Nonlinear dynamics
in the Einstein-Gauss-Bonnet gravity

Hisaaki Shinkai
(Osaka Inst. Tech., Japan)
http://www.oit.ac.jp/is/~shinkai/

- 4-dim, 5-dim, 6-dim,
  ··· how dimensionality affects to dynamics?
- Gauss-Bonnet terms
  ··· how higher-order curvature terms affects to dynamics?
- 2 models
  Colliding scalar pulses / Fate of wormholes
- Ref: HS & Torii , PRD 96(2017)044009 [arXiv:1706.02070]

2018/7/2 @ MG15, Rome

Dynamics in Gauss-Bonnet gravity?

- Action

$$S = \int_{\mathcal{M}} d^n x \sqrt{-g} \left[\frac{1}{2\kappa^2} (\alpha_{\rm GR} \mathcal{R} + \alpha_{\rm GB} \mathcal{L}_{\rm GB}) + \mathcal{L}_{\rm matter} \right]$$

where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

- Field equation

$$\alpha_1 G_{\mu\nu} + \alpha_2 H_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu}$$

where $H_{\mu\nu} = 2[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}] - \frac{1}{2} g_{\mu\nu}\mathcal{L}_{GB}$

- **has the simplest leading terms from String Theory**
- **has two solution branches (GR/non-GR).**
- **has minimum mass for static spherical BH solution**
  T Torii & H Maeda, PRD 71 (2005) 124002
  W-K Ahn, B Gwak, B-H Lee, W Lee, Eur. Phys. J. C75 (2015) 372
- **is expected to have singularity avoidance feature.**
  (but has never been demonstrated in full gravity.)
- **new topic in numerical relativity.**
  S Golod & T Piran, PRD 85 (2012) 104015
  N Deppe+, PRD 86 (2012) 104011
  F Izaurieta & E Rodriguez, 1207.1496
- **much attentions in WH community**
  H Maeda & M Nozawa, PRD 78 (2008) 024005
  P Kanti, B Kleihaus & J Kunz, PRL 107 (2011) 271101
  P Kanti, B Kleihaus & J Kunz, PRD 85 (2012) 044007

Introduction (2)

Formulation for evolution [N+1]

PHYSICAL REVIEW D **78,** 084037 (2008)

$N + 1$ formalism in Einstein-Gauss-Bonnet gravity

Takashi Torii[1,*] and Hisa-aki Shinkai[2,+]

[1]*Department of General Education, Osaka Institute of Technology, Omiya, Asahi-ku, Osaka 535-8585, Japan*

[2]*Department of Information Systems, Osaka Institute of Technology, Kitayama, Hirakata, Osaka 573-0196, Japan*

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Towards the investigation of the full dynamics in a higher-dimensional and/or a stringy gravitational model, we present the basic equations of the Einstein-Gauss-Bonnet gravity theory. We show the $(N + 1)$-dimensional version of the Arnowitt-Deser-Misner decomposition including Gauss-Bonnet terms, which shall be the standard approach to treat the space-time as a Cauchy problem. Because of the quasilinear property of the Gauss-Bonnet gravity, we find that the evolution equations can be in a treatable form in numerics. We also show the conformally transformed constraint equations for constructing the initial data. We discuss how the constraints can be simplified by tuning the powers of conformal factors. Our equations can be used both for timelike and spacelike foliations.

- **Initial Value Construction via Conformal approach**

  Black hole initial data: H Yoshino , PRD 83 (2011) 104010

- **Set of Equations**

  ready, but complicated

Formulation for evolution [dual null]

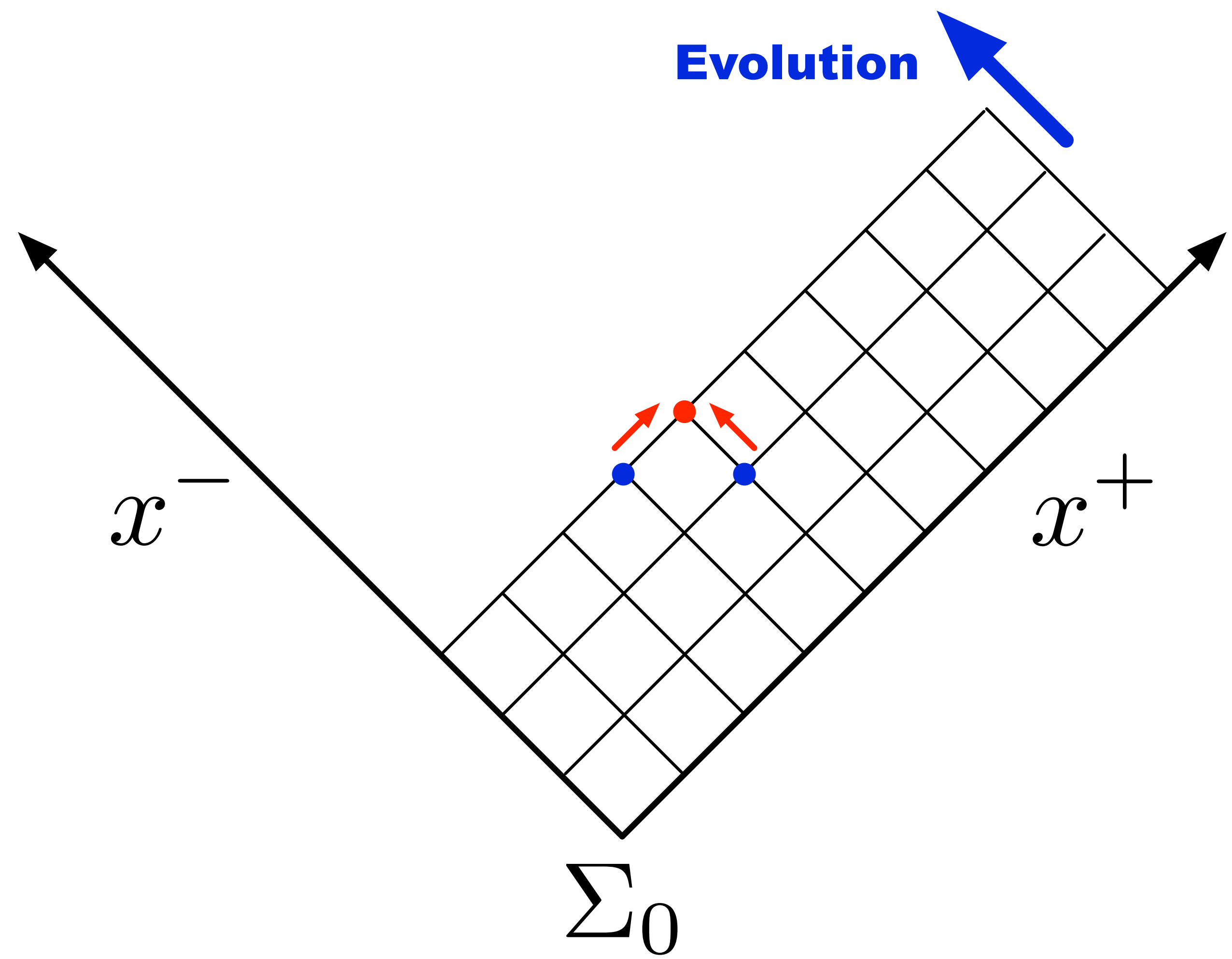
Metric n-dimensional, dual-null coordinate, $2+(n-2)$ decomposition

$$ds^2 = -2e^{-f(x^+,x^-)}\,dx^+\,dx^- + r^2(x^+,x^-)\gamma_{ij}dx^i dx^j \tag{1}$$

Variables

$\Omega = \frac{1}{r}$	Conformal factor
$\vartheta_\pm = (n-2)\partial_\pm r$	expansion
f	lapse function
$\nu_\pm = \partial_\pm f$	inaffinity (shift)

ψ	scalar field (normal)
$\pi_\pm = r\partial_\pm\psi$	scalar momentum
ϕ	scalar field (ghost)
$p_\pm = r\partial_\pm\phi$	scalar momentum



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Parameters

n	dimension
k	curvature
Λ	cosmological constant

For simplicity, we define

$$\tilde{\alpha} = (n-3)(n-4)\alpha_2, \quad (2)$$

$$A = \alpha_1 + 2\tilde{\alpha}\Omega^2 Z, \quad (3)$$

$$W = \frac{2e^f}{(n-2)^2}\vartheta_+\vartheta_-, \quad (4)$$

$$Z = k + W, \quad (5)$$

$$\eta = \Omega^2\frac{(n-2)(n-3)}{2}e^{-f}Z, \quad (6)$$

matter variables

normal field $\psi(u,v)$ and/or ghost field $\phi(u,v)$

$$\begin{aligned} T_{\mu\nu} &= T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi) \\ &= \left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right] + \left[-\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(-\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)\right] \end{aligned}$$

this derives Klein-Gordon equations

$$\Box\psi = \frac{dV_1}{d\psi}, \qquad \Box\phi = \frac{dV_2}{d\phi}.$$

Scalar field variables

$$\begin{aligned} \pi_\pm &\equiv r\partial_\pm\psi = \frac{1}{\Omega}\partial_\pm\psi \\ p_\pm &\equiv r\partial_\pm\phi = \frac{1}{\Omega}\partial_\pm\phi \end{aligned}$$

Klein-Gordon eqs.

$$\begin{aligned} \Box\phi &= -\frac{e^f}{r}\left(2r\phi_{uv} + (n-2)r_u\phi_v + (n-2)r_v\phi_u\right) \\ &= -2e^f\phi_{uv} - e^f\Omega^2\left(\vartheta_-p_+ + \vartheta_+p_-\right) \end{aligned}$$

Energy-momentum tensor

$$\begin{aligned} T_{++} &= \Omega^2(\pi_+^2 - p_+^2) \\ T_{--} &= \Omega^2(\pi_-^2 - p_-^2) \\ T_{+-} &= -e^{-f}\left(V_1(\psi) + V_2(\phi)\right) \\ T_{zz} &= e^f(\pi_+\pi_- - p_+p_-) - \frac{1}{\Omega^2}\left(V_1(\psi) - V_2(\phi)\right) \end{aligned}$$

evolution equations (1)

Equations for x^+ direction

$$\partial_+\Omega = -\frac{1}{n-2}\vartheta_+\Omega^2 \tag{7}$$

$$\partial_+\vartheta_+ = -\vartheta_+\nu_+ - \frac{1}{\Omega A}\kappa^2 T_{++} = -\vartheta_+\nu_+ - \frac{1}{A}\kappa^2\Omega(\pi_+^2 - p_+^2) \tag{8}$$

$$\partial_+\vartheta_- = \frac{1}{A}\frac{e^{-f}}{\Omega}\left[-\alpha_1\Omega^2\frac{(n-2)(n-3)}{2}Z + \Lambda + \kappa^2(V_1+V_2)\right] - \frac{\tilde{\alpha}}{A}\Omega^3 e^{-f}\frac{(n-2)(n-5)}{2}\left[Z^2+W\right] \tag{9}$$

$$\partial_+ f = \nu_+ \tag{10}$$

$\partial_+\nu_+ =$ no evolution eq. exists

$$\begin{aligned}\partial_+\nu_- = {} & \frac{\alpha_1}{A}Ze^{-f}\Omega^2\frac{(n-3)}{2}\left\{-\frac{\alpha_1}{A}2(n-3)+n-4\right\} \\ & +\frac{1}{A}\Omega^2 e^{-f}\kappa^2(\pi_+\pi_- - p_+p_-) + \frac{1}{A}e^{-f}\left\{\frac{\alpha_1}{A}\frac{2(n-3)}{(n-2)} - 1\right\}\{\Lambda + \kappa^2(V_1+V_2)\} \\ & -\frac{\tilde{\alpha}}{A}e^{-f}\Omega^2(n-5)\times\left[\frac{\alpha_1}{A}\Omega^2(n-3)\left\{k^2+2WZ+2Z^2\right\} + \frac{\tilde{\alpha}}{A}\Omega^4 2(n-5)\left\{k^2+2WZ\right\}Z\right] \\ & +\frac{\tilde{\alpha}}{A}e^{-f}\Omega^2(n-5)\times\left[\frac{1}{2}\Omega^2\left\{(n-2)k^2+2WZ-4Z^2\right\} + \frac{1}{A}\frac{4}{n-2}Z\left\{\Lambda+\kappa^2(V_1+V_2)\right\}\right] \\ & -\frac{\tilde{\alpha}}{A}e^{f}\Omega^2\frac{4}{(n-2)^2}\left\{\nu_+\vartheta_+(\partial_-\vartheta_-) + \nu_-\vartheta_-(\partial_+\vartheta_+) + (\partial_+\vartheta_+)(\partial_-\vartheta_-) + \nu_+\nu_-\vartheta_+\vartheta_- - (\partial_-\vartheta_+)^2\right\}\end{aligned} \tag{11}$$

$$\partial_+\psi = \Omega\pi_+ \tag{12}$$

$$\partial_+\phi = \Omega p_+ \tag{13}$$

$\partial_+\pi_+ =$ no evolution eq. exists

$$\partial_+\pi_- = \left(\frac{1}{n-2} - \frac{1}{2}\right)\Omega\vartheta_+\pi_- - \frac{1}{2}\Omega\vartheta_-\pi_+ - \frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} \tag{14}$$

$\partial_+ p_+ =$ no evolution eq. exists

$$\partial_+ p_- = \left(\frac{1}{n-2} - \frac{1}{2}\right)\Omega\vartheta_+ p_- - \frac{1}{2}\Omega\vartheta_- p_+ - \frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} \tag{15}$$

evolution equations (2)

Equations for x^- direction

$$\partial_- \Omega = -\frac{1}{n-2} \vartheta_- \Omega^2 \tag{16}$$

$$\partial_- \vartheta_+ = (9) \tag{17}$$

$$\partial_- \vartheta_- = -\vartheta_- \nu_- - \frac{1}{\Omega A} \kappa^2 T_{--} = -\vartheta_- \nu_- - \frac{1}{A} \Omega \kappa^2 (\pi_-^2 - p_-^2) \tag{18}$$

$$\partial_- f = \nu_- \tag{19}$$

$$\partial_- \nu_+ = (11) \tag{20}$$

$$\partial_- \nu_- = \text{no evolution eq. exists}$$

$$\partial_- \psi = \Omega \pi_- \tag{21}$$

$$\partial_- \phi = \Omega p_- \tag{22}$$

$$\partial_- \pi_+ = -\frac{1}{2} \Omega \vartheta_+ \pi_- + \left(\frac{1}{n-2} - \frac{1}{2} \right) \Omega \vartheta_- \pi_+ - \frac{1}{2e^f \Omega} \frac{dV_1}{d\psi} \tag{23}$$

$$\partial_- \pi_- = \text{no evolution eq. exists}$$

$$\partial_- p_+ = -\frac{1}{2} \Omega \vartheta_+ p_- + \left(\frac{1}{n-2} - \frac{1}{2} \right) \Omega \vartheta_- p_+ - \frac{1}{2e^f \Omega} \frac{dV_2}{d\phi} \tag{24}$$

$$\partial_- p_- = \text{no evolution eq. exists}$$

This constitutes the first-order dual-null form, suitable for numerical coding.

