

Controlling Constraint Violation using Adjusted ADM Systems

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Summary & Outlook

- Formulation problem of Numerical Relativity
- Longer Evolutions by Adjusted ADM Systems (ADM+Lagrange multipliers)
- Teukolsky wave, 3+1 propagation
→ Standard ADM $\times 1.5 \sim 4$ life-time
- Trying to keep the Error at small value is better than to force the Error to zero.
- Next Step: Develop Auto-Control system of Lagrange multipliers
→ Similar results must be hold either at Adjusted BSSN Systems.

Formulation Problem?

Numerical Relativity
= Necessary for unveiling the strong nature of gravity

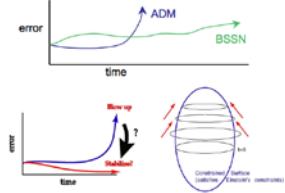
- GWs from NS-NS, BH-BH, NS-BH coalescences
- Relativistic phenomena like Cosmology, AGN, ...
- Mathematical feedbacks to Singularity, Exact Solutions, ...
- Laboratory of Gravitational theories: Higher dimensional models, ...

Current Standard Formulation

- BH-BH, NS-BH simulations (2005— Pretorius, UTB, NASA, PSU, LSU, ...)
- BSSN formulation
 - lapse function: $1+\log$ slicing
 - shift vector: Gamma-freezing driver
 - initial data: puncture initial data

This combination works, anyway.
Why? Alternatives?

blow-up



For a review, please take a look
Shinkai & Yoneda, gr-qc/0209111

Adjusted Systems

General Procedure

- prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \dots)$
- add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \dots) + F(C^a, \partial_b C^a, \dots)$
- choose appropriate $F(C^a, \partial_b C^a, \dots)$
to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \dots)$?

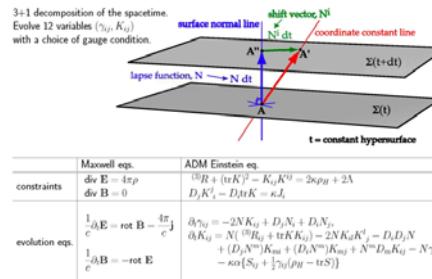
- prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \dots)$
- and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \dots) + G(C^a, \partial_b C^a, \dots)$
- Fourier transform and evaluate eigenvalues $\partial_t C^a = \langle A(\hat{C}^a) \rangle \hat{C}^a$

$\Gamma + 0\lambda$ appropriate adjustments → Better Stability

ADM vs BSSN Adjusted ADM Adjusted BSSN

The Standard ADM Formulation

(Arnowitt-Deser-Misner, 1962; York 1978)



	Maxwell eqs.	ADM Einstein eq.
constraints	$\text{div } E = -4\pi\rho$ $\text{div } B = 0$	$\partial_t K^i + (\nabla^j K^i)_j = -K^j_i K^j - 2\kappa\rho\mu + 2\lambda$ $D_j K^i - D_i K^j = \delta_{ij}$
evolution eqs.	$\frac{1}{c} \partial_t E = -\text{rot } B - \frac{4\pi}{c} J$ $\frac{1}{c} \partial_t B = -\text{rot } E$	$\partial_t K_{ij} = -2N K_{ij} + D_j N_i + D_i N_j$ $\partial_t K_{ij} = N(\partial_t R_{ij} + \text{tr} K_{ij}) - 2N K_{ik} K^k_j - D_i D_j N + (D_i N^k) K_{kj} + (D_j N^k) K_{ki} - N^k D_m K_{ij} - N \gamma_{ij}$ $\partial_t S_{ij} = -\kappa\alpha(S_{ij}) + \frac{1}{2}\gamma_{ij}(p_H - \text{tr} S)$

BSSN Formulation

(Nakamura et al. 1987; Shibata-Nakamura 1995; Baumgarte-Shapiro 1996)

