

Nonlinear Dynamics in the Einstein-Gauss-Bonnet gravity

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1. Motivation

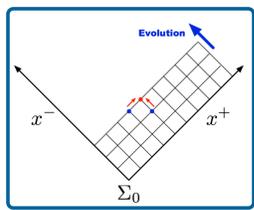
Dynamics in Gauss-Bonnet gravity?

- Action

$$S = \int_M d^{n+1}x \sqrt{-g} \left[\alpha_1 \mathcal{R} + \alpha_2 \mathcal{L}_{GB} + \mathcal{L}_{matter} \right]$$
 where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$
- Field equation

$$\alpha_1 G_{\mu\nu} + \alpha_2 H_{\mu\nu} + g_{\mu\nu} \Lambda = \kappa^2 T_{\mu\nu}$$
 where $H_{\mu\nu} = 2(R_{\mu\nu}R - 2R_{\mu\nu}R^{\rho\sigma} - 2R^{\rho\sigma}R_{\mu\nu\rho\sigma} + R_{\rho\sigma\mu\nu}R^{\rho\sigma}) - \frac{1}{2}g_{\mu\nu} \mathcal{L}_{GB}$
- has GR correction terms from String Theory
- has two solution branches (GR/non-GR).
- is expected to have singularity avoidance feature. (but has never been demonstrated.)
- new topic in numerical relativity.
 - S Golod & T Piran, PRD 85 (2012) 104015
 - N Deppe+, PRD 86 (2012) 104011
 - F Izaurieta & E Rodriguez, 1207.1496
- much attentions in WH community
 - H Maeda & M Nozawa, PRD 78 (2008) 024005
 - P Kanti, B Kleihaus & J Kunz, PRL 107 (2011) 271101
 - P Kanti, B Kleihaus & J Kunz, PRD 85 (2012) 044007

Field Eqs.



Field Equations (1)

Formulation for evolution [dual null]

Metric: n -dimensional, dual-null coordinate, $2 + (n-2)$ decomposition

$$ds^2 = -2e^{-2\psi} dt^2 + dx^2 + r^2(x^+, x^-) \gamma_{ij} dx^i dx^j \quad (1)$$

| Variables | Parameters |
|--|-----------------------|
| $\Omega = \frac{1}{r}$ | Conformal factor |
| $\partial_{x^\pm} = (n-2)\partial_{x^\pm}$ | expansion |
| $f = \partial_{x^\pm} \psi$ | lapse function |
| $\nu_{\pm} = \partial_{x^\pm} f$ | inaffinity (shift) |
| ψ | scalar field (normal) |
| $\pi_{\pm} = r\partial_{x^\pm} \psi$ | scalar momentum |
| ϕ | scalar field (ghost) |
| $p_{\pm} = r\partial_{x^\pm} \phi$ | scalar momentum |
| n | dimension |
| k | curvature |
| Λ | cosmological constant |

For simplicity, we define

$$\begin{aligned} \tilde{\alpha} &= (n-3)(n-4)\alpha_1, & (2) \\ \tilde{A} &= \alpha_1 + 2\alpha_2 \Omega^2 Z, & (3) \\ W &= \frac{2ef}{(n-2)^2} \partial_{x^\pm}, & (4) \\ Z &= k + W, & (5) \\ \eta &= \Omega^2 \frac{(n-2)(n-3)}{2} e^{-f} Z, & (6) \end{aligned}$$

Field Equations (2)

matter variables

normal field $\psi(u, v)$ and/or ghost field $\phi(u, v)$

$$T_{\mu\nu} = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi)$$

$$= \left[\partial_{x^\pm} \psi_{,\pm} - g_{\mu\nu} \left(\frac{1}{2} (\nabla\psi)^2 + V_1(\psi) \right) \right] + \left[-\partial_{x^\pm} \phi_{,\pm} - g_{\mu\nu} \left(-\frac{1}{2} (\nabla\phi)^2 + V_2(\phi) \right) \right]$$

this derives Klein-Gordon equations

$$\square\psi = \frac{dV_1}{d\psi}, \quad \square\phi = \frac{dV_2}{d\phi}$$

Klein-Gordon eqs.

