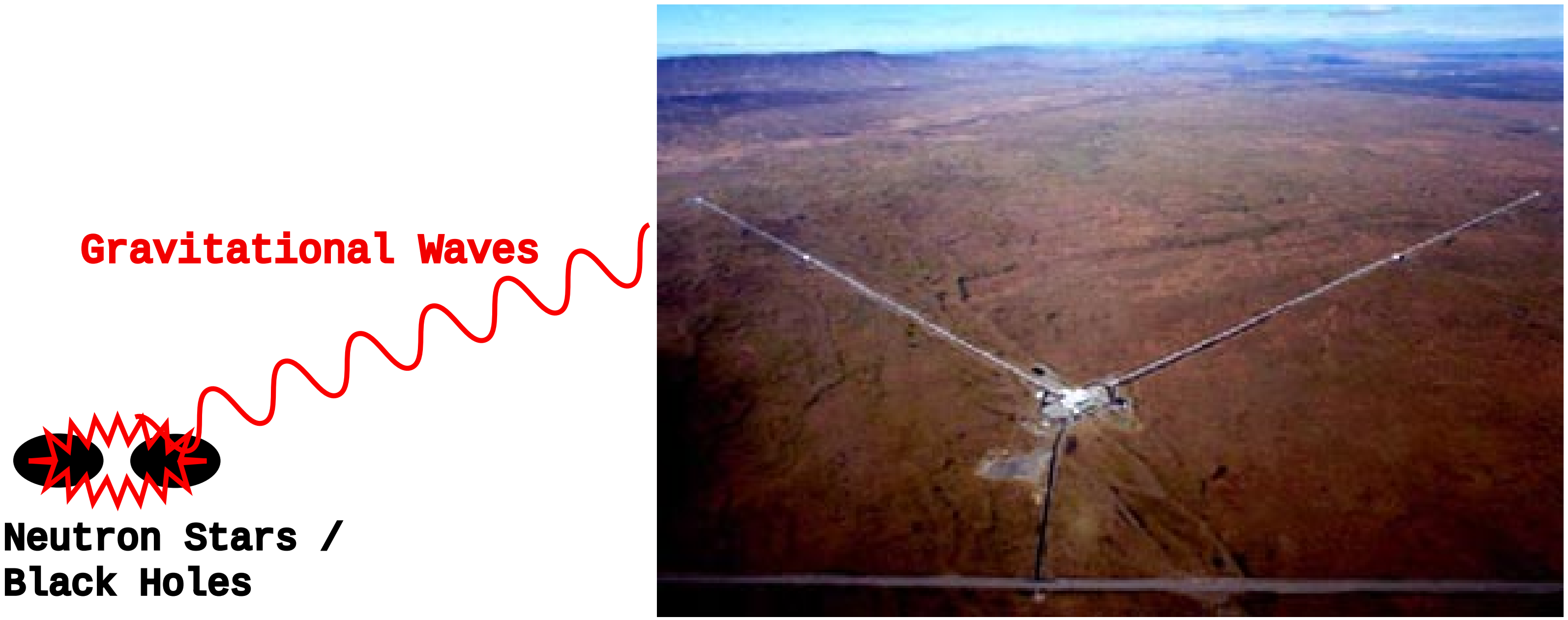
Plan of the talk

1. **Introduction**
   (0) **Arnowitt-Deser-Misner**
2. **Three approaches**
   (1) **Baumgarte-Shapiro-Shibata-Nakamura formulation**
   (2) **Hyperbolic formulations**
   (3) **Attractor systems – "Adjusted Systems"**
3. **Adjusted ADM systems**
   Flat background
   Schwarzschild background
   + Numerical Examples
4. **Adjusted BSSN systems**
   Flat background
   + Numerical Examples
5. **Summary and Future Issues**

1 Numerical Relativity and "Formulation" Problem

Numerical Relativity – Necessary for unveiling the nature of strong gravity

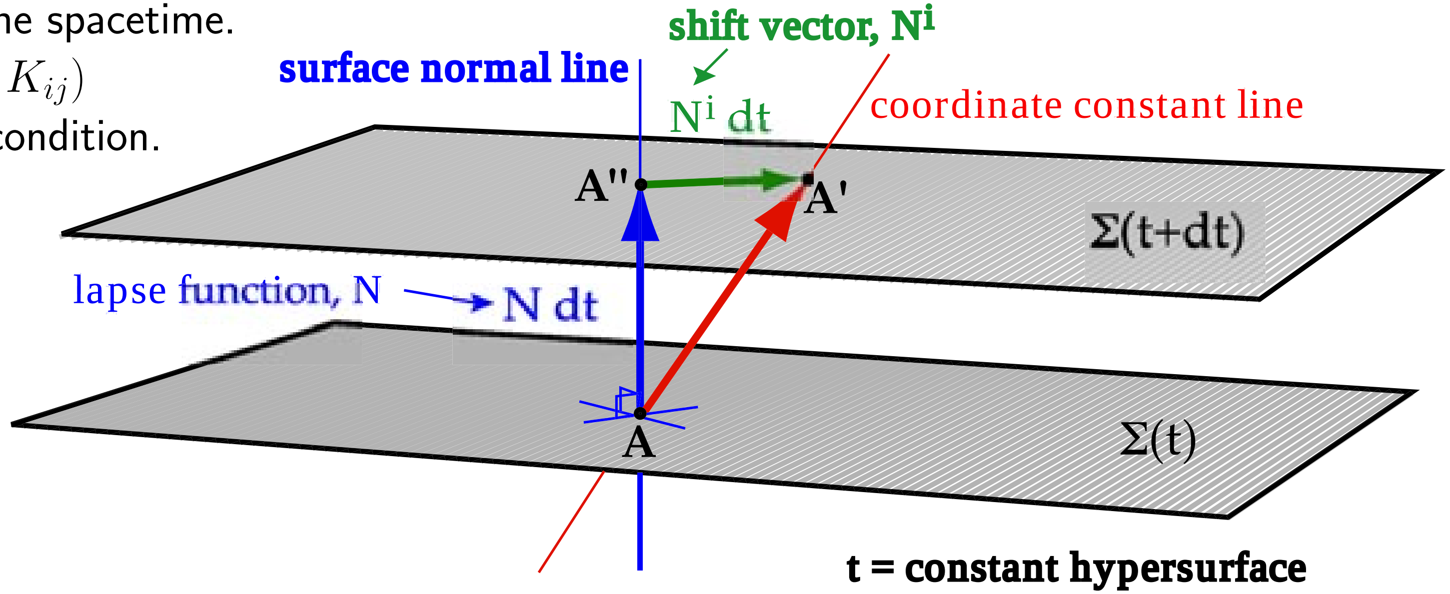
- Gravitational Wave from colliding Black Holes, Neutron Stars, Supernovae, ...
- Relativistic Phenomena like Cosmology, Active Galactic Nuclei, ...
- Mathematical feedbacks to Singularity, Exact Solutions, Chaotic behaviors, ...
- Laboratory of Gravitational theories, Higher dimensional models, ...



LIGO/VIRGO/GEO/TAMA, ...

strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

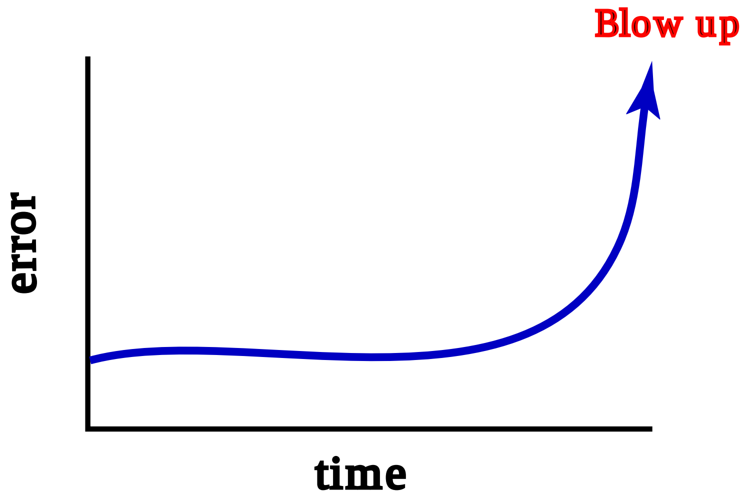
3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.

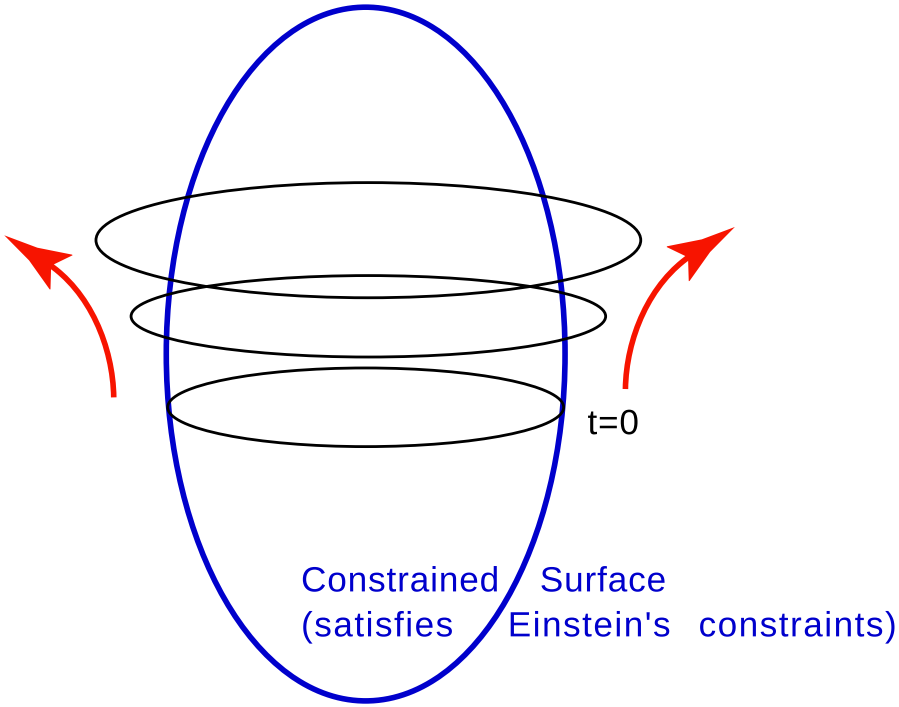


	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} =$ rot $\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -$rot $\mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

Many (too many) trials and errors, not yet a definit recipe.