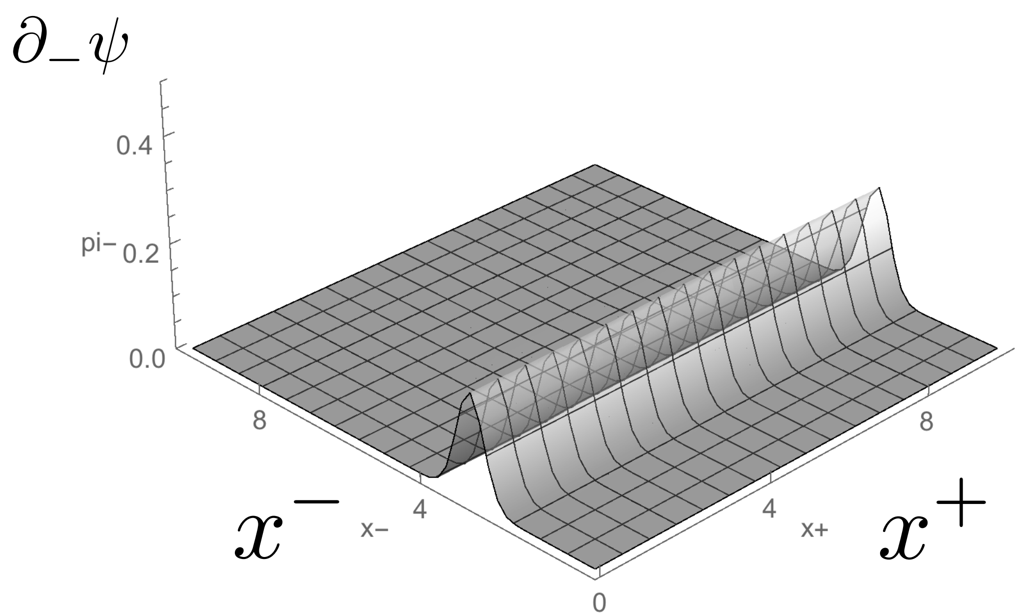
GR 5d: **small amplitude waves**

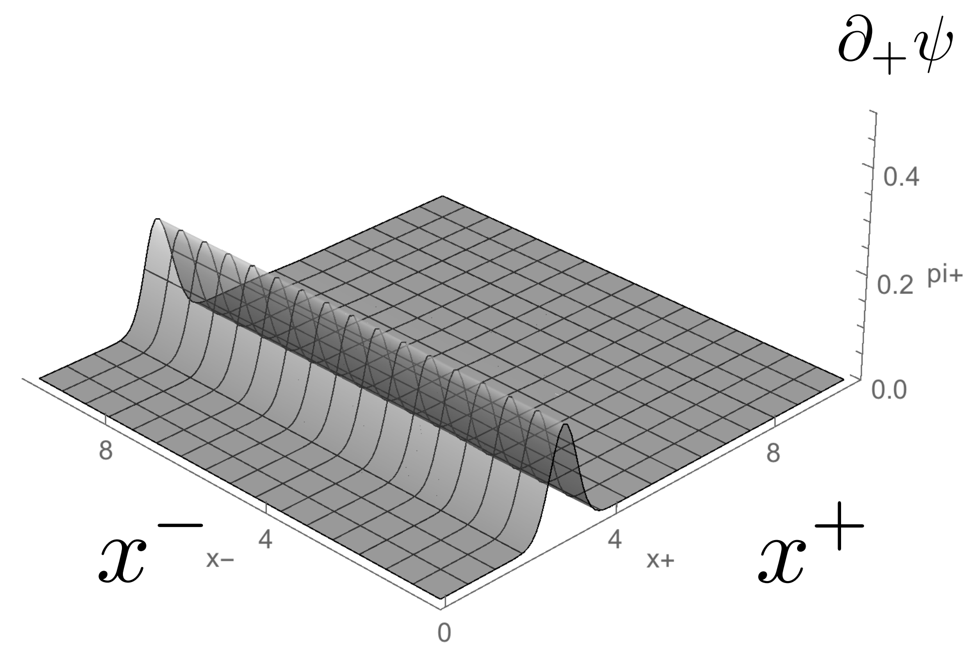
flat background, normal scalar field

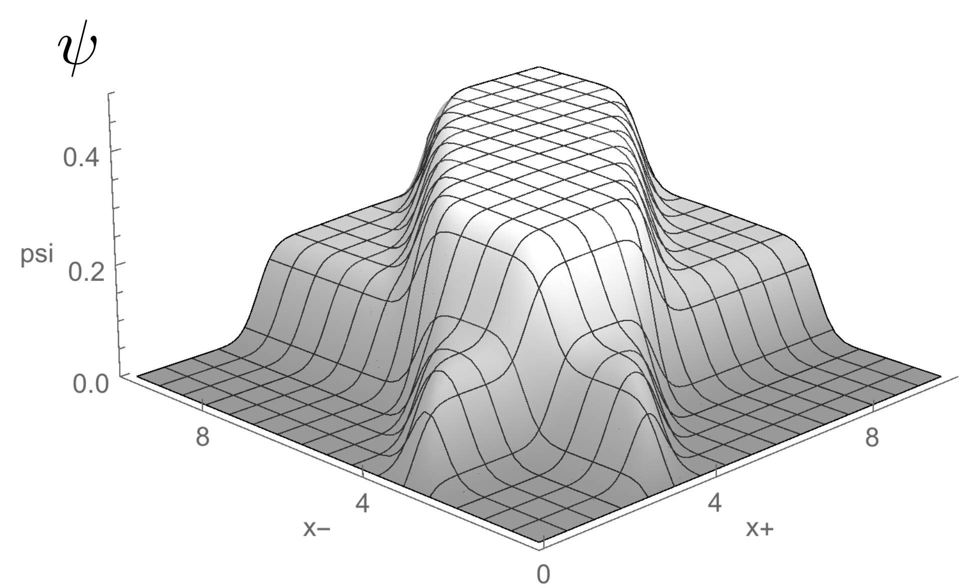
Initial data:

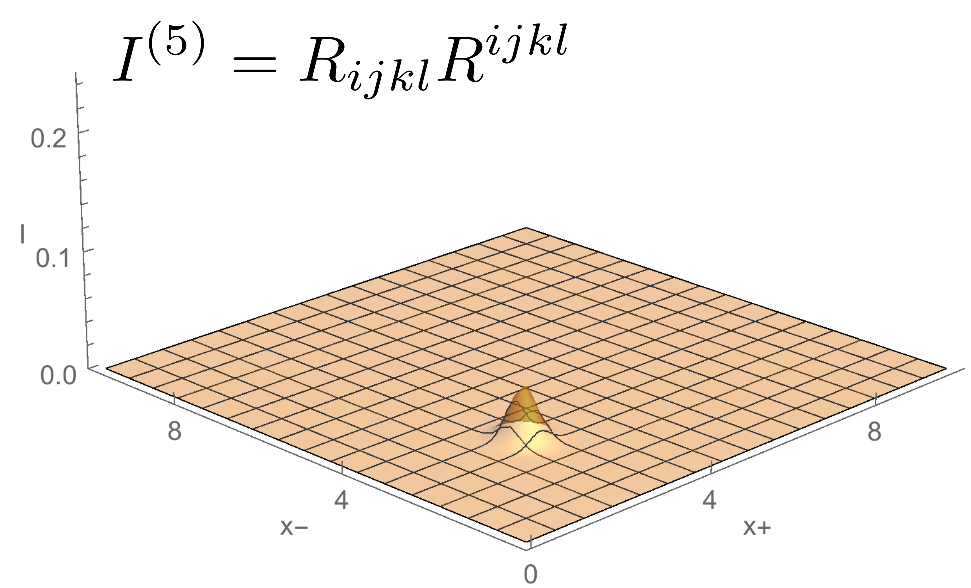
$\psi = 0,\ \pi_+ = a\exp(-b(z-c)^2)$ on $x_- = 0$ surface, where $z = x^+/\sqrt{2}$

$\psi = 0,\ \pi_- = a\exp(-b(z-c)^2)$ on $x_+ = 0$ surface, where $z = x^-/\sqrt{2}$