$$\square\psi = -\frac{e^{-f}}{r} (2r\partial_{x^\pm} + (n-2)r_{,\pm}) \partial_{x^\pm} \psi + (n-2)r_{,\pm} \psi, & (7)$$

$$\square\phi = -2e^{-f} \partial_{x^\pm} \phi - e^{-f} \Omega^2 (\partial_{x^\pm} p_{\pm} + \partial_{x^\pm} p_{\pm}) & (8)$$

Energy-momentum tensor

$$\begin{aligned} T_{++} &= \Omega^2 (\pi_+^2 - p_+^2) \\ T_{--} &= \Omega^2 (\pi_-^2 - p_-^2) \\ T_{+-} &= -e^{-f} (V_1(\psi) + V_2(\phi)) \\ T_{\pm\pm} &= e^{-f} (\pi_{\pm} p_{\pm} - p_{\pm} p_{\pm}) - \frac{1}{4r^2} (V_1(\psi) - V_2(\phi)) \end{aligned}$$

Field Equations (3)

evolution equations (1)

Equations for x^+ direction

$$\begin{aligned} \partial_t \Omega &= -\frac{1}{n-2} \partial_{x^+} \Omega^2 & (16) \\ \partial_t \psi &= \nu_+ & (17) \\ \partial_t \pi_+ &= -\partial_{x^+} \pi_+ - \frac{1}{r} \partial_{x^+} T_{++} = -\partial_{x^+} \pi_+ - \frac{1}{r} \Omega^2 (\pi_+^2 - p_+^2) & (18) \\ \partial_t p_+ &= \frac{1}{r} \Omega^2 \left[-\alpha_1 \Omega^2 \frac{(n-2)(n-3)}{2} Z + \Lambda + \alpha_2 (V_1 + V_2) \right] - \frac{1}{r} \partial_{x^+} p_+ \frac{(n-2)(n-3)}{2} (\pi_+^2 + W) & (19) \\ \partial_t f &= \nu_+ & (20) \\ \partial_t \nu_+ &= \text{no evolution eq. exists} & (21) \\ \partial_t \nu_- &= \frac{1}{4} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left[\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right. & (22) \\ & \quad \left. + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) \right. & (23) \\ & \quad \left. + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) \right] & (24) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (25) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (26) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (27) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (28) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (29) \\ & \quad + \frac{1}{r} \partial_{x^+} \nu_- \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^+} (2(n-3)) \left(\frac{1}{r} \partial_{x^+} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (30) \end{aligned}$$

Field Equations (4)

evolution equations (2)

Equations for x^- direction

$$\begin{aligned} \partial_t \Omega &= -\frac{1}{n-2} \partial_{x^-} \Omega^2 & (16) \\ \partial_t \psi &= \nu_- & (17) \\ \partial_t \pi_- &= -\partial_{x^-} \pi_- - \frac{1}{r} \partial_{x^-} T_{--} = -\partial_{x^-} \pi_- - \frac{1}{r} \Omega^2 (\pi_-^2 - p_-^2) & (18) \\ \partial_t p_- &= \frac{1}{r} \Omega^2 \left[-\alpha_1 \Omega^2 \frac{(n-2)(n-3)}{2} Z + \Lambda + \alpha_2 (V_1 + V_2) \right] - \frac{1}{r} \partial_{x^-} p_- \frac{(n-2)(n-3)}{2} (\pi_-^2 + W) & (19) \\ \partial_t f &= \nu_- & (20) \\ \partial_t \nu_- &= \text{no evolution eq. exists} & (21) \\ \partial_t \nu_+ &= \frac{1}{4} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left[\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right. & (22) \\ & \quad \left. + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) \right. & (23) \\ & \quad \left. + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) \right] & (24) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (25) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (26) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (27) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (28) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (29) \\ & \quad + \frac{1}{r} \partial_{x^-} \nu_+ \frac{(n-3)}{2} \left(\frac{1}{r} \partial_{x^-} (2(n-3)) \left(\frac{1}{r} \partial_{x^-} (2(n-3)) - 1 \right) (\Lambda + \alpha_2 (V_1 + V_2)) \right) & (30) \end{aligned}$$

This constitutes the first-order dual-null form, suitable for numerical coding.