— define new variables $(\gamma_{ij}, K_{ij}, \lambda)$ instead of the ADM's (N, K_{ij}) , where
 $\gamma_{ij} = e^{2N} K_{ij}$, $K_{ij} = e^{-2N} K_{ij} - (1/2)\delta_{ij} K_{kk}$, $\lambda = P_{ij} K_{ij}^2$.
use evolution constraint in Γ term, and impose $d\gamma_{ij} = 1$ during the evolution.
— the set of evolution equations become

$$\begin{aligned} (\partial_t - \mathcal{L}_E) \gamma_{ij} &= -2N \gamma_{ij} + \gamma_{ij}^{-1} (\gamma_{ij} K_{ij}^2 - \gamma_{ij}^2 \nabla^k \nabla_l \gamma_{kl}), \\ (\partial_t - \mathcal{L}_E) K_{ij} &= -e^{-2N} (\nabla^k \nabla_l \gamma_{kl})^2 + \gamma_{ij}^{-1} \gamma_{ij} K_{ij}^2 + e^{-2N} (\gamma_{ij} K_{ij})^2 + e^{-2N} (\gamma_{ij} K_{ij})^2 + 2\kappa\lambda \gamma_{ij} - 2\lambda \delta_{ij}, \\ \partial_t \lambda &= -2(\rho\mu) \lambda^2 - (1/2)(\kappa\alpha) \lambda^2 + (1/2) \lambda^2 (\rho\mu) - 2\kappa\lambda \gamma_{ij}^2 - \lambda^2 \gamma_{ij}^2 + 2(\rho\mu) \lambda^2 \gamma_{ij}^2 \end{aligned}$$

$$R_{ij} = -1/2(\rho\mu - \partial_t \rho\mu) + \partial_t \rho\mu \lambda^2 + 1/2 \gamma_{ij}^2 \partial_t \gamma_{ij} + 2P_{ij} \gamma_{ij}^2, \quad \gamma_{ij} = \rho\mu \gamma_{ij}^2 / \lambda$$

Guidelines for Better Formulation

Eigenvalue-analysis of Constraint Propagation eqs.

Conjecture on Constraint Amplification Factors (CAFs):

$$\partial_t \begin{pmatrix} \tilde{C}_1 \\ i \\ C_N \end{pmatrix} = \begin{pmatrix} \text{Constraint Propagation Matrix} \\ i \\ C_N \end{pmatrix} \begin{pmatrix} \tilde{C}_1 \\ i \\ C_N \end{pmatrix}$$

We see more stable evolution, if CAFs have
(A) negative real-part (the constraints are forced to be diminished), or
(B) non-zero imaginary-part (the constraints are propagating away).

Adjusted ADM ADM vs BSSN Adjusted BSSN

- Standard ADM has constraint violating mode!
- Better ADM must be available → This Work
- Better BSSN must be available

Adjusted ADM formulation (1)

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i + P_{ij} \mathcal{H} \quad (1)$$

$$+ P_{ij} \mathcal{H} + Q'_{ij} M_{ij} + P_{ij} (\nabla_i \mathcal{H}) + q''_{ij} (\nabla_i \mathcal{M}_{ij}) \quad (2)$$

$$\partial_t K_{ij} = \alpha R_{ij}^0 + \alpha K_{ij} - 2\alpha K_{ik} K^k_j - \nabla_i \nabla_j \alpha + (\nabla_i \partial_j^k) K_{kj} + (\nabla_j \partial_i^k) K_{ki} + \beta^k \nabla_k K_{ij} \quad (3)$$

$$+ R_{ij} \mathcal{H} + S'_{ij} M_{ij} + r^k_{ij} (\nabla_i \mathcal{H}) + s^k_{ij} (\nabla_i \mathcal{M}_{ij}) \quad (4)$$

with constraint equations

$$H := R_{ij}^0 + K^2 - K_{ij} K^{ij}, \quad (5)$$

$$M_{ij} := \nabla_i K_j + \nabla_j K_i. \quad (6)$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \{\text{original terms}\} + H_{ij}^{\text{min}}[2] + H_{ij}^{\text{max}}[\partial_j[2]] + H_{ij}^{\text{min}}[\partial_i[2]] + H_{ij}^{\text{max}}[4], \quad (7)$$

$$\partial_t M_{ij} = \{\text{original terms}\} + M_{ij}^{\text{min}}[2] + M_{ij}^{\text{max}}[\partial_j[2]] + M_{ij}^{\text{min}}[\partial_i[4]] + M_{ij}^{\text{max}}[\partial_j[4]]. \quad (8)$$