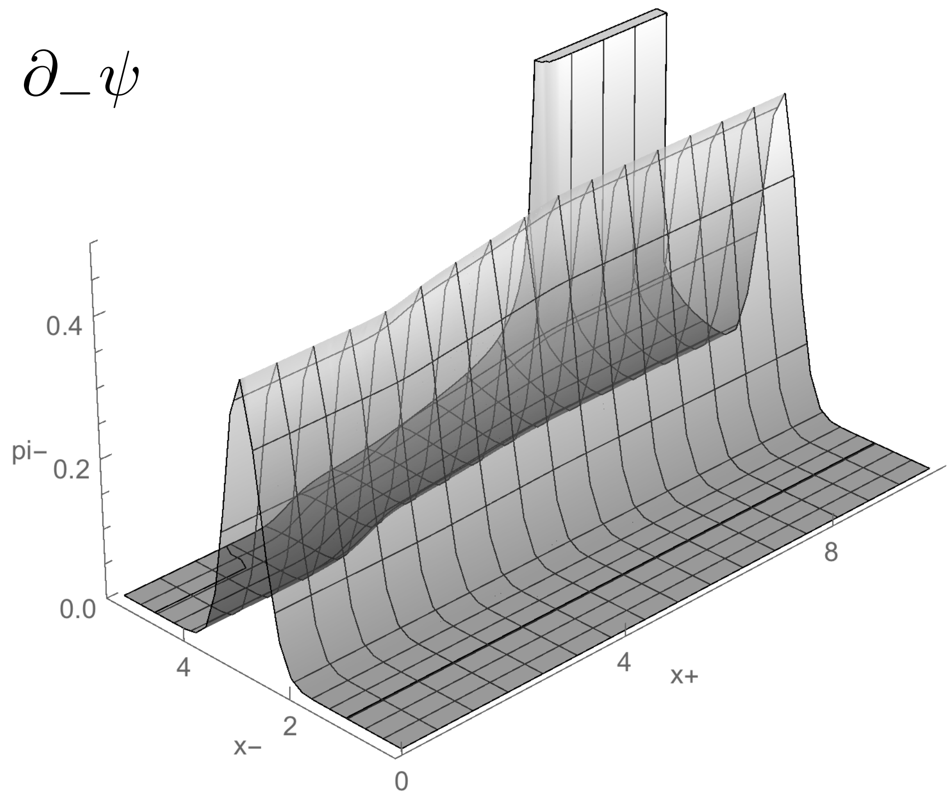


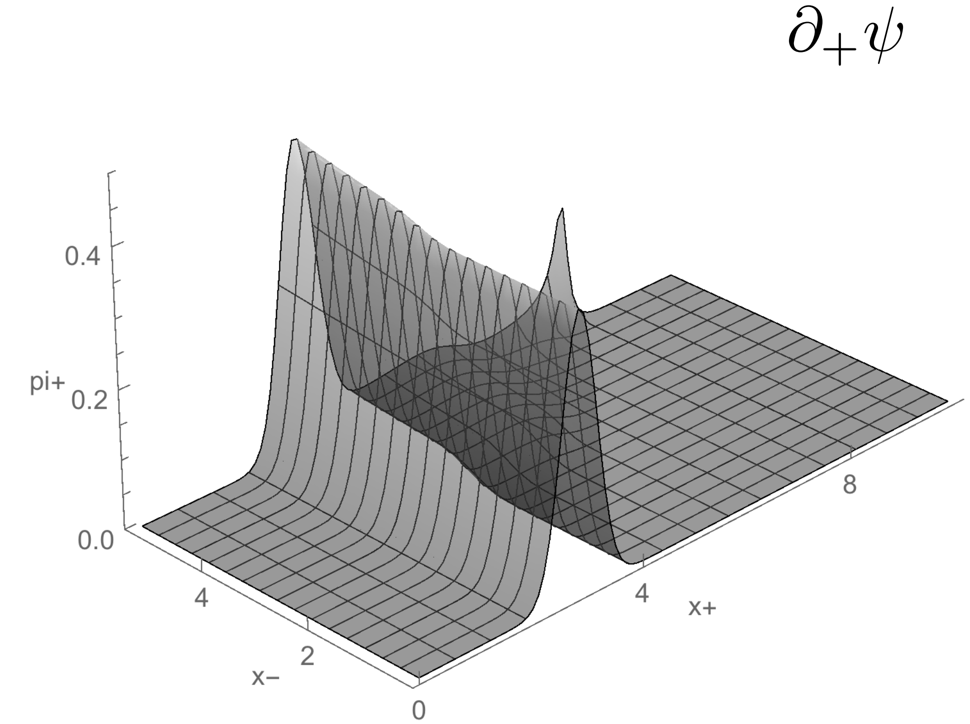


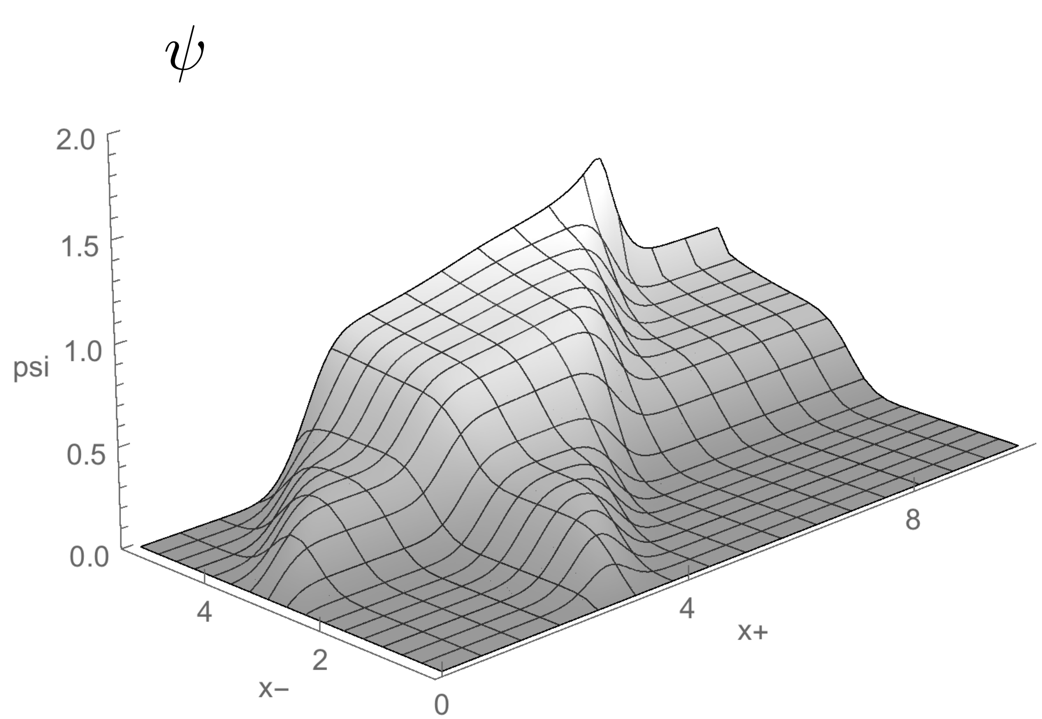


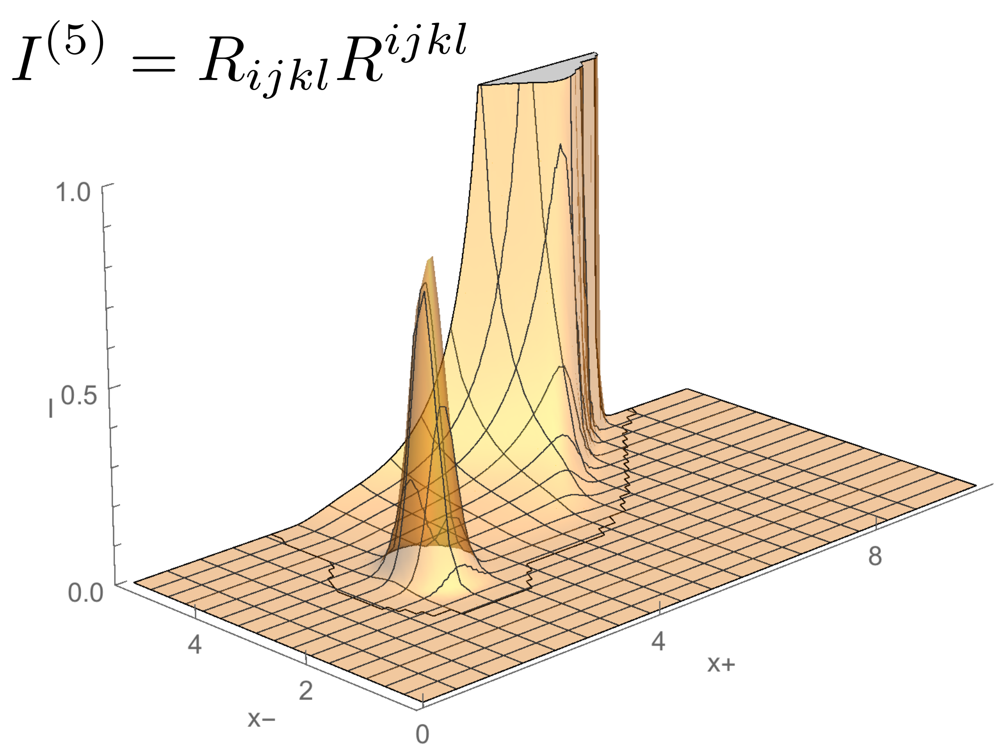


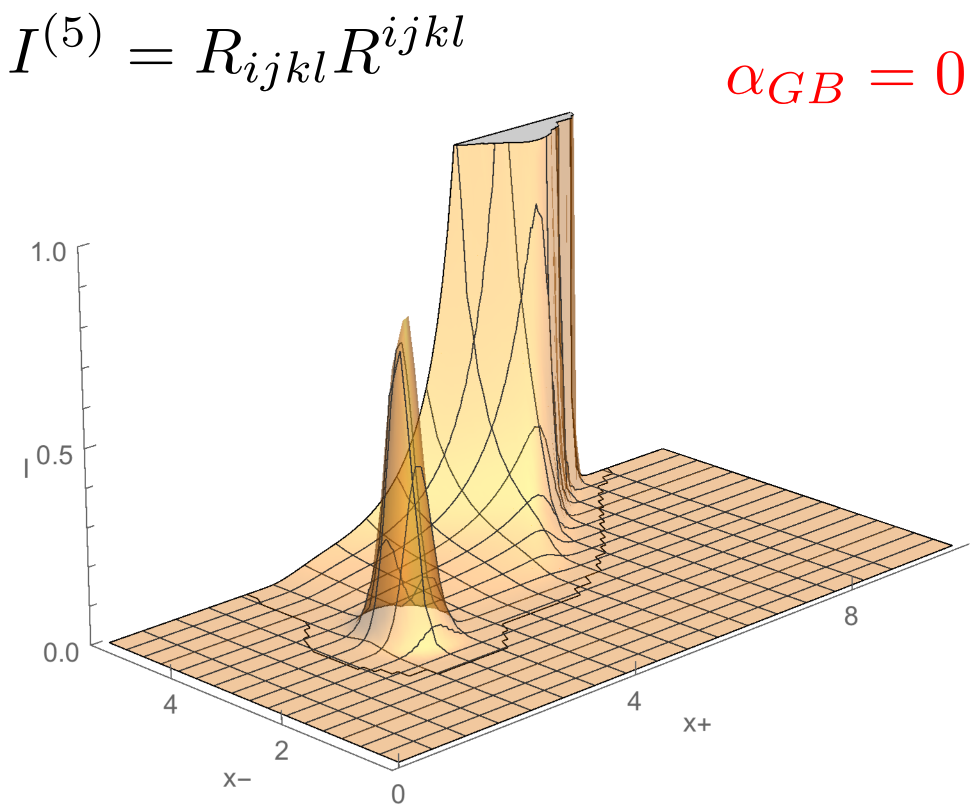
GR 5d: large amplitude waves



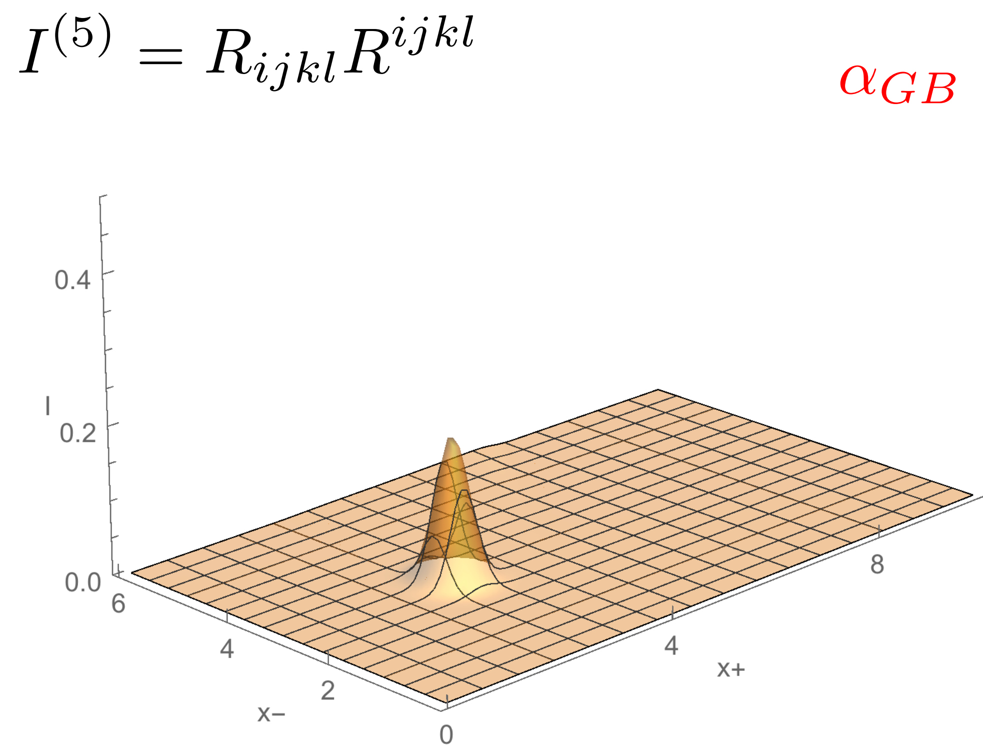




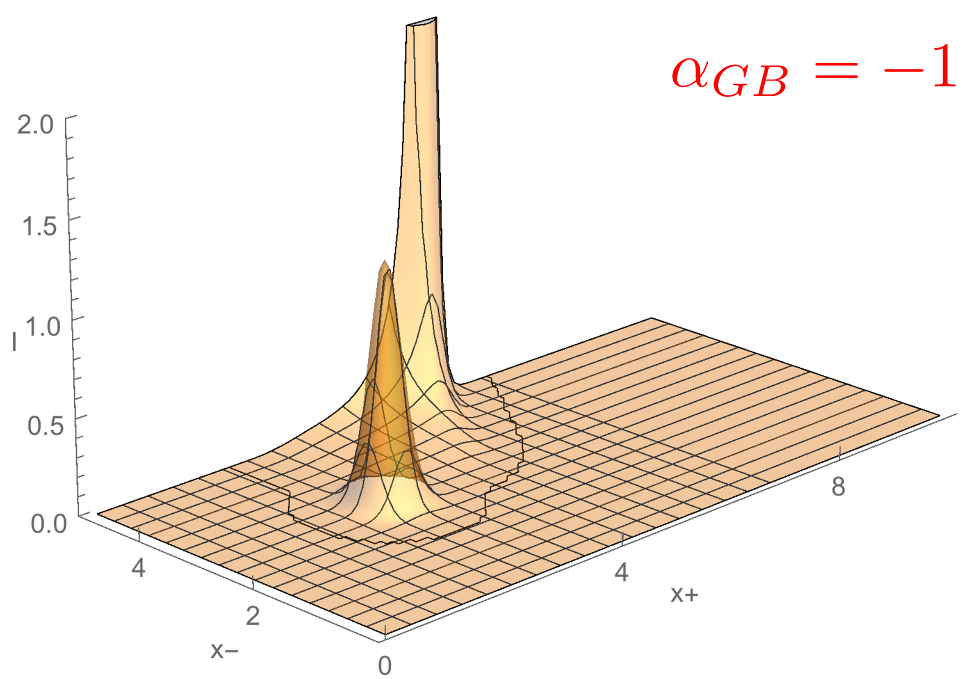




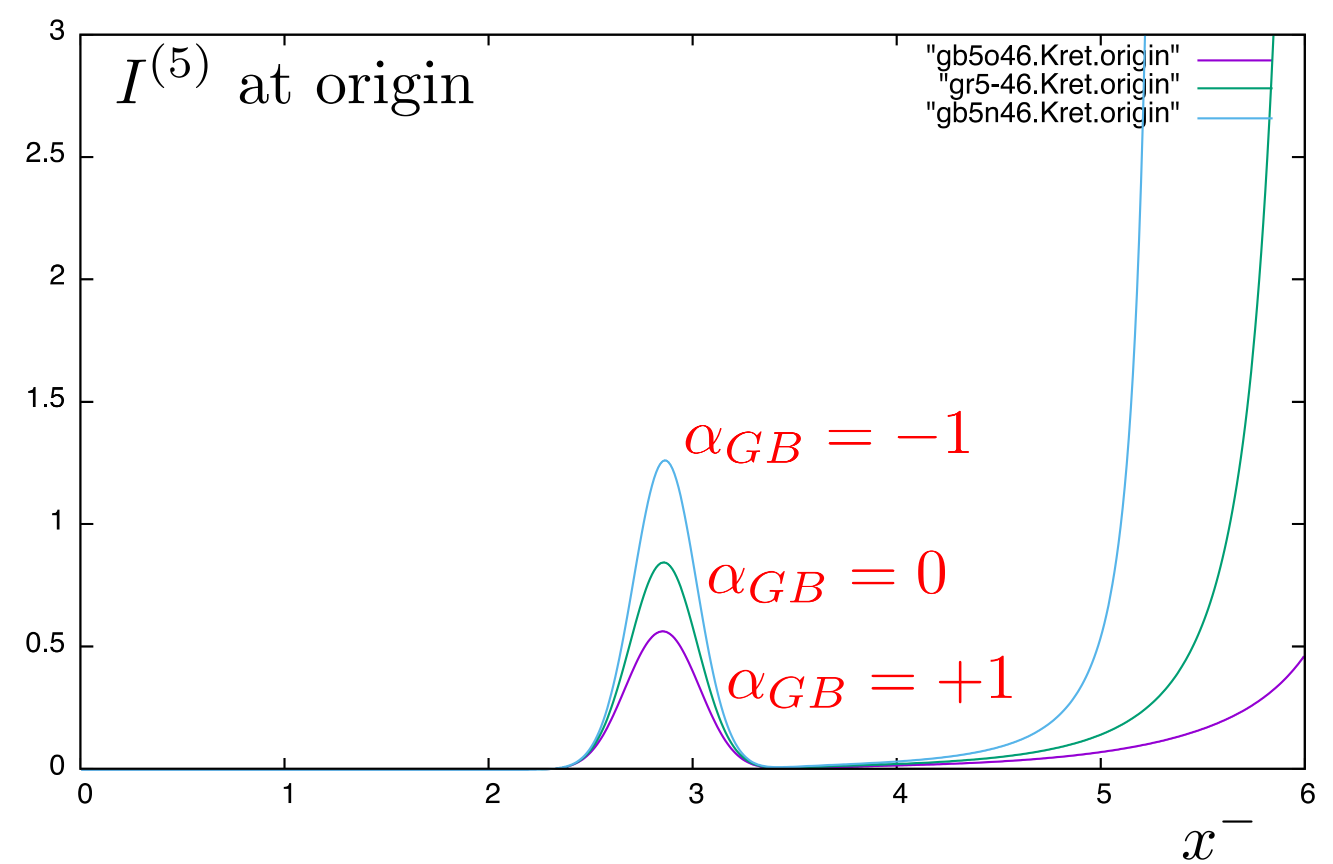
GR 5d