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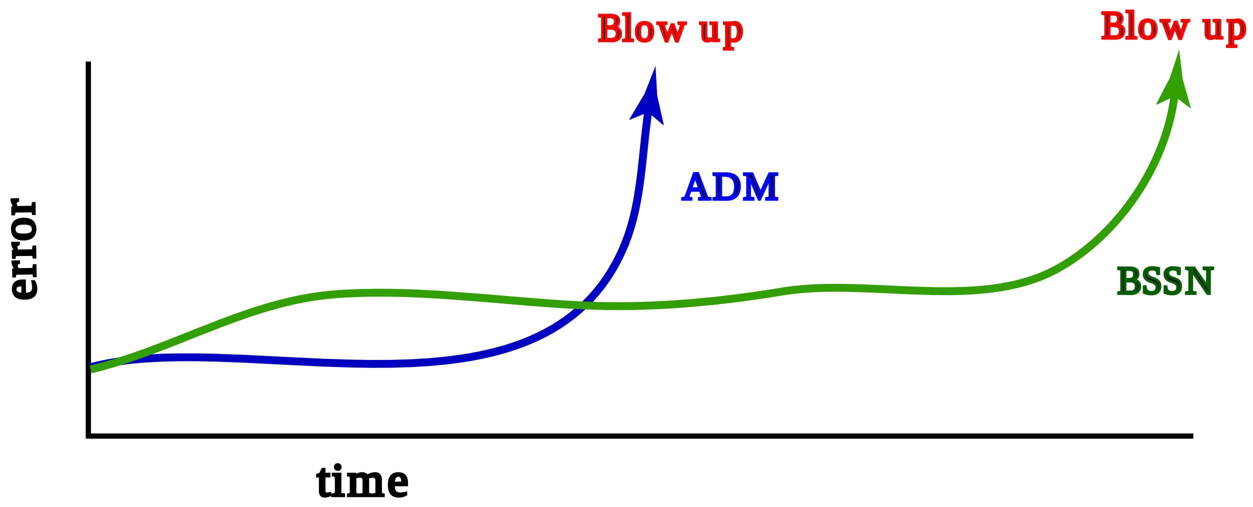


Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation

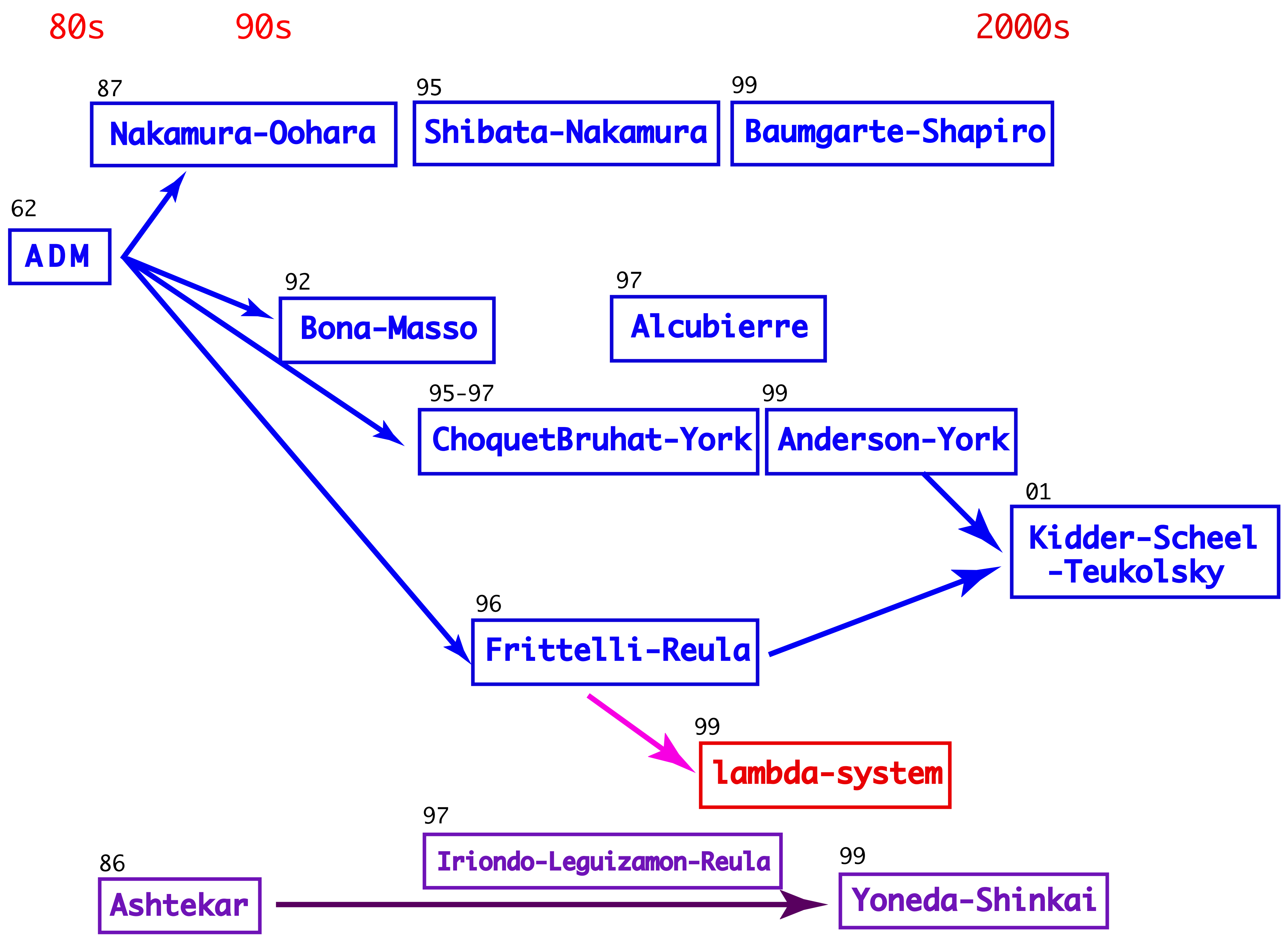
strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?

	formulations	numerical applications
(0) The standard ADM formulation		
ADM	1962 Arnowitt-Deser-Misner [12, 78]	⇒ many
(1) The BSSN formulation		
BSSN	1987 Nakamura et al [62, 63, 72]	⇒ 1987 Nakamura et al [62, 63]
		⇒ 1995 Shibata-Nakamura [72]
		⇒ 2002 Shibata-Uryu [73] etc
	1999 Baumgarte-Shapiro [15]	⇒ 1999 Baumgarte-Shapiro [15]
		⇒ 2000 Alcubierre et al [5, 7]
		⇒ 2001 Alcubierre et al [6] etc
	1999 Alcubierre et al [8]	
	1999 Frittelli-Reula [41]	
	2002 Laguna-Shoemaker [54]	⇒ 2002 Laguna-Shoemaker [54]
(2) The hyperbolic formulations		
BM	1989 Bona-Massó [17, 18, 19]	⇒ 1995 Bona et al [19, 20, 21]
		⇒ 1997 Alcubierre, Massó [2, 4]
	1997 Bona et al [20]	⇒ 2002 Bardeen-Buchman [16]
	1999 Arbona et al [11]	
CB-Y	1995 Choquet-Bruhat and York [31]	⇒ 1997 Scheel et al [69]
	1995 Abrahams et al [1]	⇒ 1998 Scheel et al [70]
	1999 Anderson-York [10]	⇒ 2002 Bardeen-Buchman [16]
FR	1996 Frittelli-Reula [40]	⇒ 2000 Hern [43]
	1996 Stewart [79]	
KST	2001 Kidder-Scheel-Teukolsky [51]	⇒ 2001 Kidder-Scheel-Teukolsky [51]
		⇒ 2002 Calabrese et al [26]
		⇒ 2002 Lindblom-Scheel [57]
	2002 Sarbach-Tiglio [68]	
CFE	1981 Friedrich[35]	⇒ 1998 Frauendiener [34]
		⇒ 1999 Hübner [45]
tetrad	1995 vanPutten-Eardley[84]	⇒ 1997 vanPutten [85]
Ashtekar	1986 Ashtekar [13]	⇒ 2000 Shinkai-Yoneda [75]
	1997 Iriondo et al [47]	
	1999 Yoneda-Shinkai [90, 91]	⇒ 2000 Shinkai-Yoneda [75, 92]
(3) Asymptotically constrained formulations		
λ-system to FR	1999 Brodbeck et al [23]	⇒ 2001 Siebel-Hübner [77]
to Ashtekar	1999 Shinkai-Yoneda [74]	⇒ 2001 Yoneda-Shinkai [92]
adjusted to ADM	1987 Detweiler [32]	⇒ 2001 Yoneda-Shinkai [93]
to ADM	2001 Shinkai-Yoneda [93, 76]	⇒ 2002 Mexico NR Workshop [58]
to BSSN	2002 Yoneda-Shinkai [94]	⇒ 2002 Mexico NR Workshop [58]
		⇒ 2002 Yo-Baumgarte-Shapiro [88]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strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables (ϕ, $\tilde\gamma_{ij}$,K,$\tilde A_{ij}$,$\tilde\Gamma^i$), instead of the ADM's (γ_{ij},K_{ij}) where

$$\tilde\gamma_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde A_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde\Gamma^i \equiv \tilde\Gamma^i_{jk}\tilde\gamma^{jk},$$

use momentum constraint in Γ^i-eq., and impose $\det\tilde\gamma_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K,\\
(\partial_t - \mathcal{L}_\beta)\tilde\gamma_{ij} &= -2\alpha\tilde A_{ij},\\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde A_{ij}\tilde A^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha),\\
(\partial_t - \mathcal{L}_\beta)\tilde A_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde A_{ij} - 2\tilde A_{ik}\tilde A^k{}_j)\\
\partial_t\tilde\Gamma^i &= -2(\partial_j\alpha)\tilde A^{ij} - (4/3)\alpha(\partial_j K)\tilde\gamma^{ij} + 12\alpha\tilde A^{ji}(\partial_j\phi) - 2\alpha\tilde A_k{}^j(\partial_j\tilde\gamma^{ik}) - 2\alpha\tilde\Gamma^k{}_{lj}\tilde A^j{}_k\tilde\gamma^{il}\\
&\quad -\partial_j\left(\beta^k\partial_k\tilde\gamma^{ij} - \tilde\gamma^{kj}(\partial_k\beta^i) - \tilde\gamma^{ki}(\partial_k\beta^j) + (2/3)\tilde\gamma^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde R_{ij} + R^\phi_{ij}\\
&\quad R^\phi_{ij} = -2\tilde D_i\tilde D_j\phi - 2\tilde g_{ij}\tilde D^l\tilde D_l\phi + 4(\tilde D_i\phi)(\tilde D_j\phi) - 4\tilde g_{ij}(\tilde D^l\phi)(\tilde D_l\phi)\\
&\quad \tilde R_{ij} = -(1/2)\tilde g^{lm}\partial_{lm}\tilde g_{ij} + \tilde g_{k(i}\partial_{j)}\tilde\Gamma^k + \tilde\Gamma^k\tilde\Gamma_{(ij)k} + 2\tilde g^{lm}\tilde\Gamma^k_{l(i}\tilde\Gamma_{j)km} + \tilde g lm\tilde\Gamma^k_{im}\tilde\Gamma_{klj}
\end{aligned}$$

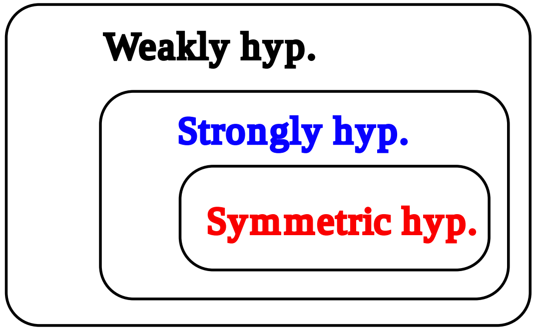
– No explicit explanations why this formulation works better.
AEI group (2000): <u>the replacement by momentum constraint is essential</u>.

strategy 2 Apply a formulation which reveals a hyperbolicity explicitly.

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvalues.
symmetric hyperbolic	if A is real and symmetric (Hermitian).



Expectations

– Wellposed behaviour

$$\text{symmetric hyperbolic system} \Longrightarrow \text{WELL-POSED}, \quad ||u(t)|| \leq e^{\kappa t}||u(0)||$$

– Better boundary treatments $\Longleftarrow \exists$ characteristic field.

– known numerical techniques in Newtonian hydrodynamics.

strategy 2 Apply a formulation which reveals a hyperbolicity explicitly. (cont.)