Adjusted ADM formulation (2)

Table 3. List of adjustments we used in the Schwarzschild spacetime. The values of adjustments are chosen in terms of (7) and (8). The values "0" and "1" are the same as in table 1. The effects to amplitude factor, which is $\epsilon = 0$, are computed for each constraint system and the end frequency per 10%, respectively. The "TNS" means that the numerical simulation is terminated when the total numerical error reaches 10% of the initial value. The "B" means that the numerical simulation is terminated when the numerical error reaches 10% of the initial value. The "S" means that the numerical simulation is stopped when the numerical error reaches 10% of the initial value. The "B" and "S" are the same as in table 1.

No.	Num. value	Adaptation	inf TNS	End	Frequency	End	Frequency
P. 1	0	no	moderate	moderate	moderate	moderate	moderate
P. 2	0	yes	moderate	moderate	moderate	moderate	moderate
P. 3	0	yes	moderate	moderate	moderate	moderate	moderate
P. 4	0	yes	moderate	moderate	moderate	moderate	moderate
P. 5	0	yes	moderate	moderate	moderate	moderate	moderate
P. 6	0	yes	moderate	moderate	moderate	moderate	moderate
P. 7	0	yes	moderate	moderate	moderate	moderate	moderate
P. 8	0	yes	moderate	moderate	moderate	moderate	moderate
P. 9	0	yes	moderate	moderate	moderate	moderate	moderate
P. 10	0	yes	moderate	moderate	moderate	moderate	moderate
P. 11	0	yes	moderate	moderate	moderate	moderate	moderate
P. 12	0	yes	moderate	moderate	moderate	moderate	moderate
P. 13	0	yes	moderate	moderate	moderate	moderate	moderate
P. 14	0	yes	moderate	moderate	moderate	moderate	moderate
P. 15	0	yes	moderate	moderate	moderate	moderate	moderate
P. 16	0	yes	moderate	moderate	moderate	moderate	moderate
P. 17	0	yes	moderate	moderate	moderate	moderate	moderate
P. 18	0	yes	moderate	moderate	moderate	moderate	moderate
P. 19	0	yes	moderate	moderate	moderate	moderate	moderate
P. 20	0	yes	moderate	moderate	moderate	moderate	moderate
P. 21	0	yes	moderate	moderate	moderate	moderate	moderate
P. 22	0	yes	moderate	moderate	moderate	moderate	moderate
P. 23	0	yes	moderate	moderate	moderate	moderate	moderate
P. 24	0	yes	moderate	moderate	moderate	moderate	moderate
P. 25	0	yes	moderate	moderate	moderate	moderate	moderate
P. 26	0	yes	moderate	moderate	moderate	moderate	moderate
P. 27	0	yes	moderate	moderate	moderate	moderate	moderate
P. 28	0	yes	moderate	moderate	moderate	moderate	moderate
P. 29	0	yes	moderate	moderate	moderate	moderate	moderate
P. 30	0	yes	moderate	moderate	moderate	moderate	moderate
P. 31	0	yes	moderate	moderate	moderate	moderate	moderate
P. 32	0	yes	moderate	moderate	moderate	moderate	moderate
P. 33	0	yes	moderate	moderate	moderate	moderate	moderate
P. 34	0	yes	moderate	moderate	moderate	moderate	moderate
P. 35	0	yes	moderate	moderate	moderate	moderate	moderate
P. 36	0	yes	moderate	moderate	moderate	moderate	moderate
P. 37	0	yes	moderate	moderate	moderate	moderate	moderate
P. 38	0	yes	moderate	moderate	moderate	moderate	moderate
P. 39	0	yes	moderate	moderate	moderate	moderate	moderate
P. 40	0	yes	moderate	moderate	moderate	moderate	moderate
P. 41	0	yes	moderate	moderate	moderate	moderate	moderate
P. 42	0	yes	moderate	moderate	moderate	moderate	moderate
P. 43	0	yes	moderate	moderate	moderate	moderate	moderate
P. 44	0	yes	moderate	moderate	moderate	moderate	moderate