GaussBonnet 5d



GaussBonnet 5d (negative α)

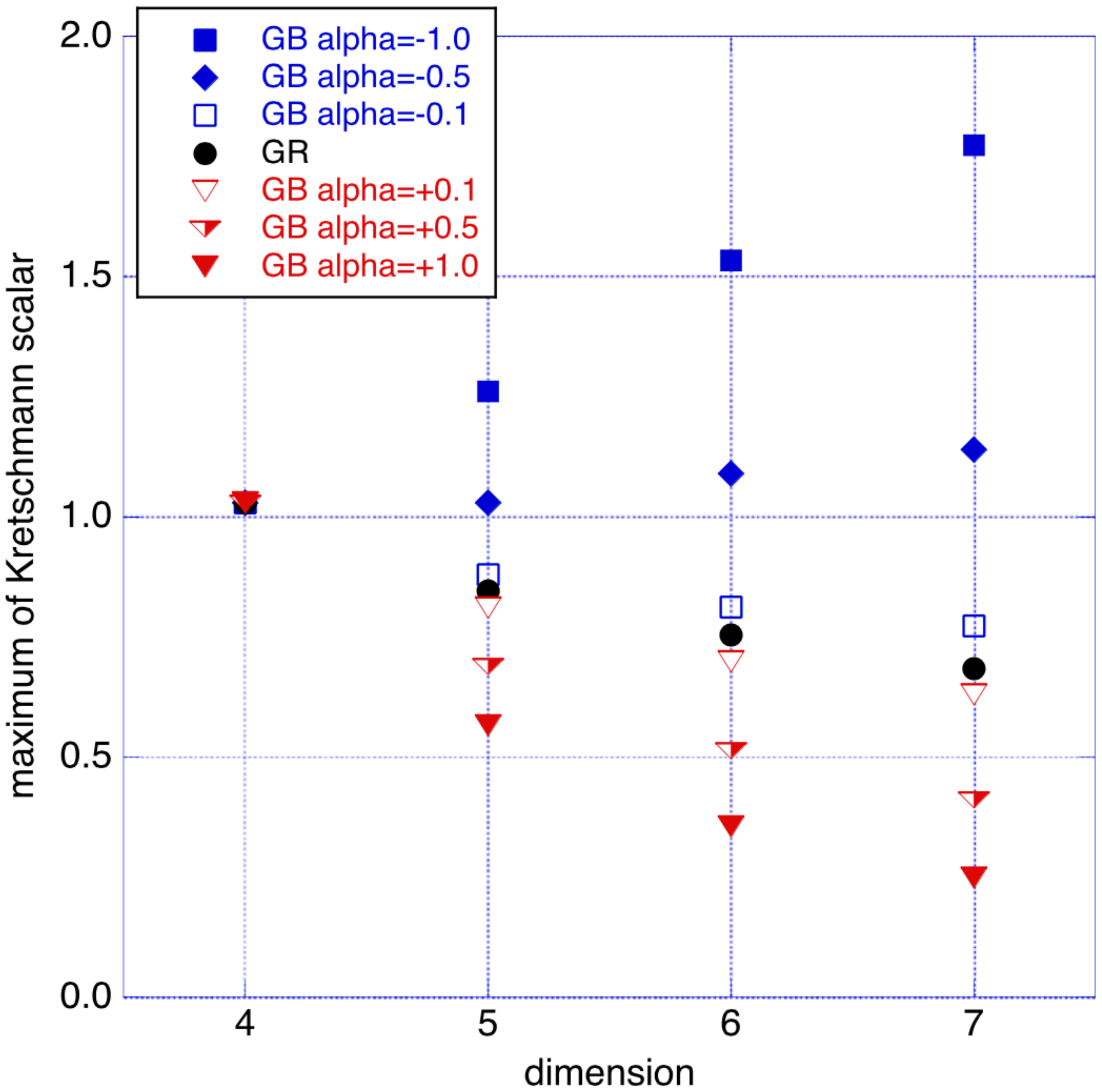


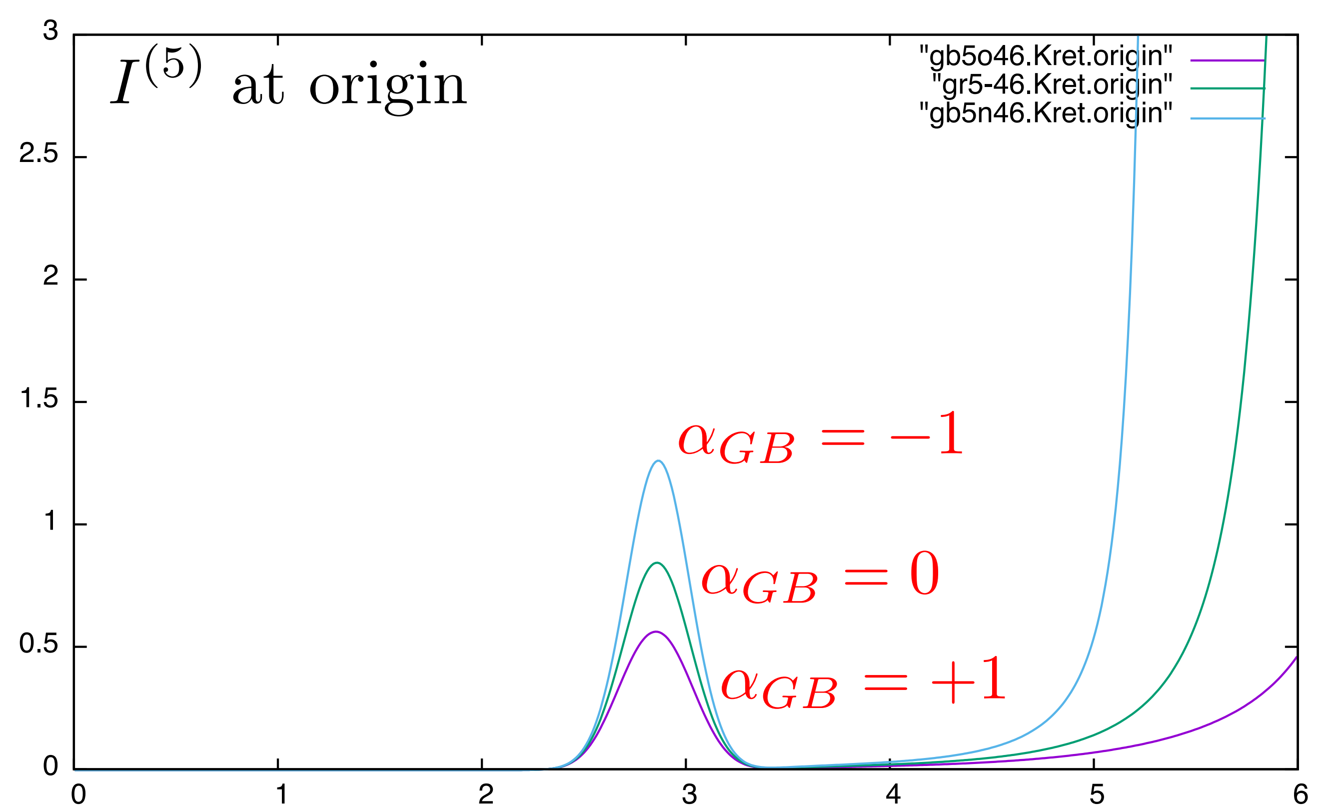
Colliding Scalar Waves

＊4dim, 5dim, 6dim,⋯ higher dim
＊Gauss-Bonnet coupling (α>0)
—> less growth of curvature

$\max\ (R_{ijkl}R^{ijkl})$

maximum of Kretschmann invariant



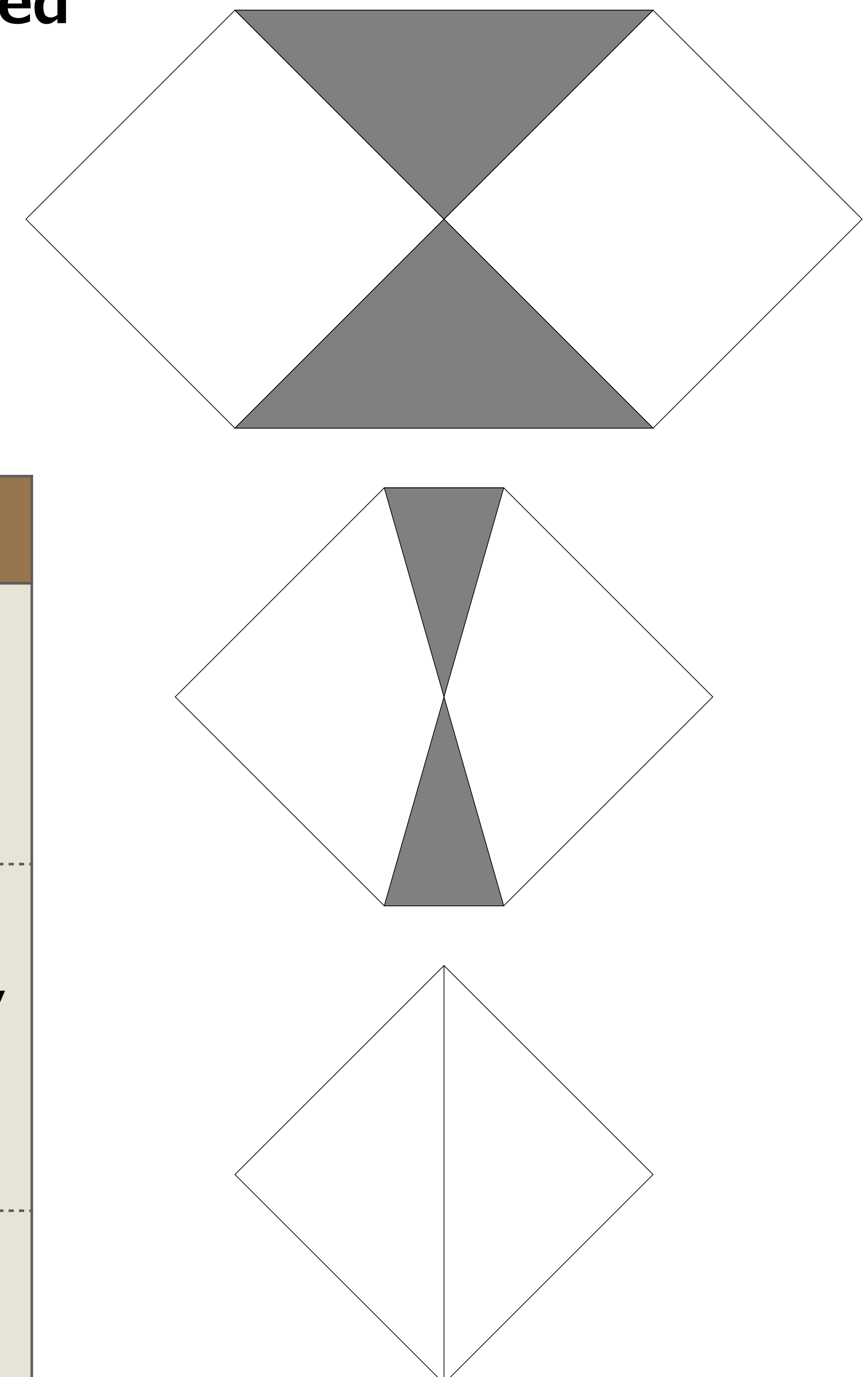


BH & WH are interconvertible?