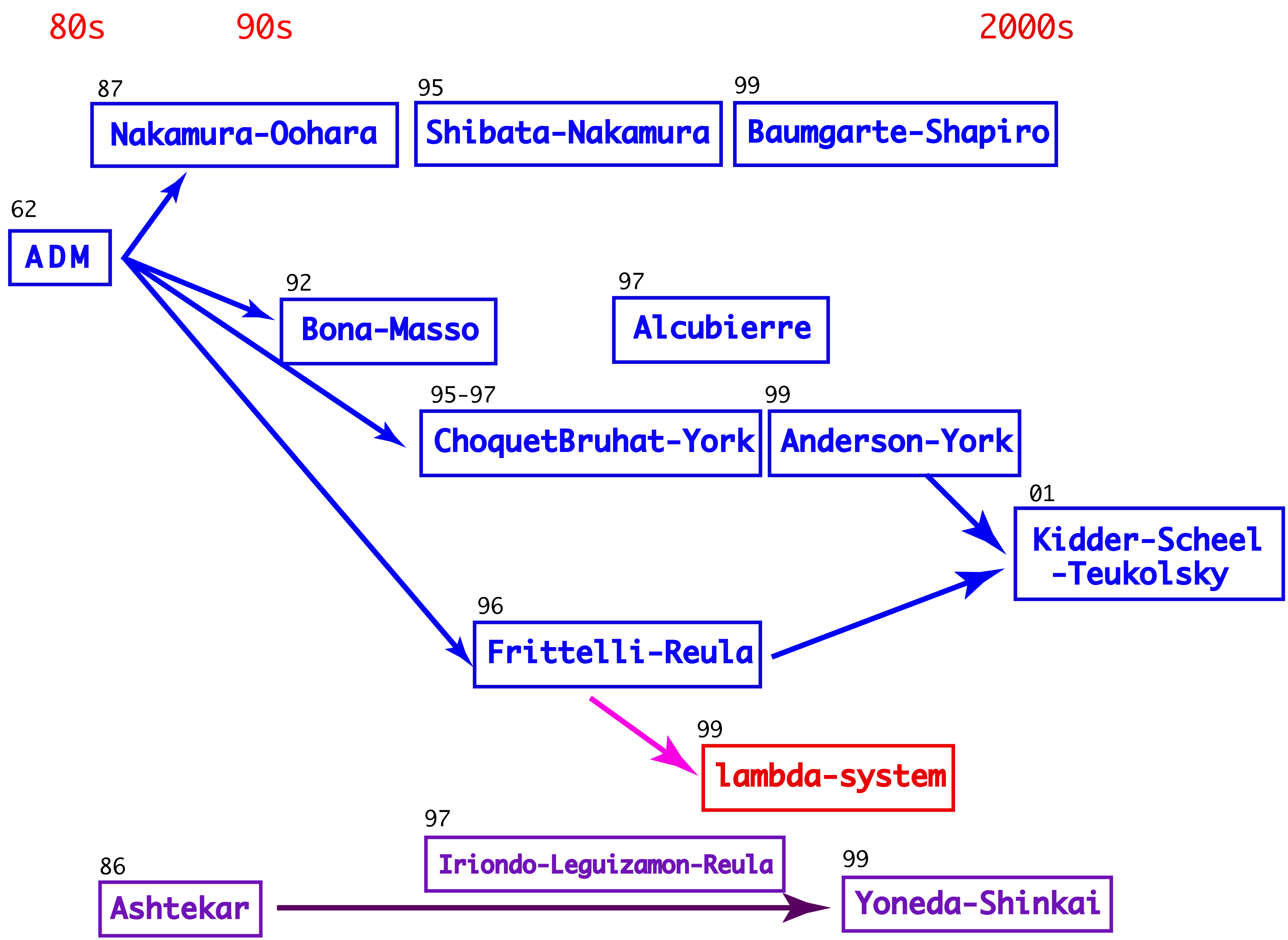
symmetric hyperbolic $\subset$ strongly hyperbolic $\subset$ weakly hyperbolic systems,

- Are they actually helpful? — if so, which level of hyperbolicity is necessary?
- Under what conditions/situations the advantages will be observed?

Unfortunately, we do not have conclusive answers to them yet.

- Several numerical experiments indicate that the direction is NOT a full of success.
  - Earlier numerical comparisons reported the advantages of hyperbolic formulations, but they were against to the standard ADM formulation. [Cornell-Illinois, NCSA, ...]
  - Numerical evolutions are always **terminated with blow-ups**.
  - If the gauge functions are evolved with hyperbolic equations, then their finite propagation speeds may cause a pathological **shock formations** [Alcubierre].
  - No drastic numerical differences **between** three hyperbolic levels [HS Yoneda, Hern].
  - Proposed symmetric hyperbolic systems were not always the best one for numerics.

Of course, these statements only casted on a particular formulation, therefore we have to be careful not to over-announce the results.



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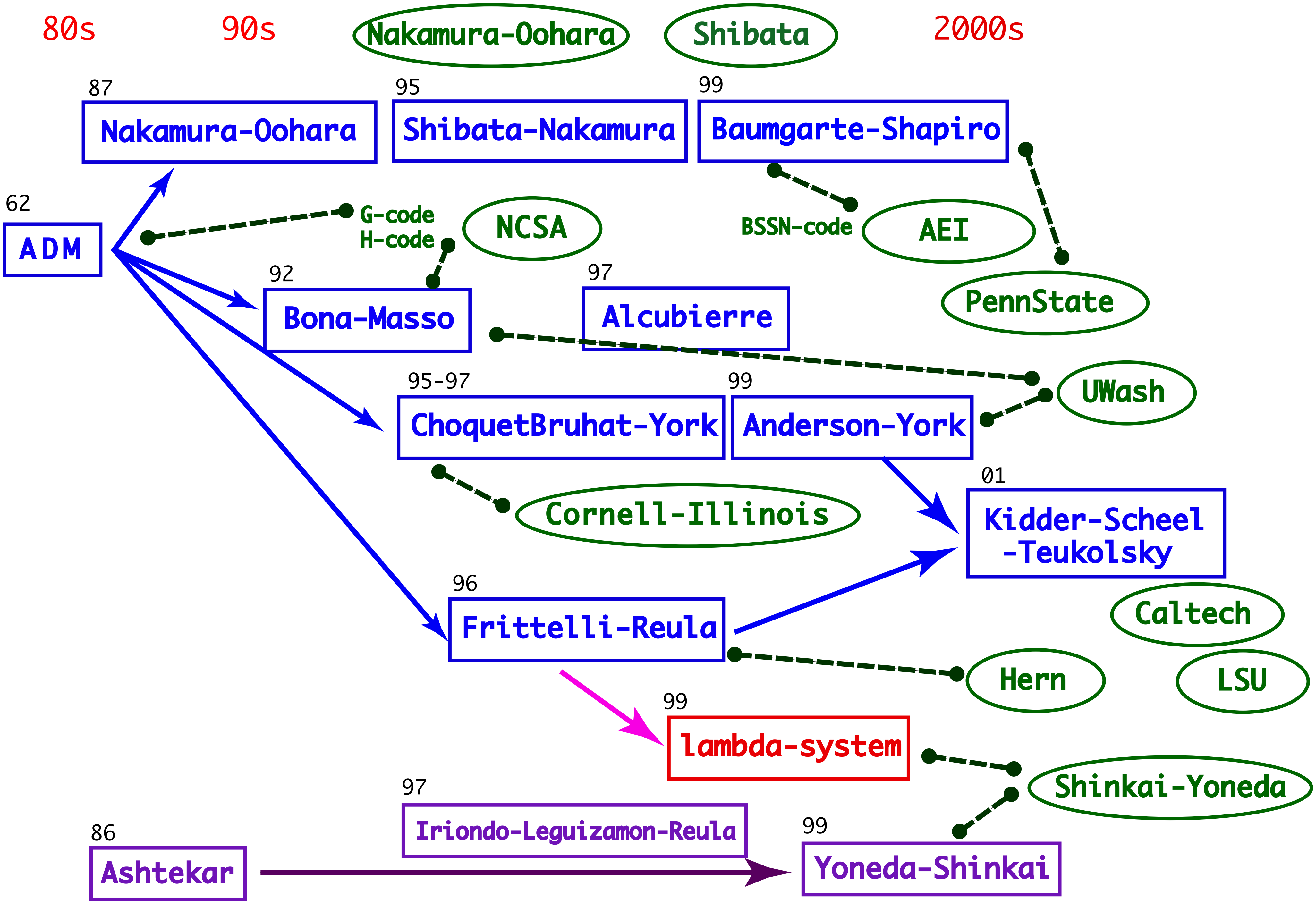

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Kidder-Scheel-Teukolsky hyperbolic formulation (Anderson-York + Frittelli-Reula)

Phys. Rev. D. 64 (2001) 064017

- Construct a First-order form using variables $(K_{ij}, g_{ij}, d_{kij})$ where $d_{kij} \equiv \partial_k g_{ij}$
  Constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{C}_{kij}, \mathcal{C}_{klij})$ where $\mathcal{C}_{kij} \equiv d_{kij} - \partial_k g_{ij}$, and $\mathcal{C}_{klij} \equiv \partial_{[k} d_{l]ij}$
- Densitize the lapse, $Q = \log(N g^{-\sigma})$
- Adjust equations with constraints

$$\begin{aligned}
\hat{\partial}_0 g_{ij} &= -2NK_{ij} \\
\hat{\partial}_0 K_{ij} &= (\cdots) + \gamma N g_{ij} \mathcal{H} + \zeta N g^{ab} \mathcal{C}_{a(ij)b} \\
\hat{\partial}_0 d_{kij} &= (\cdots) + \eta N g_{k(i} \mathcal{M}_{j)} + \chi N g_{ij} \mathcal{M}_k
\end{aligned}$$

- Re-deining the variables $(P_{ij}, g_{ij}, M_{kij})$

$$\begin{aligned}
P_{ij} &\equiv K_{ij} + \hat{z} g_{ij} K, \\
M_{kij} &\equiv (1/2)[\hat{k} d_{kij} + \hat{e} d_{(ij)k} + g_{ij}(\hat{a} d_k + \hat{b} b_k) + g_{k(i}(\hat{c} d_{j)} + \hat{d} b_{j)})], \quad d_k = g^{ab} d_{kab}, b_k = g^{ab} d_{abk}
\end{aligned}$$

The redefinition parameters

- do not change the eigenvalues of evolution eqs.
- do not effect on the principal part of the constraint evolution eqs.
- do affect the eigenvectors of evolution system.
- do affect nonlinear terms of evolution eqs/constraint evolution eqs.

strategy 2 Apply a formulation which reveals a hyperbolicity explicitly. (cont.)

- Remarks to hyperbolic formulations

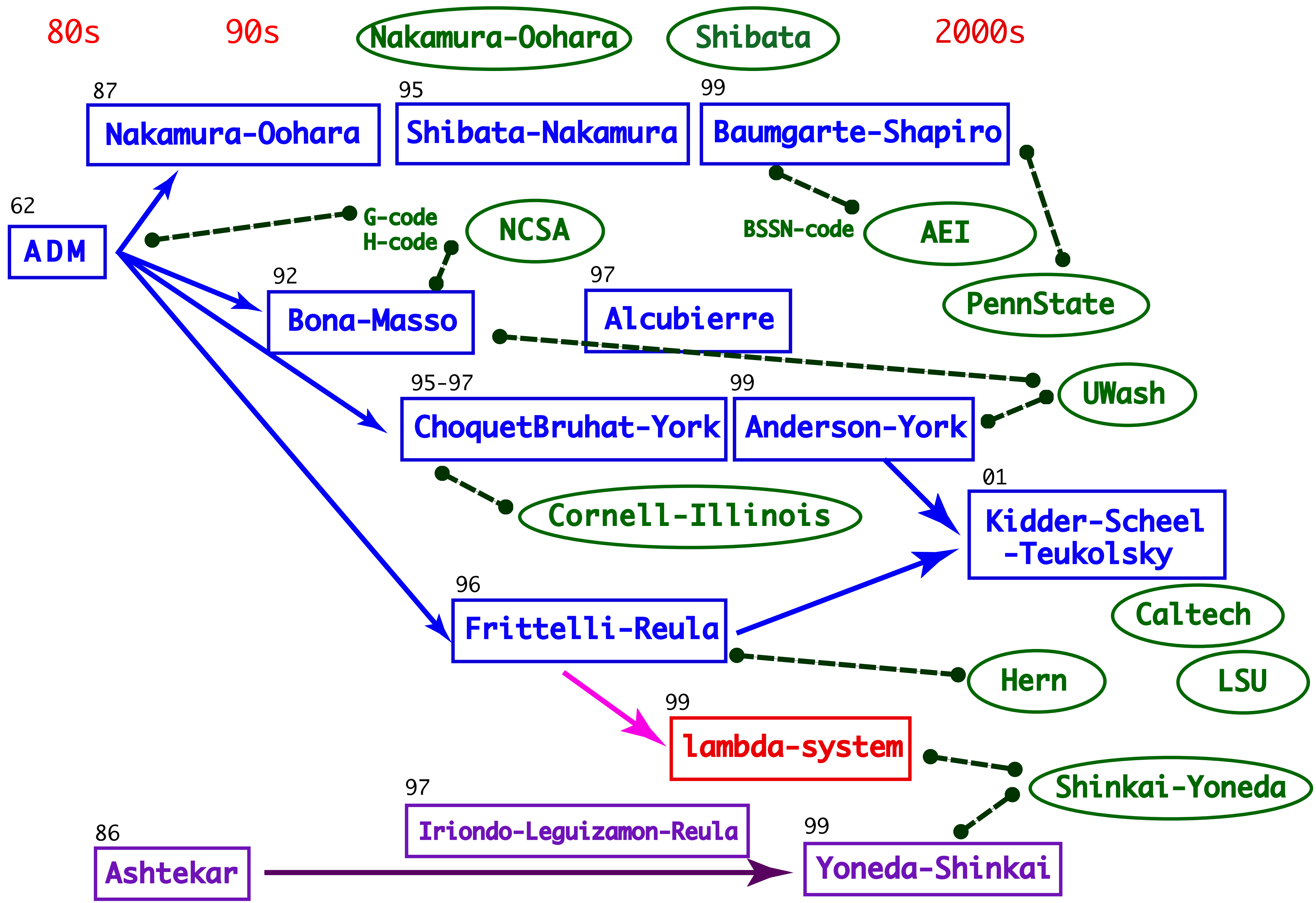
  (a) Rigorous mathematical proofs of well-posedness of PDE are mostly for a simple symmetric or strongly hyperbolic systems. If the matrix components or coefficients depend dynamical variables (like in any versions of hyperbolized Einstein equations), almost nothing was proved in its general situations.