Outline & Summary

We numerically investigated how the dynamics depends on the dimensionality and how the higher-order curvature terms affect to singularity formation in two models: (i) perturbed wormhole in spherically symmetric space-time, and (ii) colliding scalar pulses in planar space-time. Our numerical code uses dual-null formulation, and we compare the dynamics in 5, 6 and 7-dimensional General Relativity and Gauss-Bonnet (GB) gravity.

Both results suggest that **GB correction works for avoiding singularity formation in their dynamics**. We also found that **the existence of the trapped surface in GB gravity does not directly indicates formation of BH**.

Wormhole Evolutions

For wormhole dynamics, we observe that the perturbed throat will be easily enhance in the presence of GB term. If we inject large positive energy, then the throat turns to a blackhole, but that threshold of energy becomes larger for larger coupling constant, alpha, and for larger dimensions.

1. Motivation

Why Wormhole?

They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.

They are very similar to black holes—both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

Wormhole = Hypersurface foliated by marginally trapped surfaces

BH & WH are interchangeable?

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

They are very similar -- both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)

Only the causal nature of the THs differs, whether THs evolve in plus / minus density which is given locally.

| | Black Hole | Wormhole |
|--------------------|---|---|
| Locally defined by | Achronal (spatial/null) outer TH → 1-way traversable | Temporal (timelike) outer TH → 2-way traversable |
| Einstein eqs. | Positive energy density normal matter (or vacuum) | Negative energy density "exotic" matter |
| Appearance | occur naturally | Unlikely to occur naturally, but constructible?? |

initial data

- Static condition

$$\begin{aligned} \partial_t + \partial_r \Omega &= 0 \implies \partial_t + \partial_r = 0 \\ \partial_t + \partial_r \psi &= 0 \implies \pi_+ + \pi_- = 0 \\ \partial_t + \partial_r p_{\pm} &= 0 \implies p_+ + p_- = 0 \\ \partial_t + \partial_r \nu_{\pm} &= 0 \implies \nu_+ \nu_- + \frac{1}{4} \Omega^2 (\pi_+^2 - p_+^2) = \nu_+ \nu_- + \frac{1}{4} \Omega^2 (\pi_-^2 - p_-^2) \end{aligned}$$
- Solve x^+ and x^- equations with the starting condition at the throat

$$\begin{aligned} \psi_+ = \psi_- = 0 \\ \nu_+ = \nu_- = 0 \\ -\kappa^2 \Omega (\pi_+^2 - p_+^2) e^{-f} = -\frac{1}{\Omega} \left[-\alpha_1 \Omega^2 \frac{(n-2)(n-3)}{2} k + \Lambda + \kappa^2 (V_1 + V_2) \right] + \alpha_2 \Omega^2 \frac{(n-2)(n-5)}{2} k^2 \end{aligned}$$
- If we assume only ghost field ϕ , then

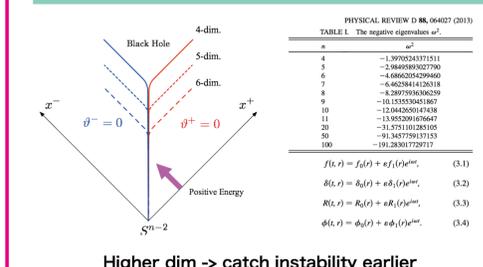
$$p_+ = -p_- = \frac{1}{\sqrt{\kappa^2 \Lambda}} \left[\alpha_1 \frac{(n-2)(n-3)}{2} k - \frac{1}{\Omega^2} (\Lambda + \kappa^2 V_2) + \alpha_2 \Omega^2 \frac{(n-2)(n-5)}{2} k^2 \right]$$
- add perturbation