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

They are very similar -- both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)

Only the causal nature of the THs differs, whether THs evolve in plus / minus density which is given locally.



	Black Hole	Wormhole
Locally defined by	Achronal (spatial/null) outer TH ⇨ 1-way traversable	Temporal (timelike) outer THs ⇨ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appear-ance	occur naturally	Unlikely to occur naturally. but constructible??

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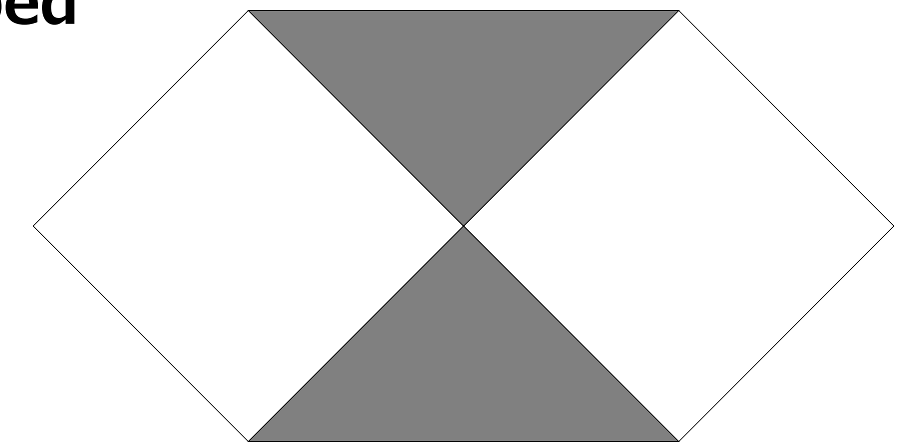

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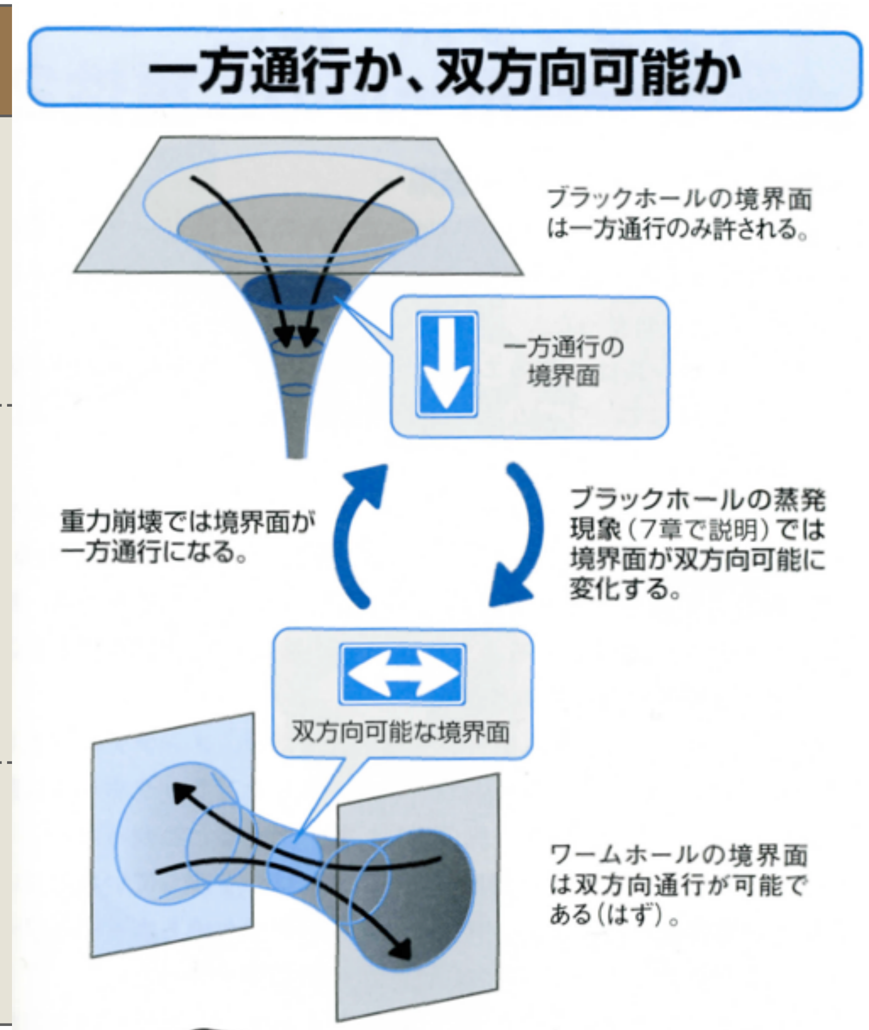
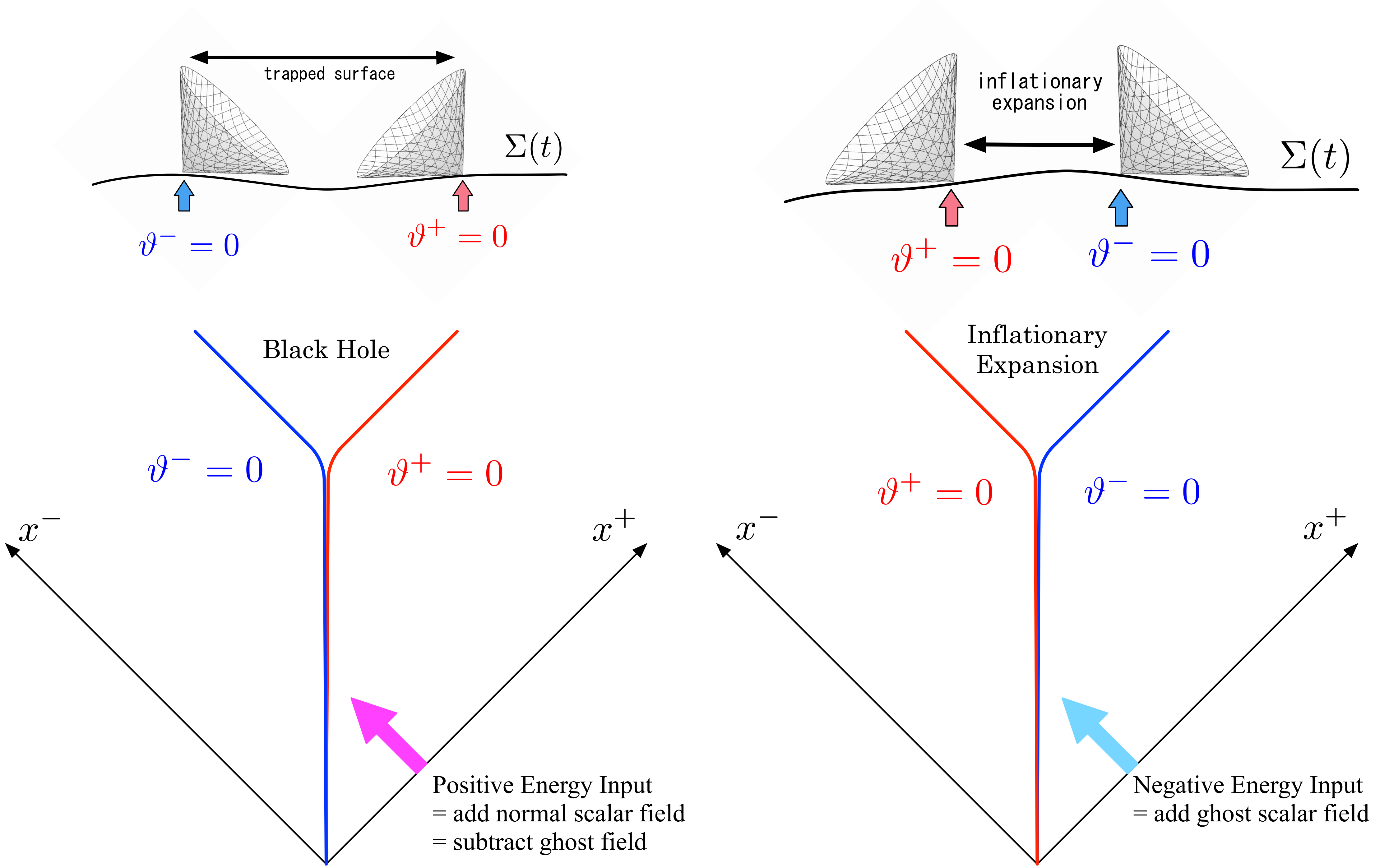


4d GR $\alpha_{\rm GB} = 0$ HS-Hayward PRD 66(2002) 044005

Wormhole evolution (known fact)



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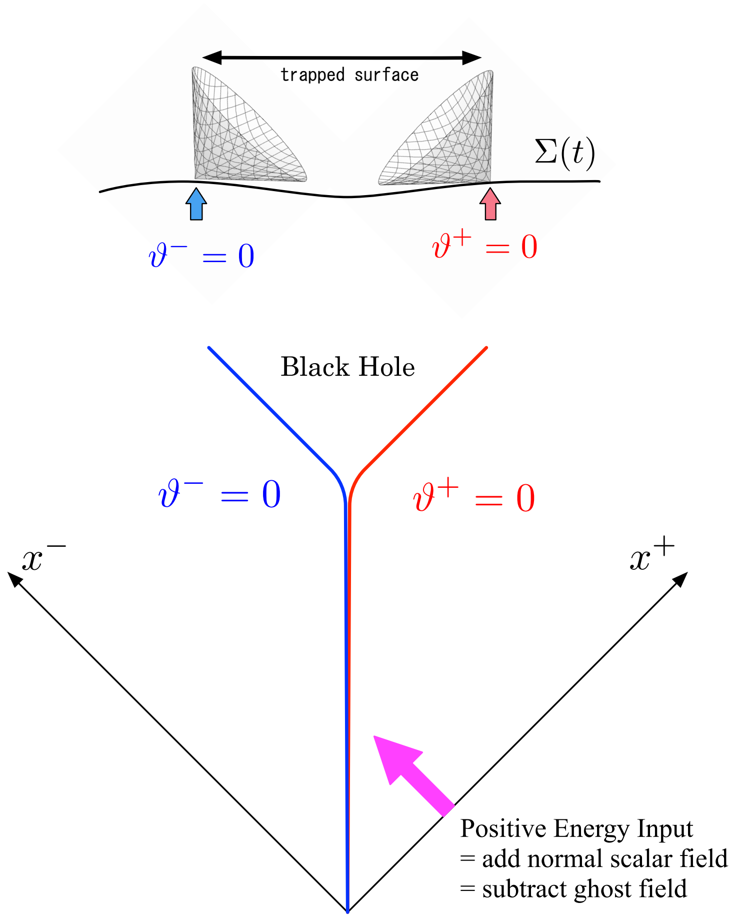