  (b) The statement of "stability" in the discussion of well-posedness means the bounded growth of the norm, and does not mean a decay of the norm in time evolution.

  (c) The discussion of hyperbolicity only uses the characteristic part of the evolution equations, and ignore the rest.

cf. Recent discussions

- KST formulation with "kinematic" parameters which enables us to reduce non-principal part.
- links to IBVP approach.
- relations between convergence behavior and levels of hyperbolicity.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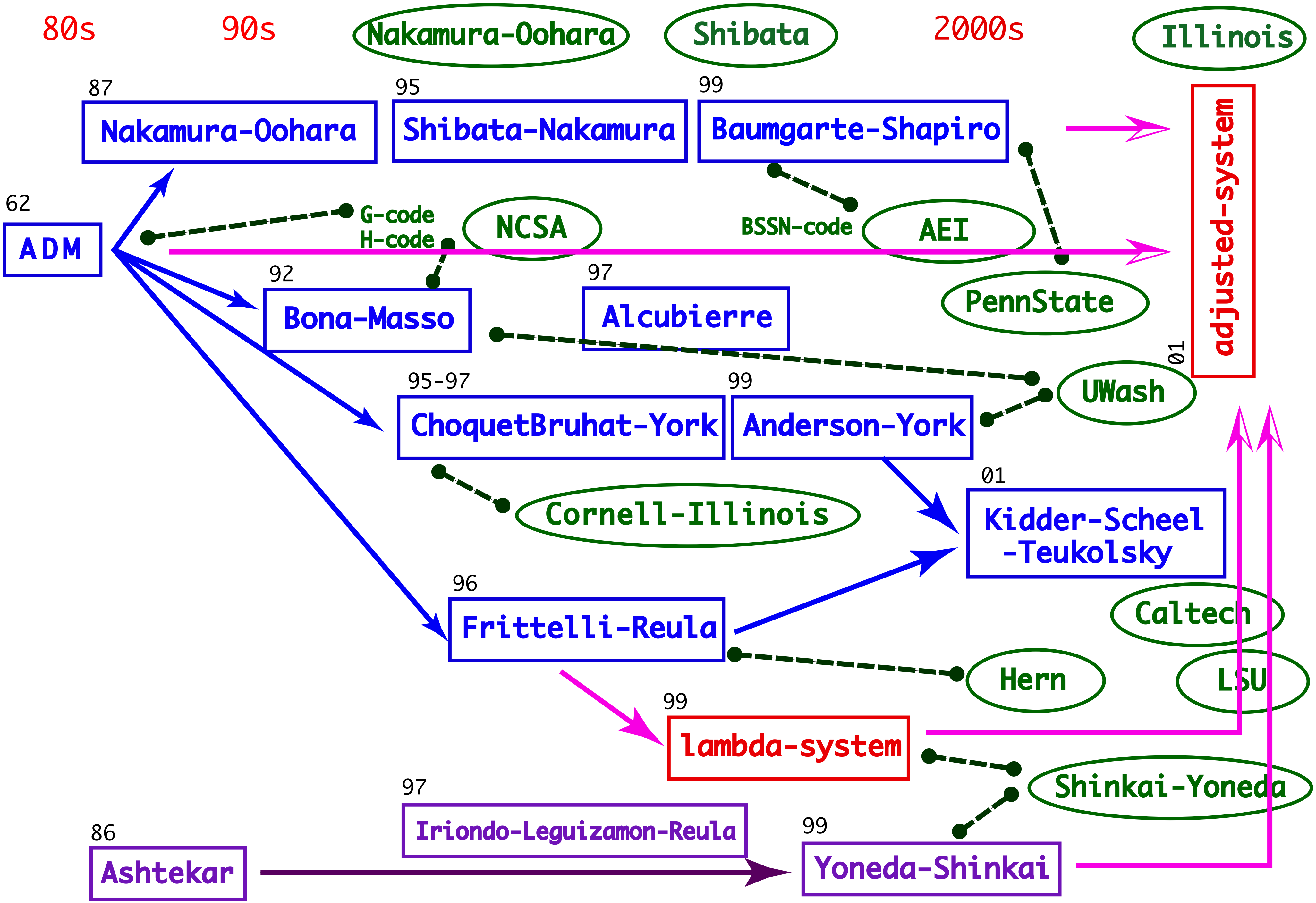
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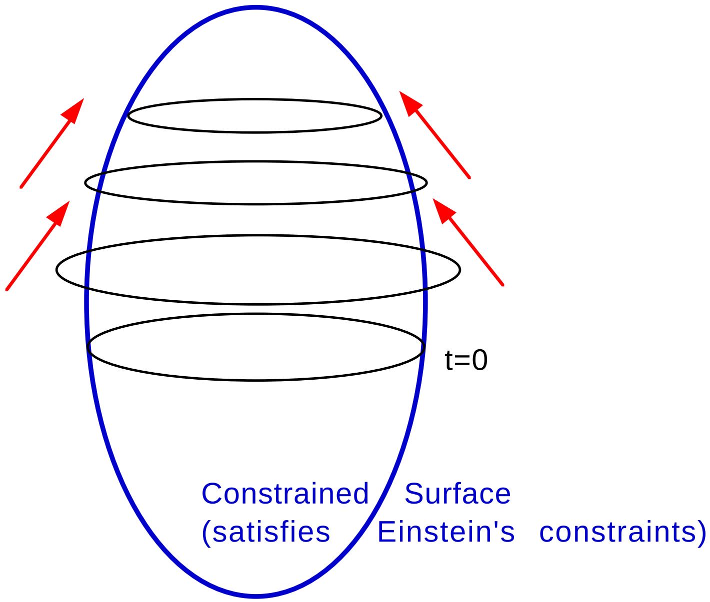
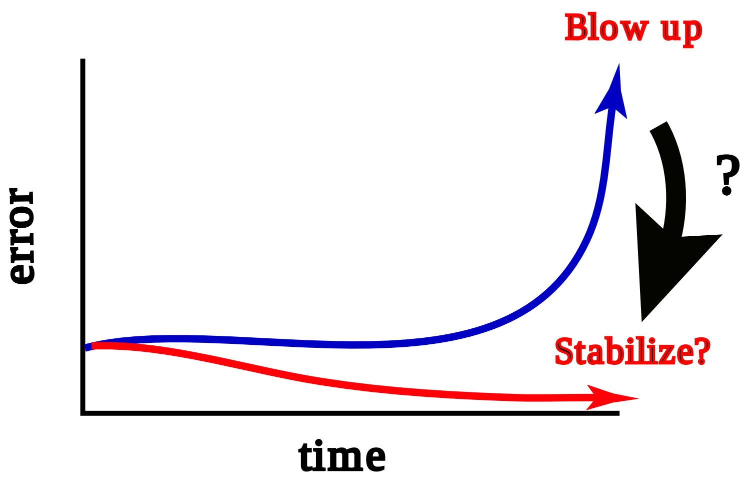
strategy 3 Formulate a system which is "asymptotically constrained" against a violation of constraints

"Asymptotically Constrained System"– Constraint Surface as an Attractor





method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM/BSSN formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\partial_t u = J\partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\partial_t \lambda = \alpha C - \beta\lambda$ $(\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]

Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)}\ \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

The Adjusted system (essentials):

Purpose:	Control the violation of constraints by reformulating the system so as to have a constrained surface an attractor.
Procedure:	Add a particular combination of constraints to the evolution equations, and adjust its multipliers.
Theoretical support:	Eigenvalue analysis of the constraint propagation equations.
Advantages:	Available even if the base system is not a symmetric hyperbolic.
Advantages:	Keep the number of the variable same with the original system.

Conjecture on Constraint Amplification Factors (CAFs):

(A) If CAF has a negative real-part (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a non-zero imaginary-part (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.

Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$\begin{aligned} \partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + Q_i C_B, \\ \partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + S_i C_B, \\ & C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0, \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

Constraint propagation equations

$$\begin{aligned} \partial_t C_E &= (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B), \\ \partial_t C_B &= (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B), \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

CAFs?

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}$$

$$\Rightarrow \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2$$

Therefore CAFs become negative-real when

$$P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0$$

Example: the Ashtekar equations

HS Yoneda, CQG 17 (2000) 4799

Adjusted dynamical equations:

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \underbrace{+X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}}_{adjust}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \underbrace{+Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}}_{adjust}$$

Adjusted and linearized:

$$X = Y = Z = 0,\ P^{ia}_b = \kappa_1(iN^i\delta^a_b),\ Q^a_i = \kappa_2(e^{-2}\underset{\sim}{N}\tilde{E}^a_i),\ R^{aj}{}_i = \kappa_3(-ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c)$$

Fourier transform and extract 0th order of the characteristic matrix:

$$\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \end{pmatrix} = \begin{pmatrix} 0 & i(1+2\kappa_3)k_j & 0 \\ i(1-2\kappa_2)k_i & \kappa_3 \epsilon^{kj}{}_i k_k & 0 \\ 0 & 2\kappa_3 \delta^j_a & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \end{pmatrix}$$

Eigenvalues:

$$\left(0, 0, 0, \pm\kappa_3\sqrt{-kx^2 - ky^2 - kz^2}, \pm\sqrt{(-1+2\kappa_2)(1+2\kappa_3)(kx^2+ky^2+kz^2)}\right)$$

In order to obtain non-positive real eigenvalues:

$$(-1+2\kappa_2)(1+2\kappa_3) < 0$$

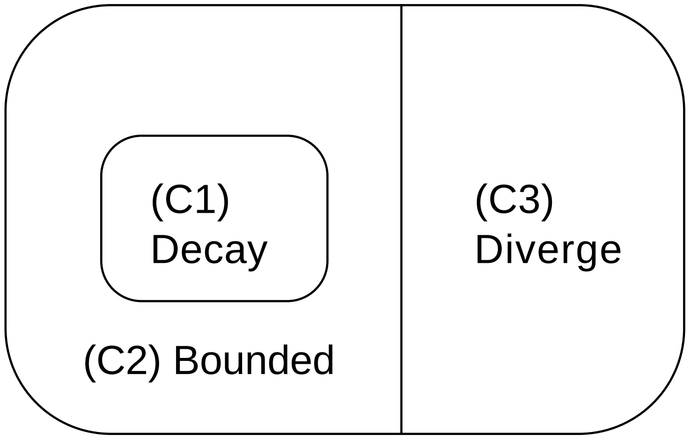
A Classification of Constraint Propagations

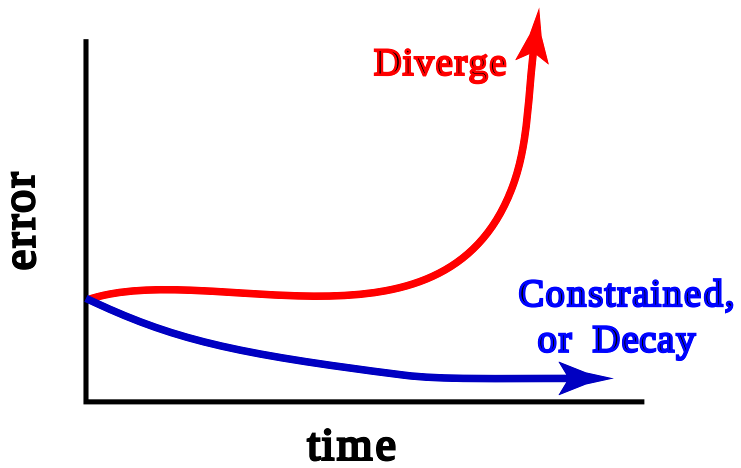
(C1) **Asymptotically constrained :**
Violation of constraints decays (converges to zero).