$$p_{\pm}(x^{\pm} = x, x^{\mp} = 0) = p_{\pm}(\text{solution}) + \alpha \exp(-100(x - 0.5)^2)$$

in GR

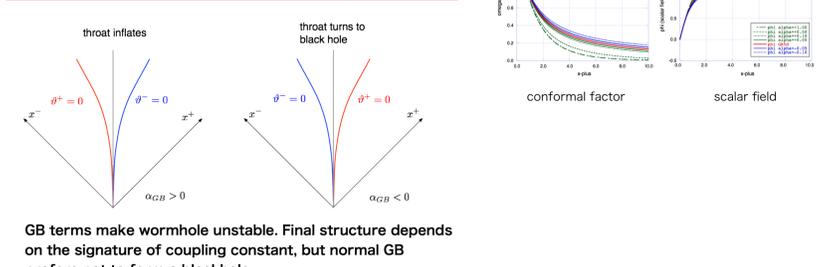
n-dim GR $\alpha_{GB} = 0$ Torii-HS PRD 88 (2013) 064027

Wormhole evolution in n-dim.

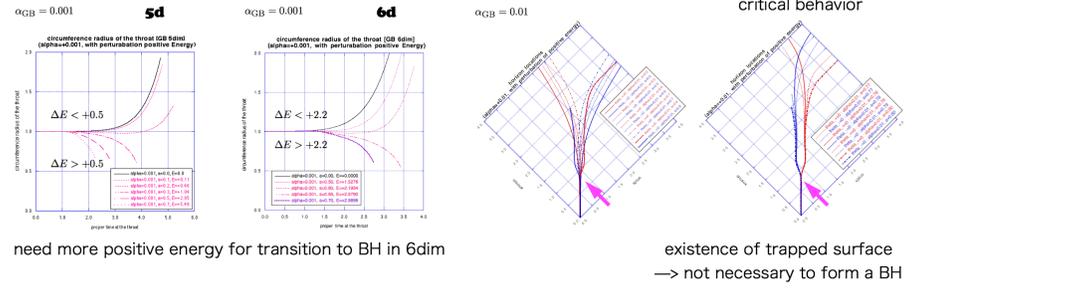


in GB

5d GR vs Gauss-Bonnet WH : instability appears



5d, 6d Gauss-Bonnet WH



Colliding Scalar Waves

For scalar wave collisions, we observe that curvature (Kretschmann invariant) evolves milder in the presence of GB term and/or in higher-dimensional space-time, while the singularity formation is inevitable.

Colliding Scalar Waves

GR 5d: small amplitude waves

flat background, normal scalar field

Initial data: $\psi = 0, \pi_{\pm} = \alpha \exp(-|z - c|^2)$ on $x_{\pm} = 0$ surface, where $z = x^i / \sqrt{2}$

$\psi = 0, \pi_{\pm} = \alpha \exp(-|z - c|^2)$ on $x_{\pm} = 0$ surface, where $z = x^i / \sqrt{2}$

Colliding Scalar Waves

GR 5d: large amplitude waves

$\partial_{x^+} \psi$

$\partial_{x^-} \psi$

ψ

$I^{(5)} = R_{ijkl} R^{ijkl}$

GR & GB 5d

$I^{(5)}$ at origin

$\alpha_{GB} = -1$

$\alpha_{GB} = 0$

$\alpha_{GB} = +1$

GR & GB 6d

$I^{(6)}$ at origin

$\alpha_{GB} = -1$

$\alpha_{GB} = 0$

$\alpha_{GB} = +1$

GR & GB 4d

$I^{(4)}$ at origin

$\alpha_{GB} = -1$

$\alpha_{GB} = 0$

$\alpha_{GB} = +1$

GR 4d-7d

$I^{(4)}, I^{(5)}, I^{(6)}, I^{(7)}$ at origin

GR4d, GR5d, GR6d, GR7d

maximum of Kretschmann invariant

$\alpha_{GB} = -1$

$\alpha_{GB} = 0$

$\alpha_{GB} = +1$