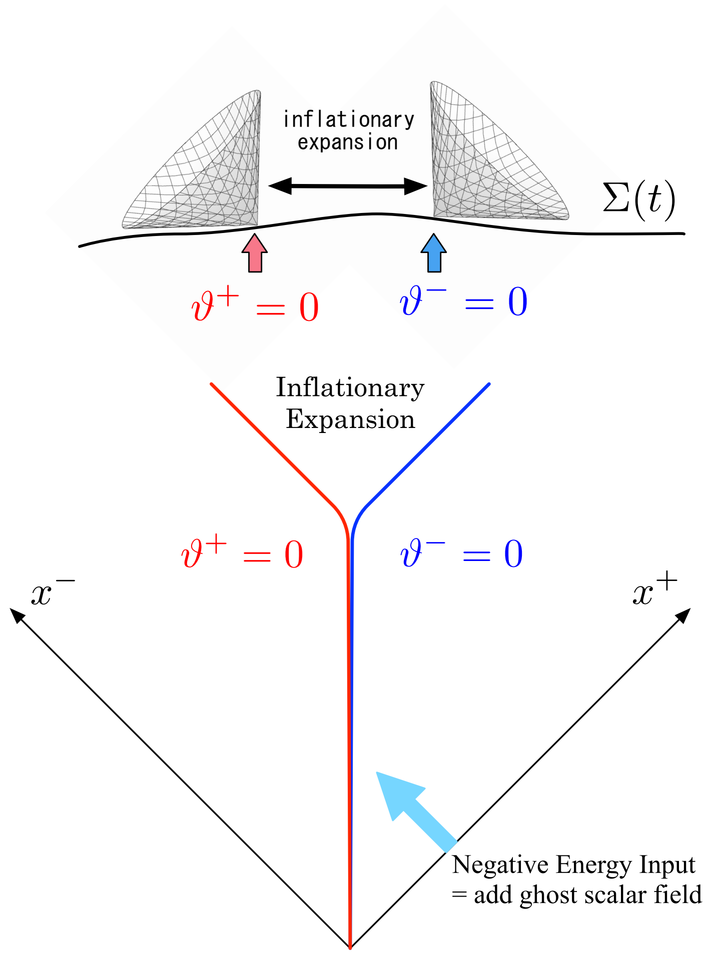

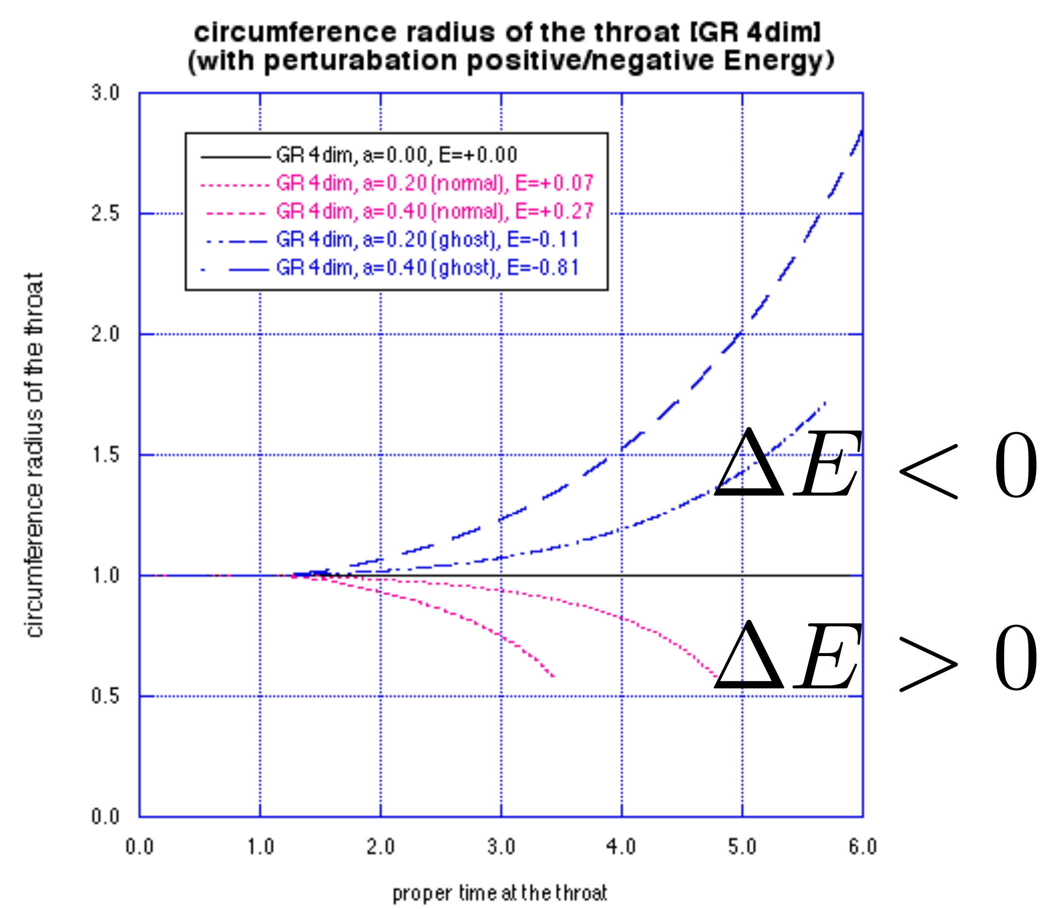

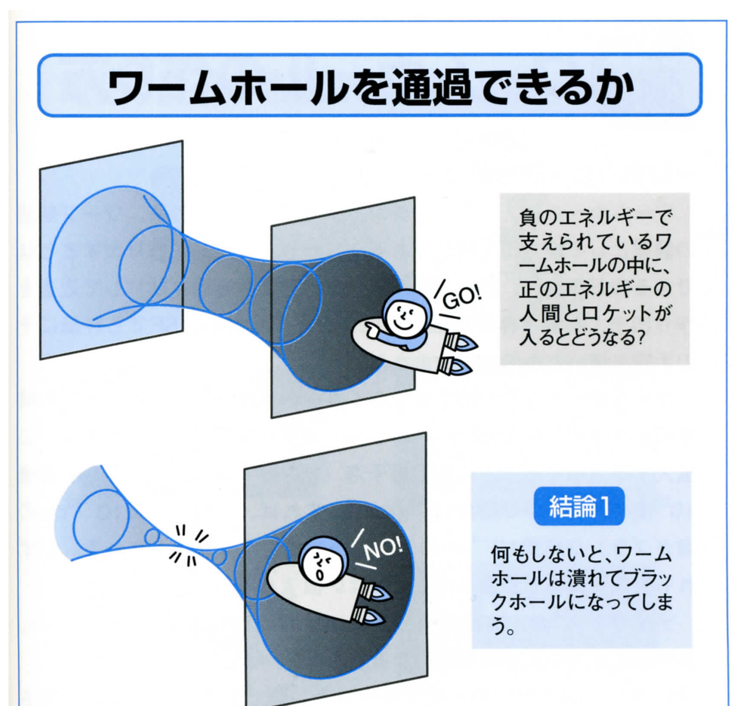

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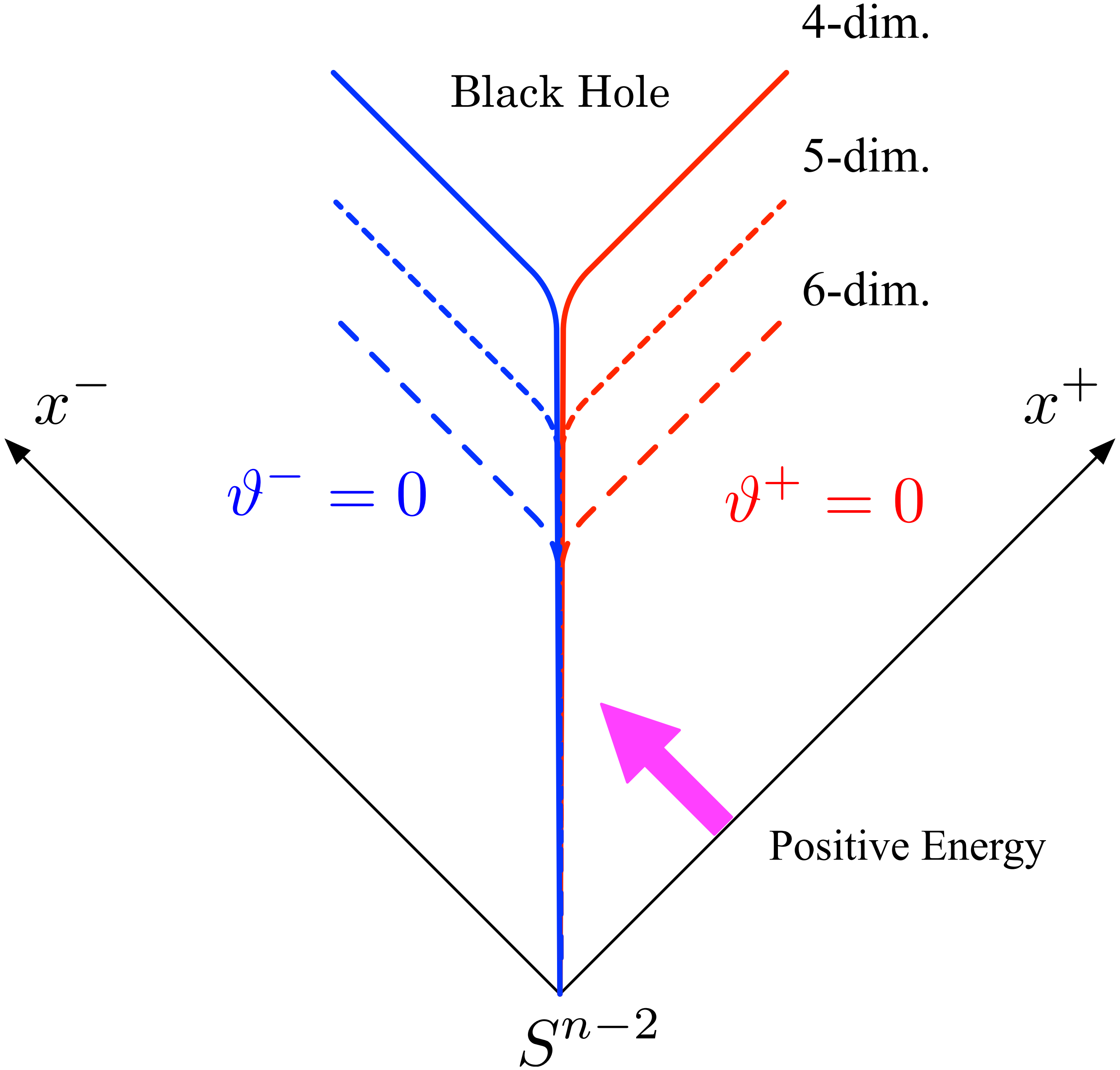
Wormhole evolution in n-dim (known fact)



PHYSICAL REVIEW D **88,** 064027 (2013)

TABLE I. The negative eigenvalues ω^2.

n	ω^2
4	−1.39705243371511
5	−2.98495893027790
6	−4.68662054299460
7	−6.46258414126318
8	−8.28975936306259
9	−10.1535530451867
10	−12.0442650147438
11	−13.9552091676647
20	−31.5751101285105
50	−91.3457759137153
100	−191.283017729717

$$f(t, r) = f_0(r) + \varepsilon f_1(r) e^{i\omega t}, \tag{3.1}$$

$$\delta(t, r) = \delta_0(r) + \varepsilon \delta_1(r) e^{i\omega t}, \tag{3.2}$$

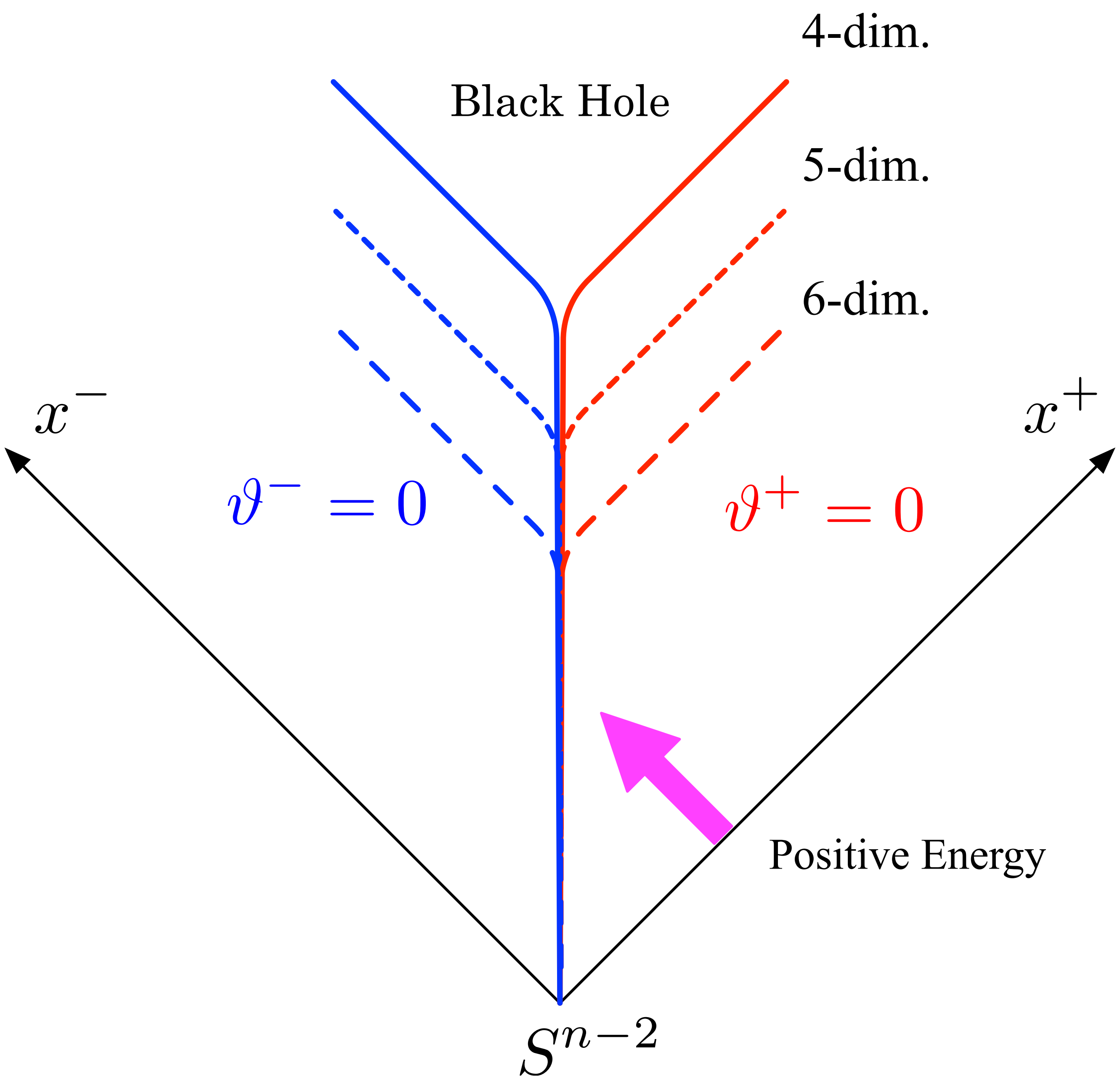
$$R(t, r) = R_0(r) + \varepsilon R_1(r) e^{i\omega t}, \tag{3.3}$$

$$\phi(t, r) = \phi_0(r) + \varepsilon \phi_1(r) e^{i\omega t}. \tag{3.4}$$

In higher dim, large instability.
(linear perturbation analysis)

n-dim GR $\alpha_{\rm GB} = 0$ Torii-HS PRD 88 (2013) 064027

Wormhole evolution in n-dim (known fact)