(C2) **Asymptotically bounded :**
Violation of constraints is bounded at a certain value.

(C3) **Diverge :**
At least one constraint will diverge.

Note that (C1) $\subset$ (C2).





A Classification of Constraint Propagations (cont.)

gr-qc/0209106

(C1) Asymptotically constrained :

Violation of constraints decays (converges to zero).

> $\Leftrightarrow$ All the real parts of CAFs are negative.

(C2) Asymptotically bounded :

Violation of constraints is bounded at a certain value.

> $\Leftrightarrow$
>
> (a) All the real parts of CAFs are not positive, and
>
> (b1) the CP matrix $M^{\alpha}{}_{\beta}$ is diagonalizable, or
>
> (b2) the real part of the degenerated CAFs is not zero.

(C3) Diverge :

At least one constraint will diverge.

A flowchart to classify the fate of constraint propagation.

Q1: Is there a CAF which real part is positive?

NO / YES ⟶ **Diverge**

↓

Q2: Are all the real parts of CAFs negative?

NO / YES ⟶ **Asymptotically Constrained**

↓

Q3: Is the constraint propagation matrix diagonalizable?

NO / YES ⟶ **Asymptotically Bounded**

↓

Q4: Is a real part of the degenerated CAFs is zero?

YES / NO ⟶ **Asymptotically Bounded**

↓

Q5: Is the associated Jordan matrix diagonal?

NO / YES ⟶ **Asymptotically Bounded**

↓

Diverge

Reformulating the Einstein eqs for Stable Numerical Simulations
– Formulation Problem in Numerical Relativity –

Hisaaki Shinkai

1. Introduction
   (0) Arnowitt-Deser-Misner
2. Three approaches
   (1) Baumgarte-Shapiro-Shibata-Nakamura formulation
   (2) Hyperbolic formulations
   (3) Attractor systems – "Adjusted Systems"
3. Adjusted ADM systems
   Flat background
   Schwarzschild background
   \+ Numerical Examples
4. Adjusted BSSN systems
   Flat background
   \+ Numerical Examples
5. Summary and Future Issues

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}
$$

with constraint equations

$$
\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij}K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}
$$

We can write the adjusted constraint propagation equations as

$$
\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}
$$

Original ADM The original construction by ADM uses the pair of (h_{ij}, π^{ij}).

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} = \dfrac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\dfrac{\delta \mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N({}^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad +\sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

Standard ADM (by York) NRists refer ADM as the one by York with a pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

3 Constraint propagation of ADM systems

3.1 Original ADM vs Standard ADM

Try the adjustment $\underline{\; R_{ij} = \kappa_1 \alpha \gamma_{ij} \;\text{ and other multiplier zero,}}$ where $\kappa_1 = \begin{cases} 0 & \text{the standard ADM} \\ -1/4 & \text{the original ADM} \end{cases}$

- The constraint propagation eqs keep the first-order form (cf Frittelli, PRD55(97)5992):

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l{}_i - \delta^l_i R & \beta^l \delta^j_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \tag{5}$$

  The eigenvalues of the characteristic matrix:

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \sqrt{\alpha^2 \gamma^{ll}(1 + 4\kappa_1)})$$

  The hyperbolicity of (5): $\begin{cases} \text{symmetric hyperbolic} & \text{when } \kappa_1 = 3/2 \\ \text{strongly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) > 0 \\ \text{weakly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) \geq 0 \end{cases}$

- On the Minkowskii background metric, the linear order terms of the Fourier-transformed constraint propagation equations gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1 + 4\kappa_1)}).$$

  That is, $\begin{cases} \text{(two 0s, two pure imaginary)} & \text{for the standard ADM} \\ \text{(four 0s)} & \text{for the original ADM} \end{cases}$ BETTER STABILITY

4 Constraint propagations in spherically symmetric spacetime

4.1 The procedure

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics.

$$C_\mu = \sum_{l,m} \left(A^{lm}(t,r) a_{lm}(\theta,\varphi) + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{1}$$

where we choose the basis of the vector harmonics as

$$a_{lm} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_{lm} = \begin{pmatrix} 0 \\ Y_{lm} \\ 0 \\ 0 \end{pmatrix}, c_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ \partial_\theta Y_{lm} \\ \partial_\varphi Y_{lm} \end{pmatrix}, d_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sin\theta}\partial_\varphi Y_{lm} \\ \sin\theta\, \partial_\theta Y_{lm} \end{pmatrix}.$$

The basis are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^\pi C_\mu^* C_\rho \, \eta^{\nu\rho} \sin\theta d\theta,$$

where $\eta^{\nu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.}$$

We also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t)\, e^{ikr} \quad \text{etc.} \tag{2}$$

So that we will be able to obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

4.2 Constraint propagations in Schwarzschild spacetime

1. the standard Schwarzschild coordinate

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)}$$

2. the isotropic coordinate, which is given by, $r = (1 + M/2r_{iso})^2 r_{iso}$:

$$ds^2 = -(\frac{1 - M/2r_{iso}}{1 + M/2r_{iso}})^2 dt^2 + (1 + \frac{M}{2r_{iso}})^4 [dr_{iso}^2 + r_{iso}^2 d\Omega^2], \qquad \text{(the isotropic expression)}$$

3. the ingoing Eddington-Finkelstein (iEF) coordinate, by $t_{iEF} = t + 2M \log(r - 2M)$:

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r} dt_{iEF} dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)}$$

4. the Painlevé-Gullstrand (PG) coordinates,

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right) dt_{PG}^2 + 2\sqrt{\frac{2\,M}{r}} dt_{PG}\, dr + dr^2 + r^2 d\Omega^2 \ , \qquad \text{(the PG expression)}$$

which is given by $t_{PG} = t + \sqrt{8Mr} - 2M \log\{(\sqrt{r/2M} + 1)/(\sqrt{r/2M} - 1)\}$

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

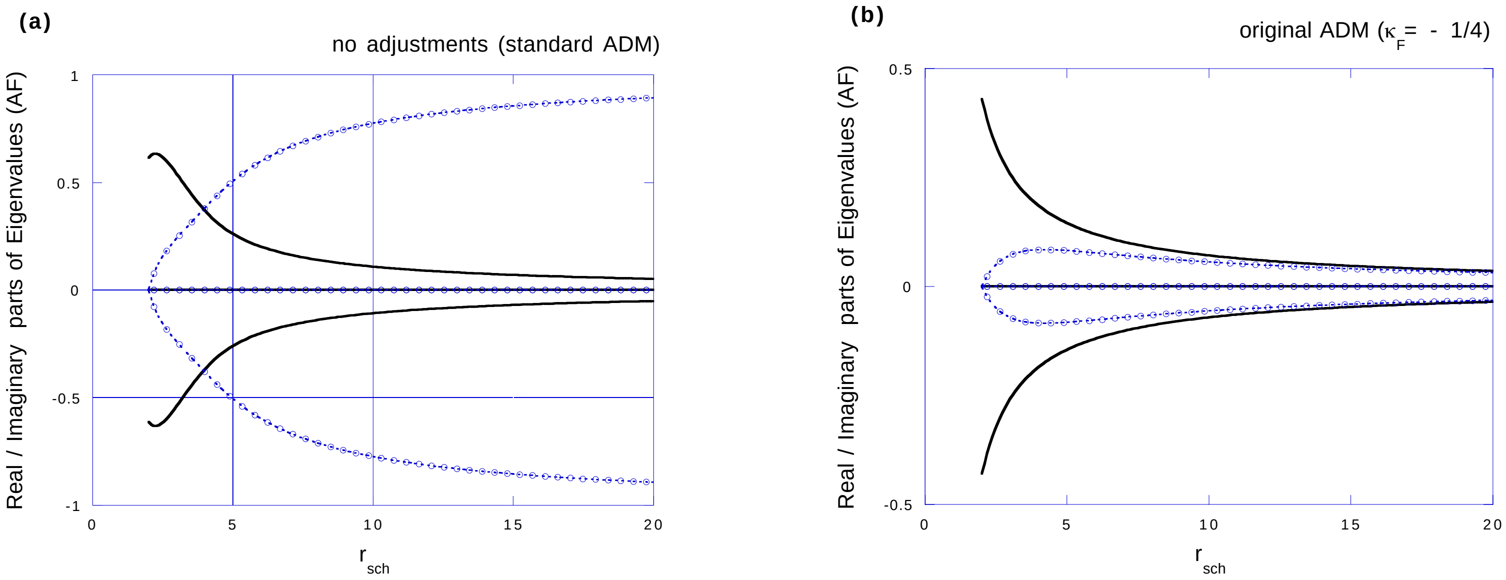


Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

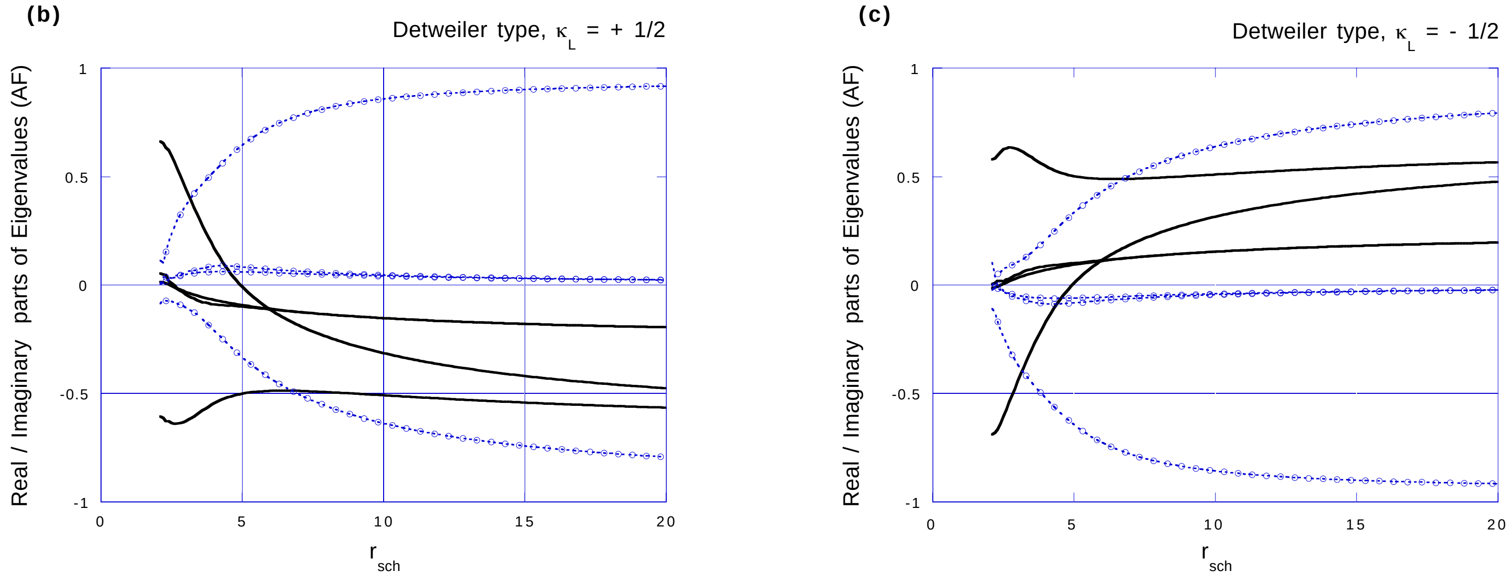


Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= \text{(original terms)} + P_{ij}\mathcal{H}, \\
\partial_t K_{ij} &= \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
& \text{where} \quad P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij}), \\
& \qquad\quad S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}
$$

Example 3: standard ADM (in isotropic/iEF coord.)