P. 45	0	yes	moderate	moderate	moderate	moderate	moderate
P. 46	0	yes	moderate	moderate	moderate	moderate	moderate
P. 47	0	yes	moderate	moderate	moderate	moderate	moderate
P. 48	0	yes	moderate	moderate	moderate	moderate	moderate
P. 49	0	yes	moderate	moderate	moderate	moderate	moderate
P. 50	0	yes	moderate	moderate	moderate	moderate	moderate
P. 51	0	yes	moderate	moderate	moderate	moderate	moderate
P. 52	0	yes	moderate	moderate	moderate	moderate	moderate
P. 53	0	yes	moderate	moderate	moderate	moderate	moderate
P. 54	0	yes	moderate	moderate	moderate	moderate	moderate
P. 55	0	yes	moderate	moderate	moderate	moderate	moderate
P. 56	0	yes	moderate	moderate	moderate	moderate	moderate
P. 57	0	yes	moderate	moderate	moderate	moderate	moderate
P. 58	0	yes	moderate	moderate	moderate	moderate	moderate
P. 59	0	yes	moderate	moderate	moderate	moderate	moderate
P. 60	0	yes	moderate	moderate	moderate	moderate	moderate
P. 61	0	yes	moderate	moderate	moderate	moderate	moderate
P. 62	0	yes	moderate	moderate	moderate	moderate	moderate
P. 63	0	yes	moderate	moderate	moderate	moderate	moderate
P. 64	0	yes	moderate	moderate	moderate	moderate	moderate
P. 65	0	yes	moderate	moderate	moderate	moderate	moderate
P. 66	0	yes	moderate	moderate	moderate	moderate	moderate
P. 67	0	yes	moderate	moderate	moderate	moderate	moderate
P. 68	0	yes	moderate	moderate	moderate	moderate	moderate
P. 69	0	yes	moderate	moderate	moderate	moderate	moderate
P. 70	0	yes	moderate	moderate	moderate	moderate	moderate
P. 71	0	yes	moderate	moderate	moderate	moderate	moderate
P. 72	0	yes	moderate	moderate	moderate	moderate	moderate
P. 73	0	yes	moderate	moderate	moderate	moderate	moderate
P. 74	0	yes	moderate	moderate	moderate	moderate	moderate
P. 75	0	yes	moderate	moderate	moderate	moderate	moderate
P. 76	0	yes	moderate	moderate	moderate	moderate	moderate
P. 77	0	yes	moderate	moderate	moderate	moderate	moderate
P. 78	0	yes	moderate	moderate	moderate	moderate	moderate
P. 79	0	yes	moderate	moderate	moderate	moderate	moderate
P. 80	0	yes	moderate	moderate	moderate	moderate	moderate
P. 81	0	yes	moderate	moderate	moderate	moderate	moderate
P. 82	0	yes	moderate	moderate	moderate	moderate	moderate
P. 83	0	yes	moderate	moderate	moderate	moderate	moderate
P. 84	0	yes	moderate	moderate	moderate	moderate	moderate
P. 85	0	yes	moderate	moderate	moderate	moderate	moderate
P. 86	0	yes	moderate	moderate	moderate	moderate	moderate
P. 87	0	yes	moderate	moderate	moderate	moderate	moderate
P. 88	0	yes	moderate	moderate	moderate	moderate	moderate
P. 89	0	yes	moderate	moderate	moderate	moderate	moderate
P. 90	0	yes	moderate	moderate	moderate	moderate	moderate
P. 91	0	yes	moderate	moderate	moderate	moderate	moderate
P. 92	0	yes	moderate	moderate	moderate	moderate	moderate
P. 93	0	yes	moderate	moderate	moderate	moderate	moderate
P. 94	0	yes	moderate	moderate	moderate	moderate	moderate
P. 95	0	yes	moderate	moderate	moderate	moderate	moderate
P. 96	0	yes	moderate	moderate	moderate	moderate	moderate
P. 97	0	yes	moderate	moderate	moderate	moderate	moderate
P. 98	0	yes	moderate	moderate	moderate	moderate	moderate
P. 99	0	yes	moderate	moderate	moderate	moderate	moderate
P. 100	0	yes	moderate	moderate	moderate	moderate	moderate
P. 101	0	yes	moderate	moderate	moderate	moderate	moderate
P. 102	0	yes	moderate	moderate	moderate	moderate	moderate