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In higher dim, large instability.
(linear perturbation analysis)

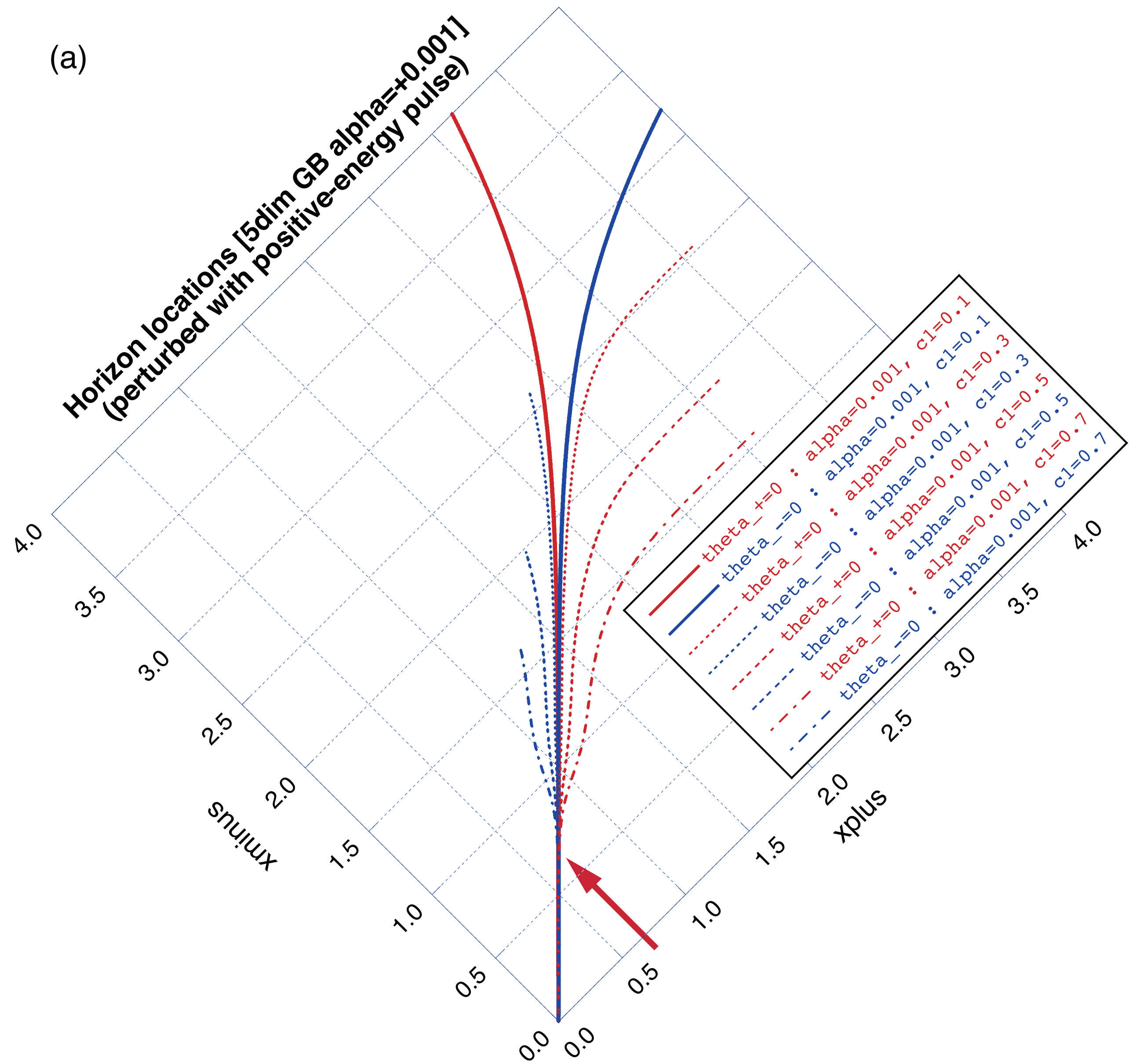
Confirmed Numerically

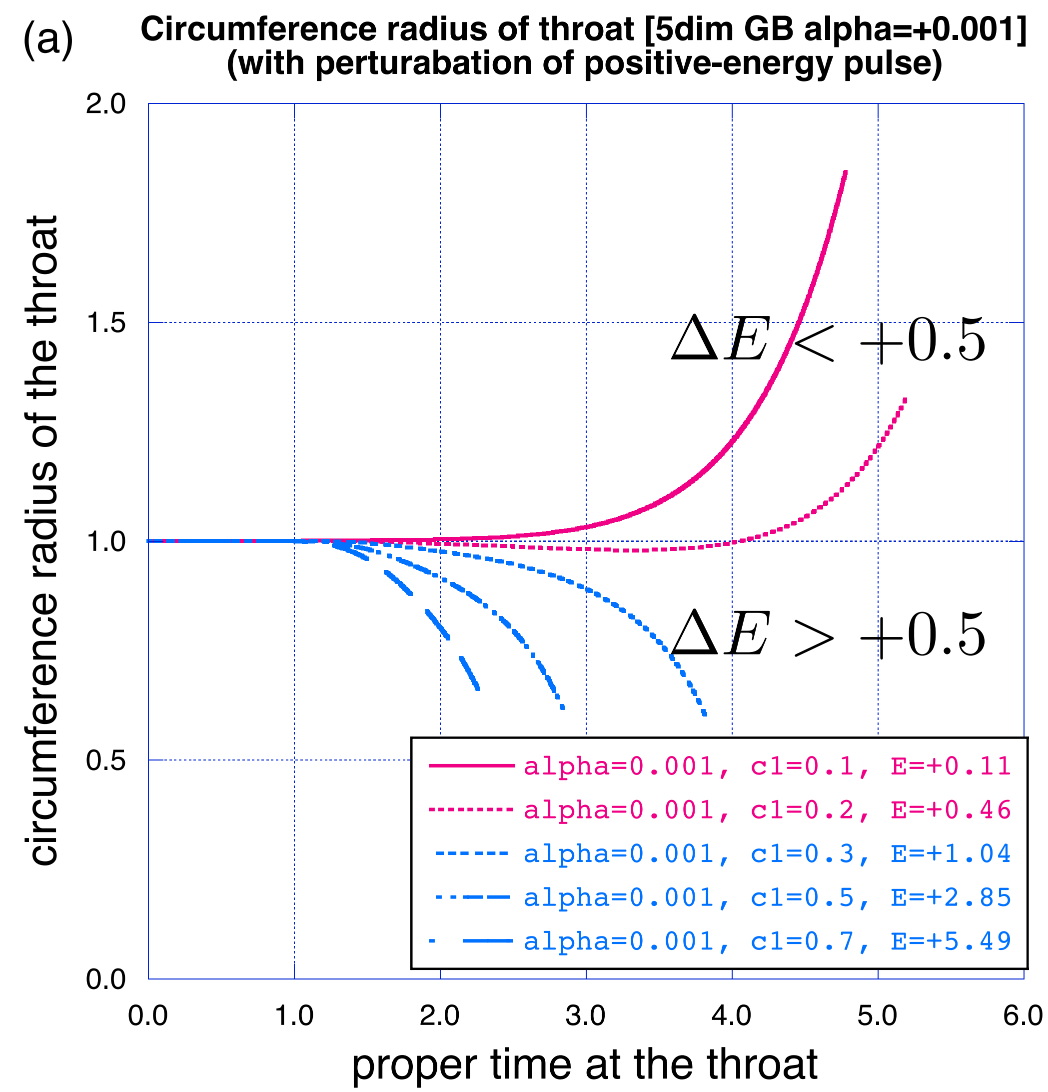
5d Gauss-Bonnet WH : positive energy injection

$$\alpha_{\rm GB} = +0.001$$

$$m = \frac{(n-2)V^k_{n-2}}{2\kappa_n^2} r^{n-3} \left[-\tilde{\Lambda} r^2 + \left(k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right) + \tilde{\alpha} r^{-2} \left(k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right)^2 \right]$$

MSmass, H.Maeda-Nozawa, PRD77 (2008) 063031





$\Delta E > \Delta E_5 > 0$ --> BH collapse

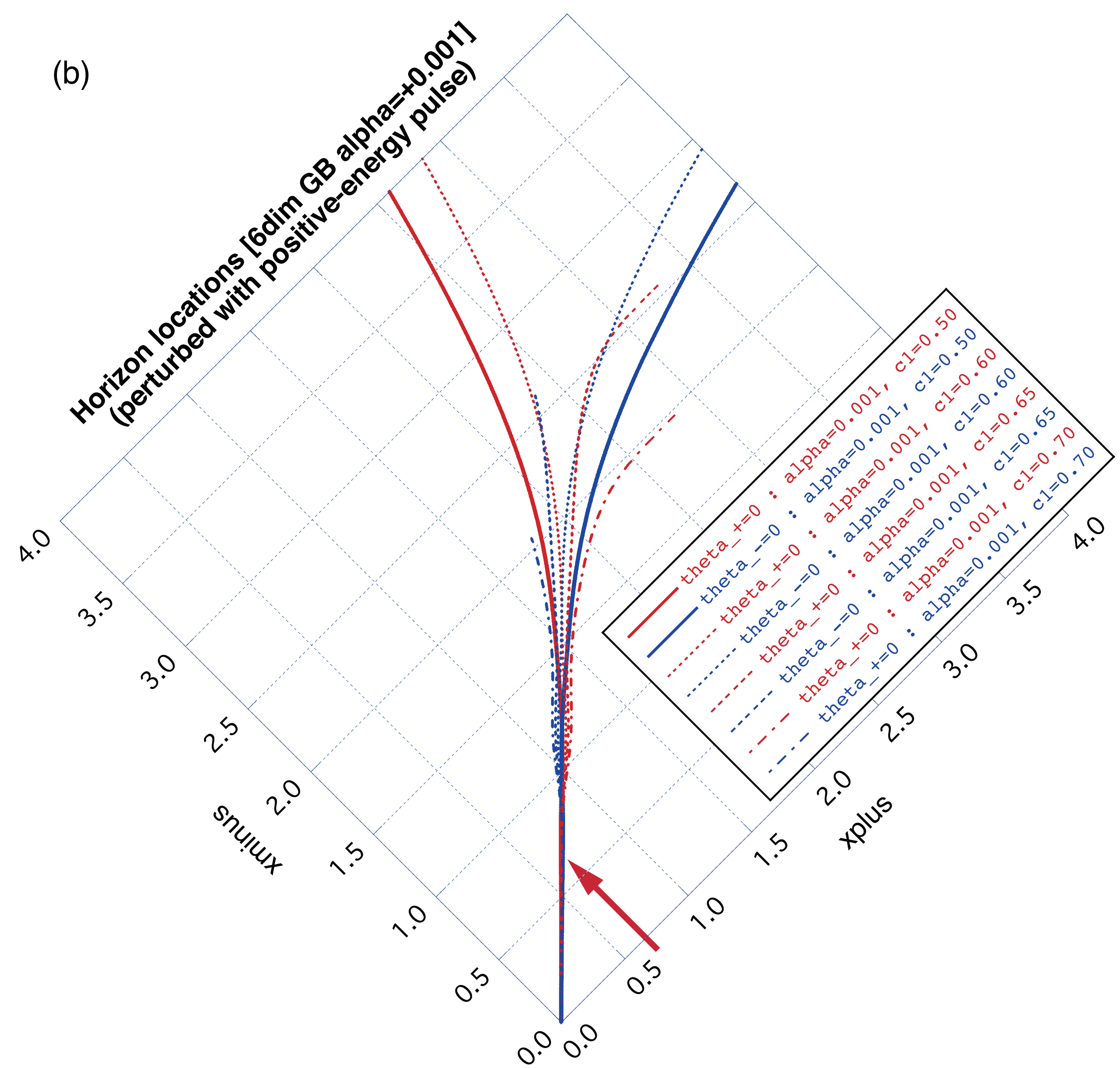
$\Delta E < \Delta E_5$ --> Inflationary expansion