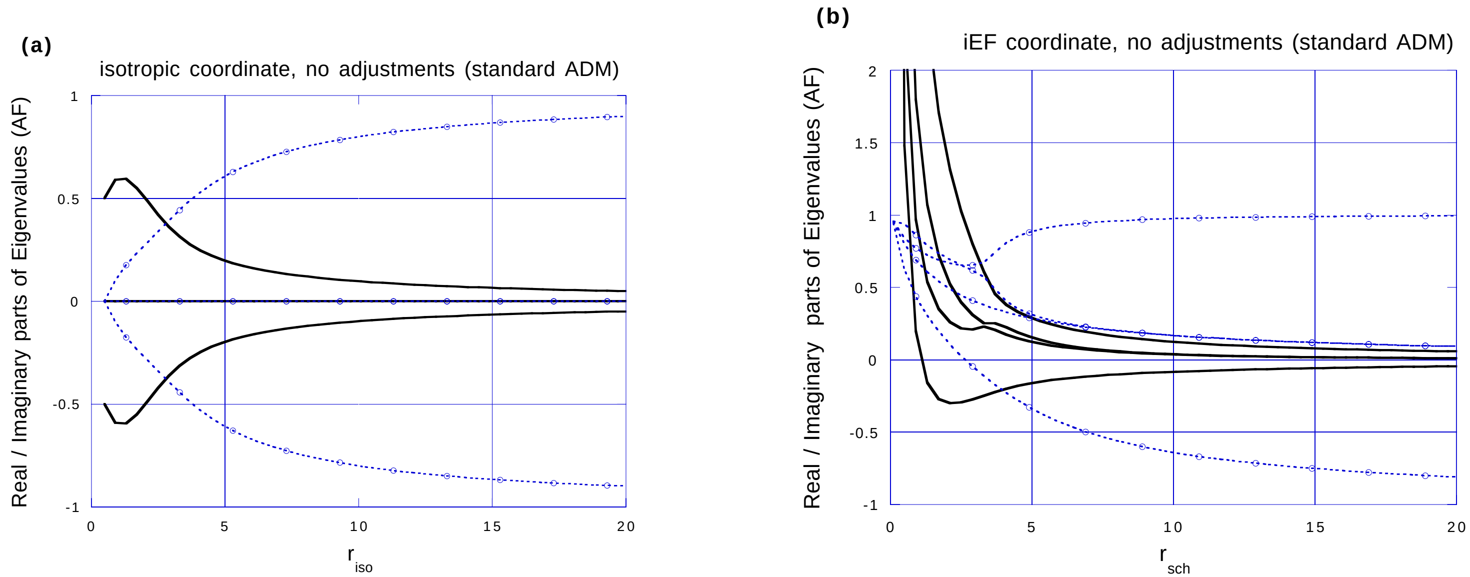


Figure 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \leq r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

Example 4: Detweiler-type adjusted (in iEF/PG coord.)

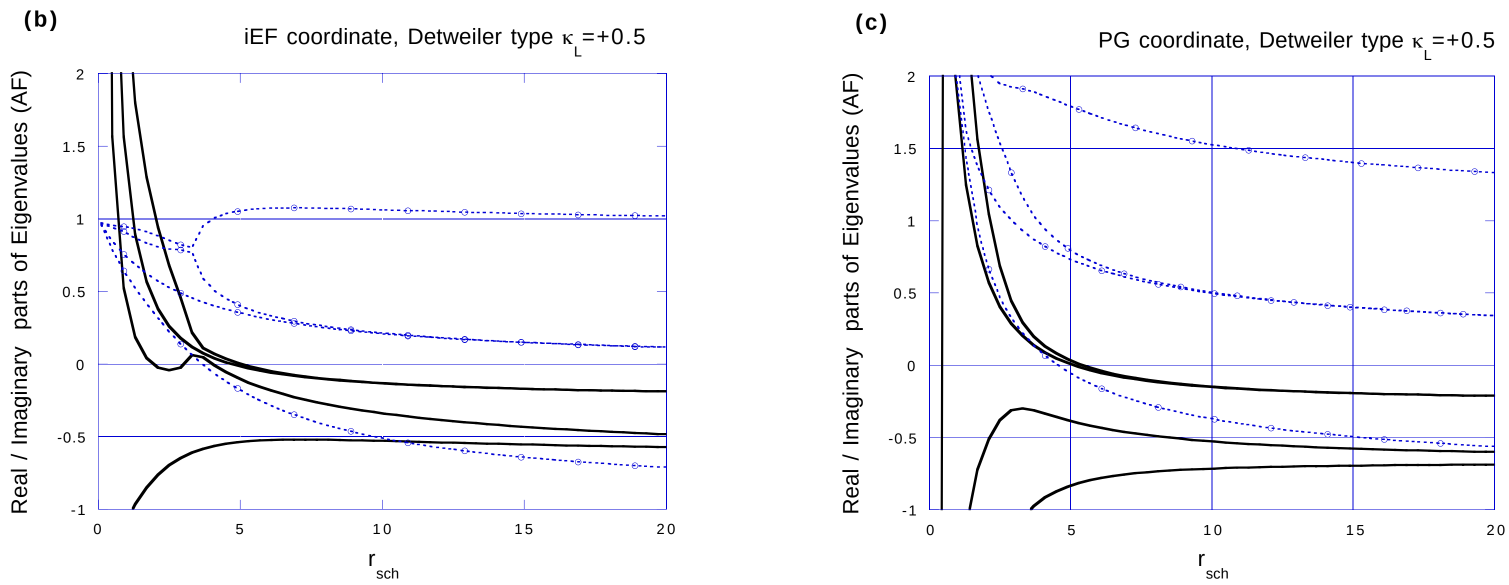


Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1		Adjustment	1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa\gamma_{ij}$ or $Q^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa\gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu\alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa\gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa\alpha\gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

3.2.2 Numerical demonstration

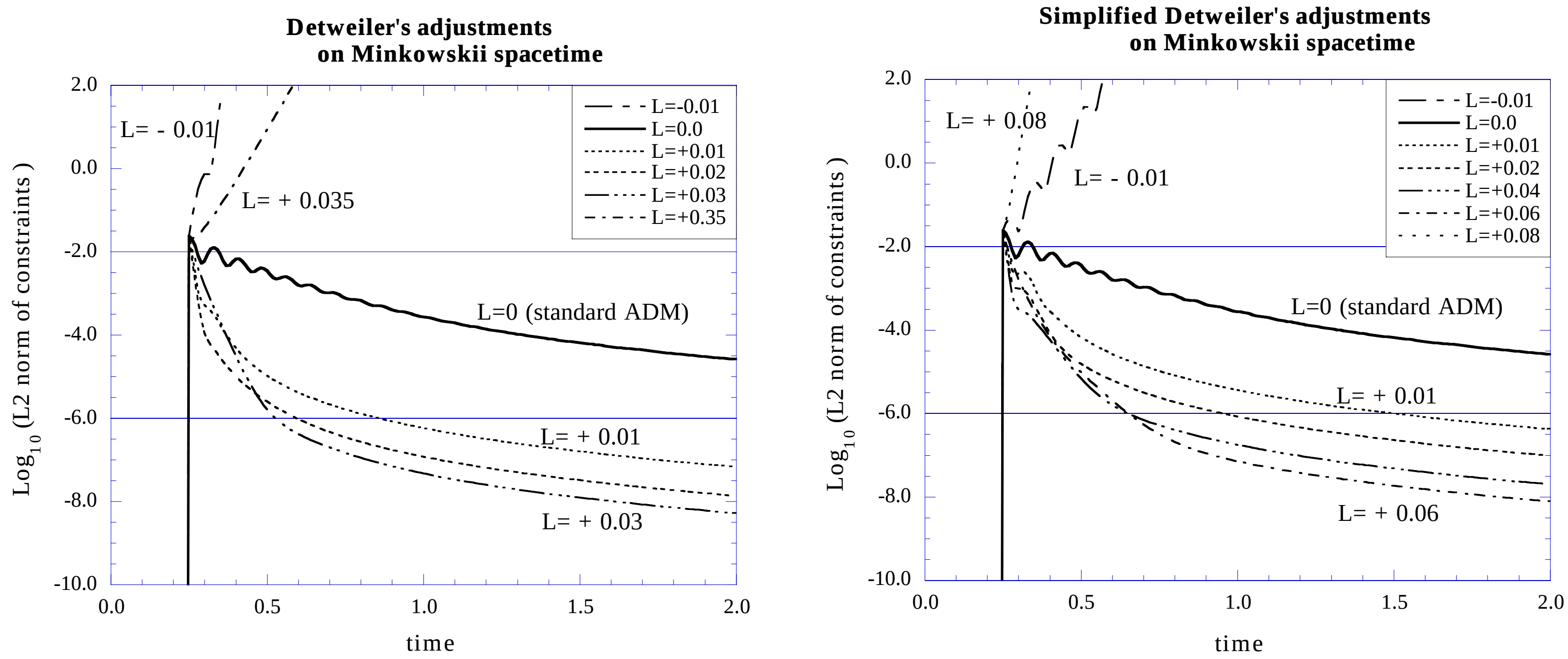


Figure 1: We confirmed numerically, using Minkowskii perturbation, that Detweiler's system presents better accuracy than the standard ADM, but only form small positive L.

Comparisons of Adjusted ADM systems (Teukolsky wave)
3-dim, harmonic slice, periodic BC **HS original Cactus/GR code**

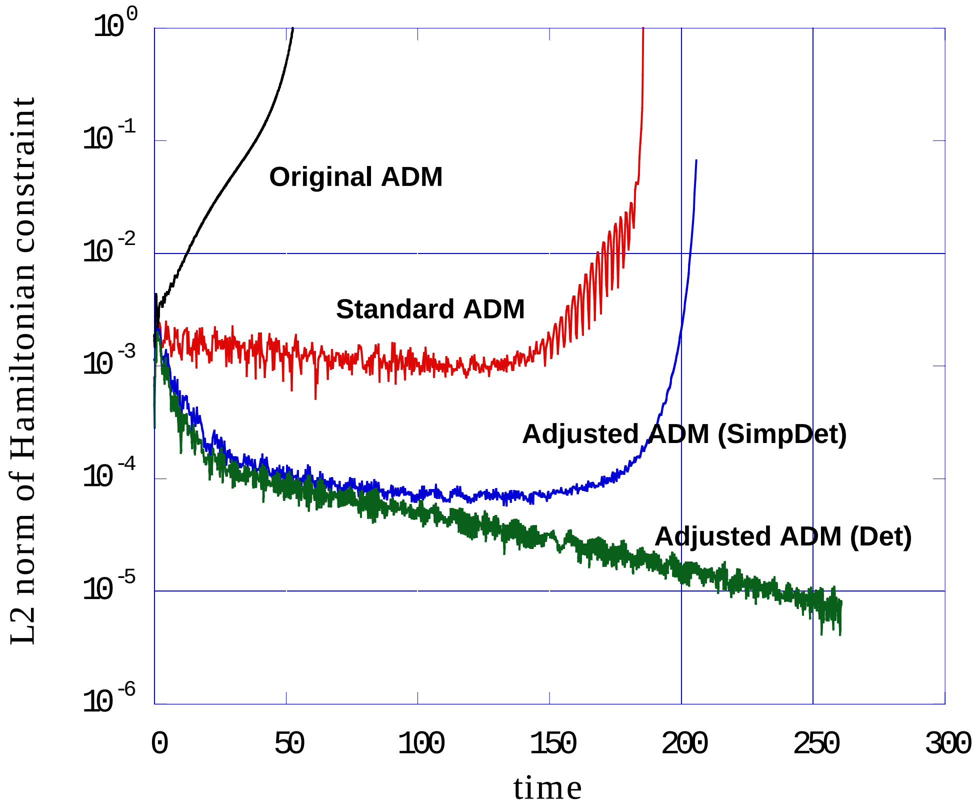


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid = 24^3, $\Delta x = 0.25$, iterative Crank-Nicholson method.

HS original GR code (2002/October)

Use the Cactus-code base structure (`http://cactuscode.org`)

-- parallelize, parameter control, I/O, PUGH, elliptic solvers, ...

-- original module for all GR part (initial data/ADMevolution)

`Cactus/arrangements/GR/`

PC cluster with 4 (2002/September)

Pentium 4, 2.53GHz, 2GB each, 80 GHD each, gigabit ether

-- about 1.2 x 10^6 yen for total parts

-- TurboLinux 7, Intel Fortran compiler, MPICH, ... all free

-- possible up to 120^3 grid full GR simulation

Grant-in-Aid for Scientific Research Fund of JSPS, No.14740179 (2002-2005)

"Einstein equations" are time-reversal invariant. So ...

Why all negative amplification factors (AFs) are available?

Explanation by the time-reversal invariance (TRI)

- the adjustment of the system I,

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{K_{ij}}_{(-)} = \kappa_1 \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

preserves TRI. ... so the AFs remain zero (unchange).

- the adjustment by (a part of) Detweiler

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{\gamma_{ij}}_{(+)} = -L \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

violates TRI. ... so the AFs can become negative.

Therefore

$$\boxed{\text{We can break the time-reversal invariant feature of the "ADM equations".}}$$

Reformulating the Einstein eqs for Stable Numerical Simulations
– Formulation Problem in Numerical Relativity –

Hisaaki Shinkai

1. **Introduction**
   (0) **Arnowitt-Deser-Misner**
2. **Three approaches**
   (1) **Baumgarte-Shapiro-Shibata-Nakamura formulation**
   (2) **Hyperbolic formulations**
   (3) **Attractor systems – "Adjusted Systems"**
3. **Adjusted ADM systems**
   Flat background
   Schwarzschild background
   + Numerical Examples
4. **Adjusted BSSN systems**
   Flat background
   + Numerical Examples
5. **Summary and Future Issues**

strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables ($\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i$), instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
&\quad R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
&\quad \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

– No explicit explanations why this formulation works better.
AEI group (2000): the replacement by momentum constraint is essential.

Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{1}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{2}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{3}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{4}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{5}$$

Adjustments in evolution equations

$$\partial_t^B \varphi = \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, \tag{6}$$

$$\partial_t^B \tilde{\gamma}_{ij} = \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{7}$$

$$\partial_t^B K = \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{8}$$

$$\begin{aligned}\partial_t^B \tilde{A}_{ij} = {} & \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + ((1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij}) - (1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k \\ & +\alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k)\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_t^B \tilde{\Gamma}^i = {} & \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} - (2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) \\ & +2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) + (1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) \\ & +(5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) + (1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}).\end{aligned} \tag{10}$$

A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS} \begin{pmatrix} \mathcal{H}^{BS} \\ \mathcal{M}_i \\ \mathcal{G}^i \\ \mathcal{S} \\ \mathcal{A} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ -(1/3)(\partial_i \alpha) + (1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25} \\ 0 & \alpha \tilde{\gamma}^{ij} & 0 & A_{34} & A_{35} \\ 0 & 0 & 0 & \beta^k(\partial_k \mathcal{S}) & -2\alpha\tilde{\gamma} \\ 0 & 0 & 0 & 0 & \alpha K + \beta^k \partial_k \end{pmatrix} \begin{pmatrix} \mathcal{H}^{BS} \\ \mathcal{M}_j \\ \mathcal{G}^j \\ \mathcal{S} \\ \mathcal{A} \end{pmatrix}$$

$$\begin{aligned}
A_{11} &= +(2/3)\alpha K + (2/3)\alpha\mathcal{A} + \beta^k\partial_k \\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde{\gamma}^{kj} - 2e^{-4\varphi}(\partial_k\alpha)\tilde{\gamma}^{jk} \\
A_{13} &= -2\alpha e^{-4\varphi}\tilde{A}^k{}_j\partial_k - \alpha e^{-4\varphi}(\partial_j\tilde{A}_{kl})\tilde{\gamma}^{kl} - e^{-4\varphi}(\partial_j\alpha)\mathcal{A} - e^{-4\varphi}\beta^k\partial_k\partial_j - (1/2)e^{-4\varphi}\beta^k\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\partial_k \\
&\quad +(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S}) - (2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j \\
A_{14} &= 2\alpha e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{lk}(\partial_l\varphi)\mathcal{A}\partial_k + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\mathcal{A})\tilde{\gamma}^{lk}\partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\alpha)\tilde{\gamma}^{lk}\mathcal{A}\partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m\tilde{\gamma}^{lk}\partial_m\partial_l\partial_k \\
&\quad -(5/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^m\tilde{\gamma}^{lk}(\partial_m\mathcal{S})\partial_l\partial_k + e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m(\partial_m\tilde{\gamma}^{lk})\partial_l\partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^i(\partial_j\partial_i\tilde{\gamma}^{jk})\partial_k \\
&\quad +(3/4)e^{-4\varphi}\tilde{\gamma}^{-3}\beta^i\tilde{\gamma}^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k - (3/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^i(\partial_i\tilde{\gamma}^{jk})(\partial_j\mathcal{S})\partial_k + (1/3)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{pj}(\partial_j\beta^k)\partial_p\partial_k \\
&\quad -(5/12)e^{-4\varphi}\tilde{\gamma}^{-2}\tilde{\gamma}^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j + (1/3)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\beta^k)\partial_i - (1/6)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{mk}(\partial_k\partial_l\beta^l)\partial_m \\
A_{15} &= (4/9)\alpha K\mathcal{A} - (8/9)\alpha K^2 + (4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde{\gamma}^{ij} + (8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde{\gamma}^{lk}) + \alpha e^{-4\varphi}(\partial_j\tilde{\gamma}^{jk})\partial_k \\
&\quad +8\alpha e^{-4\varphi}\tilde{\gamma}^{jk}(\partial_j\varphi)\partial_k + \alpha e^{-4\varphi}\tilde{\gamma}^{jk}\partial_j\partial_k + 8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde{\gamma}^{lk} + e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde{\gamma}^{lk}) + 2e^{-4\varphi}(\partial_l\alpha)\tilde{\gamma}^{lk}\partial_k \\
&\quad +e^{-4\varphi}\tilde{\gamma}^{lk}(\partial_l\partial_k\alpha) \\
A_{23} &= \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)(\partial_j\tilde{\gamma}_{mi}) - (1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}(\partial_j\tilde{\gamma}_{mi}) \\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_k\partial_j\tilde{\gamma}_{mi}) + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S}) - (1/4)\alpha e^{-4\varphi}(\partial_i\tilde{\gamma}_{kl})(\partial_j\tilde{\gamma}^{kl}) + \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)\tilde{\gamma}_{ji}\partial_m \\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i - (1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}\tilde{\gamma}_{ji}\partial_m + \alpha e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\Gamma}_{ijk}\partial_m + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{lk}\tilde{\gamma}_{ji}\partial_k\partial_l \\
&\quad +(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_j\tilde{\gamma}_{im})(\partial_k\alpha) + (1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i + (1/2)e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\gamma}_{ji}(\partial_k\alpha)\partial_m \\
A_{25} &= -\tilde{A}^k{}_i(\partial_k\alpha) + (1/9)(\partial_i\alpha)K + (4/9)\alpha(\partial_i K) + (1/9)\alpha K\partial_i - \alpha\tilde{A}^k{}_i\partial_k \\
A_{34} &= -(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-2}(\partial_l\mathcal{S})\partial_k - (1/2)(\partial_l\beta^i)\tilde{\gamma}^{lk}\tilde{\gamma}^{-1}\partial_k + (1/3)(\partial_l\beta^l)\tilde{\gamma}^{ik}\tilde{\gamma}^{-1}\partial_k - (1/2)\beta^l\tilde{\gamma}^{in}(\partial_l\tilde{\gamma}_{mn})\tilde{\gamma}^{mk}\tilde{\gamma}^{-1}\partial_k \\
&\quad +(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-1}\partial_l\partial_k \\
A_{35} &= -(\partial_k\alpha)\tilde{\gamma}^{ik} + 4\alpha\tilde{\gamma}^{ik}(\partial_k\varphi) - \alpha\tilde{\gamma}^{ik}\partial_k
\end{aligned}$$

BSSN Constraint propagation analysis in flat spacetime

- The set of the constraint propagation equations, $\partial_t(\mathcal{H}^{BSSN}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$?
- For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first order perturbation equations of (6)-(10):

$$\begin{aligned}
\partial_t {}^{(1)}\varphi &= -(1/6)^{(1)}K + (1/6)\kappa_\varphi {}^{(1)}\mathcal{A} && (11)\\
\partial_t {}^{(1)}\tilde{\gamma}_{ij} &= -2^{(1)}\tilde{A}_{ij} - (2/3)\kappa_{\tilde{\gamma}}\delta_{ij}{}^{(1)}\mathcal{A} && (12)\\
\partial_t {}^{(1)}K &= -(\partial_j\partial_j {}^{(1)}\alpha) + \kappa_{K1}\partial_j {}^{(1)}\mathcal{G}^j - \kappa_{K2}{}^{(1)}\mathcal{H}^{BSSN} && (13)\\
\partial_t {}^{(1)}\tilde{A}_{ij} &= {}^{(1)}(R_{ij}^{BSSN})^{TF} - {}^{(1)}(\tilde{D}_i\tilde{D}_j\alpha)^{TF} + \kappa_{A1}\delta_{k(i}(\partial_{j)}{}^{(1)}\mathcal{G}^k) - (1/3)\kappa_{A2}\delta_{ij}(\partial_k {}^{(1)}\mathcal{G}^k) && (14)\\
\partial_t {}^{(1)}\tilde{\Gamma}^i &= -(4/3)(\partial_i {}^{(1)}K) - (2/3)\kappa_{\tilde{\Gamma}1}(\partial_i {}^{(1)}\mathcal{A}) + 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i && (15)
\end{aligned}$$

We express the adjustements as

$$\kappa_{adj} := (\kappa_\varphi, \kappa_{\tilde{\gamma}}, \kappa_{K1}, \kappa_{K2}, \kappa_{A1}, \kappa_{A2}, \kappa_{\tilde{\Gamma}1}, \kappa_{\tilde{\Gamma}2}). \quad (16)$$

- Constraint propagation equations at the first order in the flat spacetime:

$$\begin{aligned}
\partial_t {}^{(1)}\mathcal{H}^{BSSN} &= (\kappa_{\tilde{\gamma}} - (2/3)\kappa_{\tilde{\Gamma}1} - (4/3)\kappa_\varphi + 2)\,\partial_j\partial_j {}^{(1)}\mathcal{A} + 2(\kappa_{\tilde{\Gamma}2} - 1)(\partial_j {}^{(1)}\mathcal{M}_j), && (17)\\
\partial_t {}^{(1)}\mathcal{M}_i &= (-(2/3)\kappa_{K1} + (1/2)\kappa_{A1} - (1/3)\kappa_{A2} + (1/2))\,\partial_i\partial_j {}^{(1)}\mathcal{G}^j \\
&\quad +(1/2)\kappa_{A1}\partial_j\partial_j {}^{(1)}\mathcal{G}^i + ((2/3)\kappa_{K2} - (1/2))\,\partial_i {}^{(1)}\mathcal{H}^{BSSN}, && (18)\\
\partial_t {}^{(1)}\mathcal{G}^i &= 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i + (-(2/3)\kappa_{\tilde{\Gamma}1} - (1/3)\kappa_{\tilde{\gamma}})(\partial_i {}^{(1)}\mathcal{A}), && (19)\\
\partial_t {}^{(1)}\mathcal{S} &= -2\kappa_{\tilde{\gamma}}{}^{(1)}\mathcal{A}, && (20)\\
\partial_t {}^{(1)}\mathcal{A} &= (\kappa_{A1} - \kappa_{A2})(\partial_j {}^{(1)}\mathcal{G}^j). && (21)
\end{aligned}$$