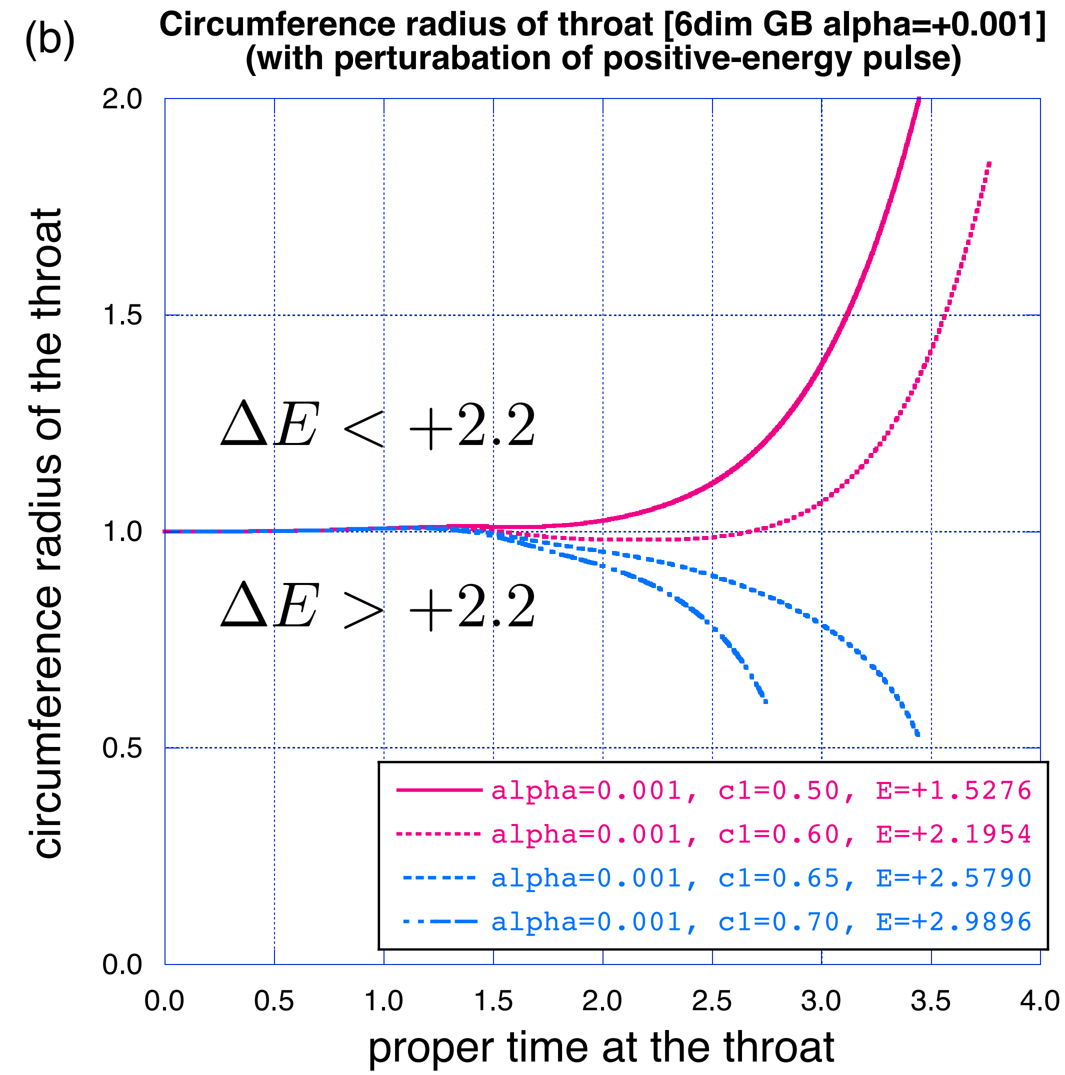
6d Gauss-Bonnet WH : positive energy injection

$$\alpha_{\rm GB} = +0.001$$

$$m = \frac{(n-2)V^k_{n-2}}{2\kappa_n^2} r^{n-3} \left[-\tilde{\Lambda} r^2 + \left(k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right) + \tilde{\alpha} r^{-2} \left(k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right)^2 \right]$$

MSmass, H.Maeda-Nozawa, PRD77 (2008) 063031





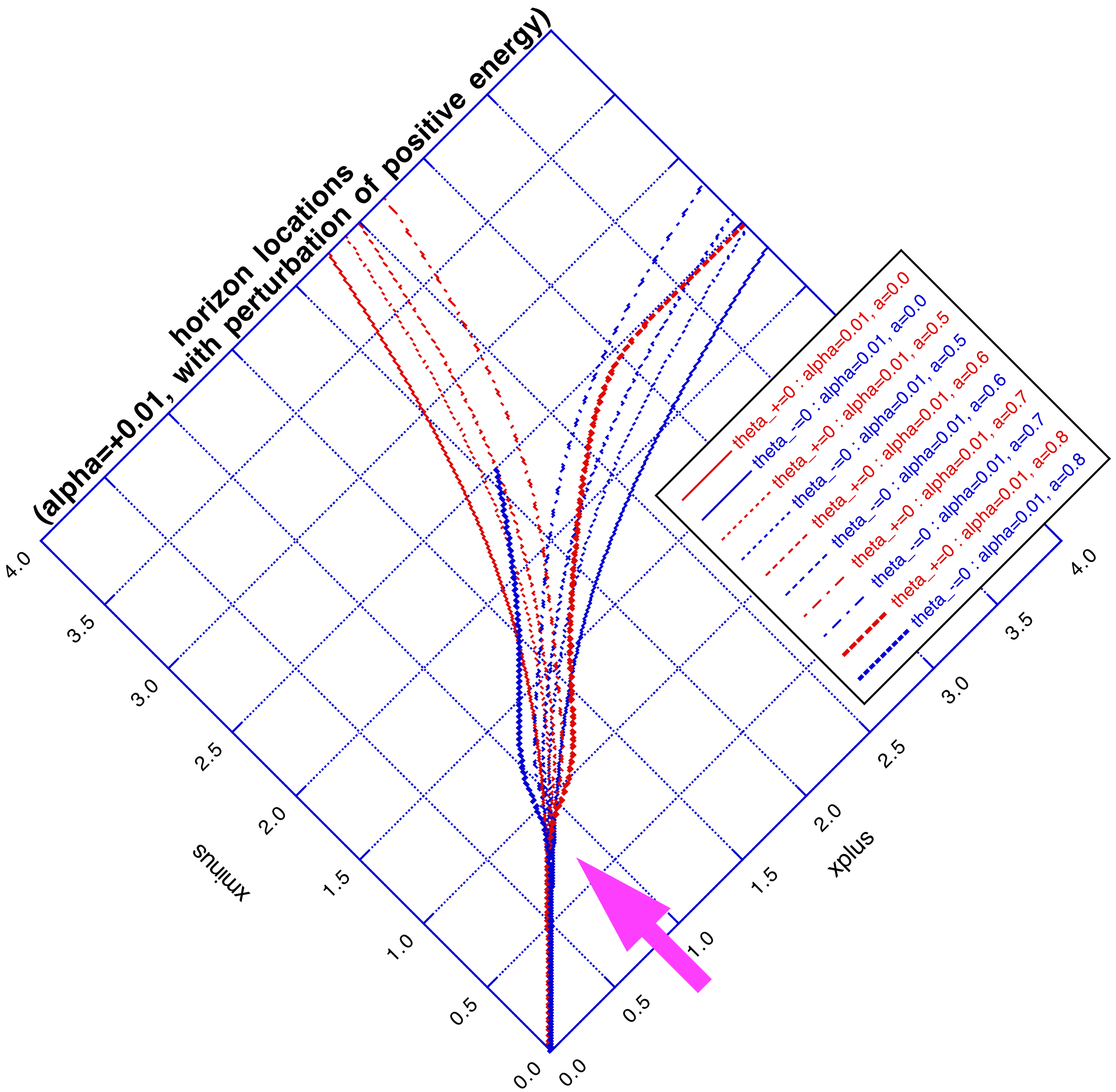
$\Delta E > \Delta E_6 > \Delta E_5 > 0$ --> BH collapse

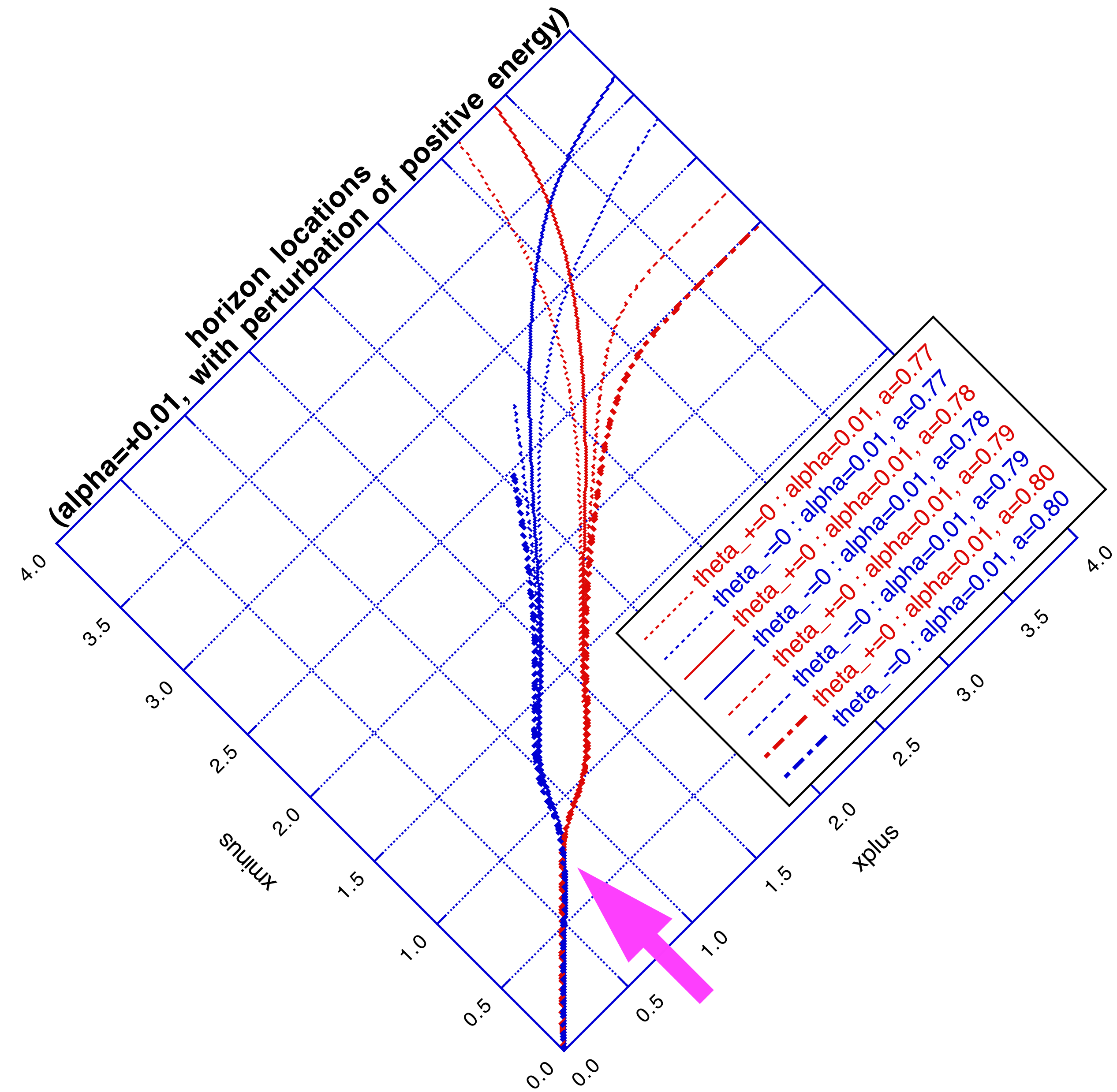
$\Delta E < \Delta E_5 < \Delta E_6$ --> Inflationary expansion

5d Gauss-Bonnet WH : trapped surface

$\alpha_{\rm GB} = 0.01$

critical behavior





existence of trapped surface

—> not necessary to form a BH

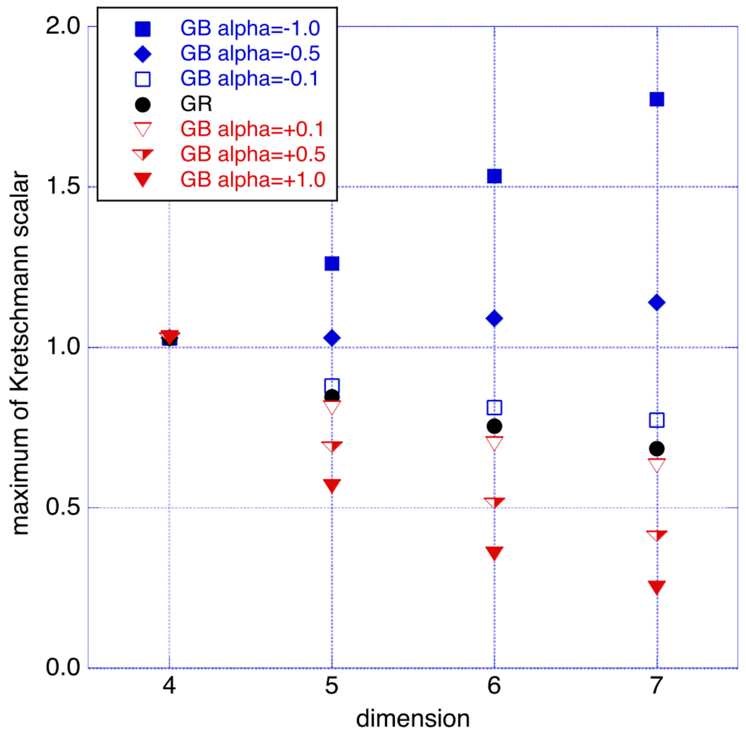
Summary

$$S = \int_{\mathcal{M}} d^n x \sqrt{-g}\Big[\frac{1}{2\kappa^2}(\alpha_{\rm GR}\mathcal{R} + \alpha_{\rm GB}\mathcal{L}_{\rm GB}) + \mathcal{L}_{\rm matter}\Big]$$

$$\mathcal{L}_{\rm GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

Colliding Scalar Waves

$\max\,(R_{ijkl}R^{ijkl})$



Wormhole Evolution

$\Delta E > \Delta E_6 > \Delta E_5 > 0$ --> BH collapse

$\Delta E < \Delta E_5 < \Delta E_6$ --> Inflationary expansion

We found that in the critical situation for forming a BH, the existence of the trapped region in the Einstein-GB gravity does not directly indicate a formation of a BH.

For both models, the normal corrections ($\alpha_{GB} > 0$) work for avoiding the appearance of singularity, although it is inevitable.