Effect of adjustments

No.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	diag?	Constr. Amp. Factors in Minkowskii background
0.	standard ADM	use	use	-	-	-	yes	$(0, 0, \Im, \Im)$
1.	BSSN no adjustment	use	use	use	use	use	yes	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$ Growing modes
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
8.	ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$	use+adj	use+adj	-	-	-	no	$(0, 0, 0, 0)$
9.	ignore $\mathcal{G}^i$, $\mathcal{A}$	use+adj	use+adj	use+adj	-	-	yes	$(0, \Im, \Im, \Im, \Im, \Im, \Im)$
10.	ignore $\mathcal{G}^i$	use+adj	use+adj	-	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0)$
11.	ignore $\mathcal{A}$	use+adj	use+adj	use+adj	-	use+adj	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
12.	ignore $\mathcal{S}$	use+adj	use+adj	use+adj	use+adj	-	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$

New Proposals :: Improved (adjusted) BSSN systems

TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t\tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{A}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{A}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{A}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{A}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}$$

or in the flat background

$$\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{A}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{A}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{A}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{A}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}$$

Constraint Amplification Factors with each adjustment

adjustment		CAFs	diag?	effect of the adjustment	
$\partial_t \phi$	$\kappa_{\phi\mathcal{H}}\,\alpha\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$	no	$\kappa_{\phi\mathcal{H}}<0$ makes 1 Neg.	
$\partial_t \phi$	$\kappa_{\phi\mathcal{G}}\,\alpha\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}}<0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{SD}\,\alpha\tilde{\gamma}_{ij}\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), (3/2)\kappa_{SD}k^2)$	yes	$\kappa_{SD}<0$ makes 1 Neg.	Case (B)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\,\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1}>0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$	(0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2}\pm\sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2}<0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1}<0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2}>0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\,\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2)$, $(1/3)\kappa_{K\mathcal{M}}k^2\pm(1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}}<0$ makes 2 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\,\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1}k^2)$	yes	$\kappa_{A\mathcal{M}1}>0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\,\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$	(0,0, $-k^2\kappa_{A\mathcal{M}2}/4\pm\sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2}>0$ makes 7 Neg	Case (D)
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1}<0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$	yes	$\kappa_{A\mathcal{A}2}>0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\,\alpha\tilde{D}^i\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}}>0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\,\alpha\mathcal{G}^i$	$(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1}<0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2}>0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\,\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3}>0$ makes 2 Neg. 1 Pos.	

gr-qc/0204002 (PRD in print)

Comparisons of Adjusted BSSN systems (linear wave)

Mexico NR 2002 Workshop participants

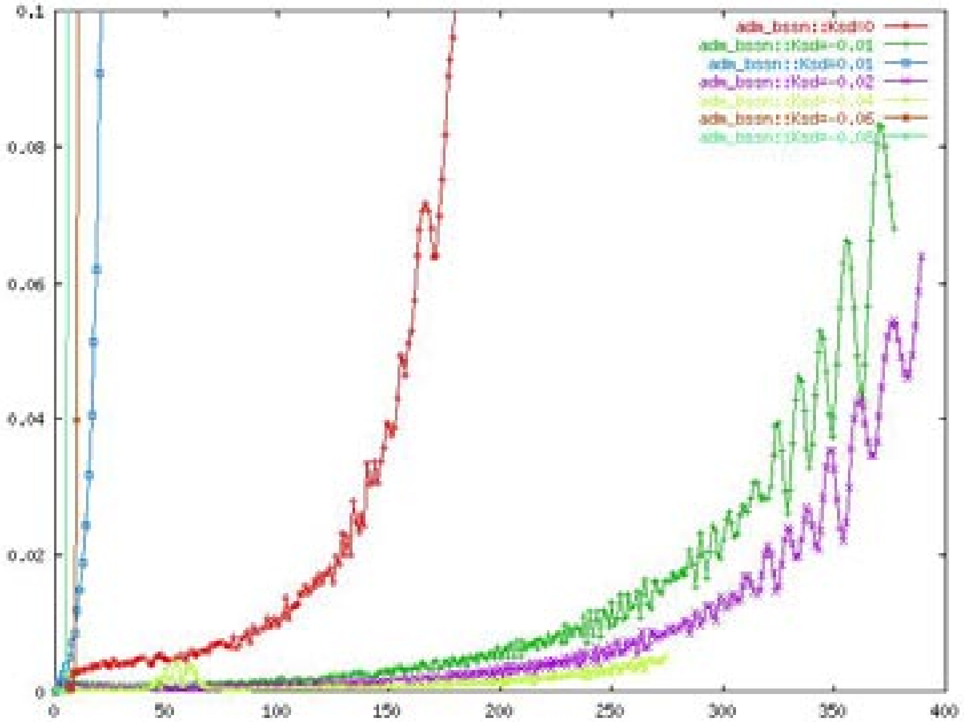
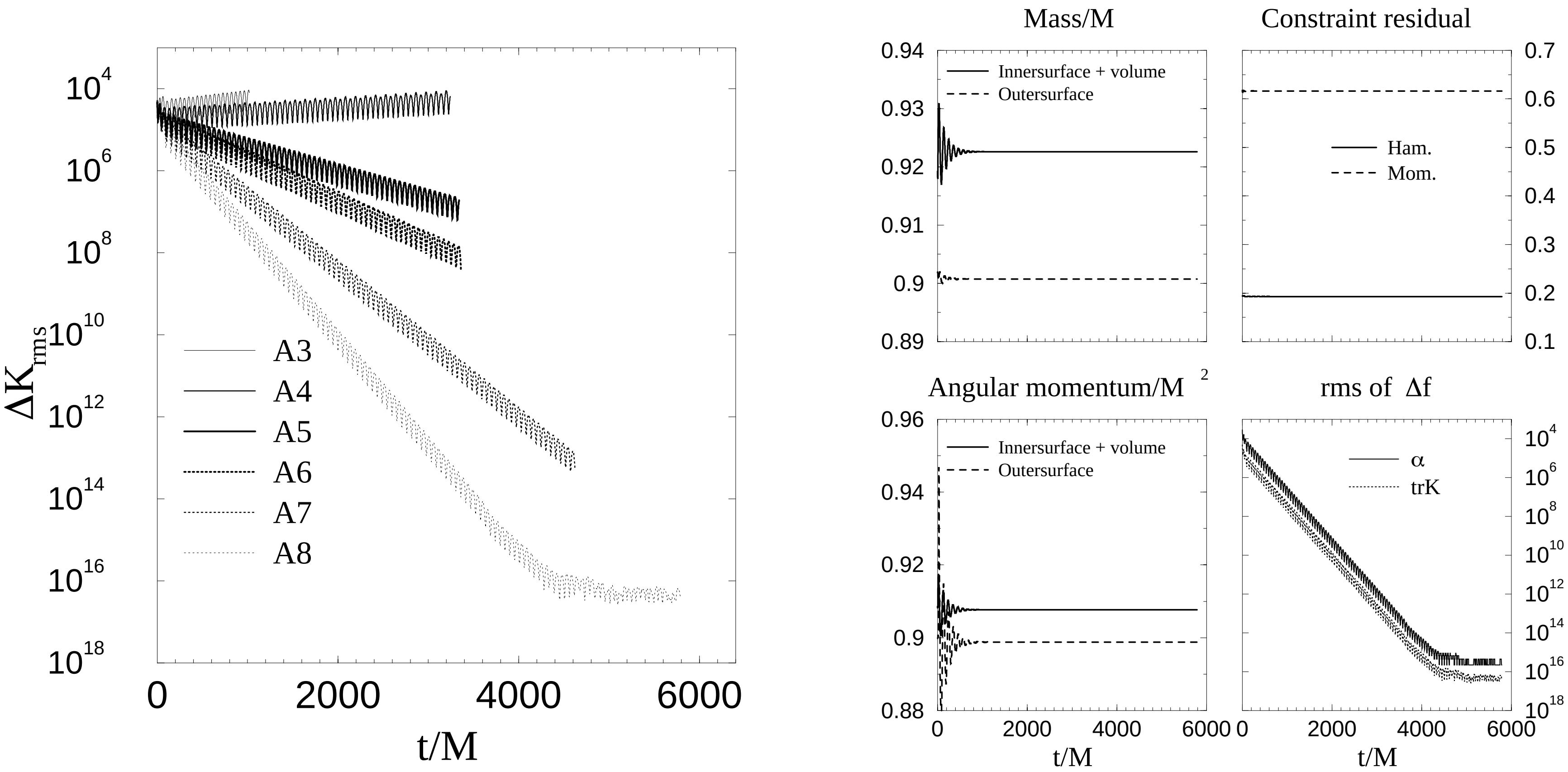


Figure 2: Violation of Hamiltonian constraints versus time: Adjusted BSSN systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/AEIThorns/BSSN code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method. Courtesy of N. Dorband and D. Pollney (AEI).

An Evolution of Adjusted BSSN Formulation

by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026



Kerr-Schild BH (0.9 J/M), excision with cube, $1+\log$-lapse, Γ-driver shift.

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j} \qquad \chi = 2/3 \;\; \textbf{for (A4)-(A8)}$$

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa \alpha \tilde{\gamma}_{ij} \mathcal{H} \qquad \kappa = 0.1 \sim 0.2 \;\; \textbf{for (A5), (A6) and (A8)}$$

Summary

Towards a stable and accurate formulation for numerical relativity

- **Several reports say numerical stabilities depend on the formulations to apply, although they are mathematically equivalent.**
- **status = chaotic, many trials and errors**
  We tried to understand the background in an unified way.
- **Our proposal = "Evaluate eigenvalues of constraint propagation eqns"**
  We give satisfactory conditions for stable evolutions.
  Fourier transformation allows to discuss lower-order terms.
- **Our Observation = "Stability will change by adding constraints in RHS"**
  We named "Adjusted System".
  Numerically confirmed in the Maxwell system and Ashtekar system.
- **Our Observation 2= The idea works even for the ADM formulation**
  We explain the effective parameter range of Detweiler's system (1987).
  We proposed variety of adjustments. Several numerical confirmations.
- **Our Observation 3= The idea works also for the BSSN formulation**
  We explain why adjusting momentum constraints improves the stability.
  We proposed variety of adjustments. Several numerical confirmations.

Evaluation of CAFs may be an alternative guideline to hyperbolization of the system.

Next Steps?

- Generalize the procedure to construct adjusted systems
  - dynamical and automatical determination of κ under a suitable principle.
  - target to control each constraint violation by adjusting multipliers.
  - clarify the reasons of non-linear violation in current test evolutions.
- More on hyperbolic formulations
  - effects of non-principal part?, clarify the reasons of advantages of kinematic parameters (in KST) mixed-form variables, and/or densitized lapse?
  - links to the initial-boundary value problem (IBVP).
- Alternative new ideas?
  - control theories, optimization methods (convex functional theories), mathematical programming methods, or ....
- Numerical comparisons of formulations, links to other systems, ...
  - "Comparisons of Formulations of Einstein's equations for Numerical Relativity" (Mexico NR workshop, 2002) in progress
  - with MHD people, mini-symposium at The 5th International Congress on Industrial and Applied Mathematics (Sydney, July 2003).

Reformulating the Einstein eqs for Stable Numerical Simulations – Formulation Problem in Numerical Relativity –

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Outline

Why mathematically equivalent eqs produce different numerical stability?

- **Three approaches: ADM/BSSN, hyperbolic formulation, attractor systems**
- **Proposals : A unified treatment as Adjusted Systems**

Ref: gr-qc/0209111 (review article, to be published from Nove Science Publ.)

A series of works with **Gen Yoneda** Math. Sci. Dept., Waseda Univ., Japan

@ 12th JGRG, Tokyo, November 27, 2002