filename="ADMtoNP.tex" 2003-0311 HShinkai hshinkai@postman.riken.go.jp

The published version of Gunnarsen-Shinkai-Maeda [1] has several typo in the process of printing. This note is for confirming the notations in the article, including one more variations in the signature in the metric (s in (11)).

The Newman-Penrose formulation [2, 3] has many advantages, especially for treating gravitational wave dynamics.

- Natural framework for calculations in radiative space-time.
- Variables have geometrical meanings.
- Practical advantages in treating Petrov type-D space-time.
- Closely related with spinor formalism.

Newman-Penrose's variables are based on real-valued null vectors $\mathbf{l}, \mathbf{n}$ and complex conjugate null vectors $\mathbf{m}, \overline{\mathbf{m}}$, which satisfy

$$\mathbf{l}\cdot\mathbf{n} = l^a n_a = l_a n^a = 1, \qquad \mathbf{m}\cdot\overline{\mathbf{m}} = m^a \overline{m}_a = m_a \overline{m}^a = -1, \qquad \text{and else } 0. \tag{1}$$

This set of null basis $(l^a, n^a, m^a, \overline{m}^a)$ have relations with orthogonal tetrad basis (t^a, x^a, y^a, z^a) as

$$l^a = o^A o^{A'} = (1/\sqrt{2})(t^a + z^a), \qquad m^a = o^A \iota^{A'} = (1/\sqrt{2})(x^a - iy^a), \tag{2}$$
$$n^a = \iota^A \iota^{A'} = (1/\sqrt{2})(t^a - z^a), \qquad \overline{m}^a = \iota^A o^{A'} = (1/\sqrt{2})(x^a + iy^a), \tag{3}$$

where I also put spinor basis expressions (o^A, ι^A) of those.

Metric g_{ab} will be recovered by

$$g_{ab} = \eta_{ij} e^i{}_a e^j{}_b \;=\; 2l_{(a}n_{b)} - 2m_{(a}\overline{m}_{b)} = t_a t_b - x_a x_b - y_a y_b - z_a z_b \tag{4}$$

where $\eta_{ij} = \eta^{ij} = \mathrm{diag}\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$.

The Weyl curvature C_{abcd} is defined as

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3} R g_{a[c}g_{d]b}. \tag{5}$$

The 10 components of Weyl curvature are expressed by the following 5 complex scalars [4];

$$\Psi_0 \equiv \psi_{ABCD} o^A o^B o^C o^D = C_{abcd} l^a m^b l^c m^d, \quad n^a\text{-directed transverse component, } \{4, 0\} \tag{6}$$
$$\Psi_1 \equiv \psi_{ABCD} o^A o^B o^C \iota^D = C_{abcd} l^a n^b l^c m^d, \quad n^a\text{-directed longitudinal component} \{2, 0\} \tag{7}$$
$$\Psi_2 \equiv \psi_{ABCD} o^A o^B \iota^C \iota^D = C_{abcd} l^a m^b \bar{m}^c n^d, \quad \text{'Coulomb' component, } \{0, 0\} \tag{8}$$
$$\Psi_3 \equiv \psi_{ABCD} o^A \iota^B \iota^C \iota^D = C_{abcd} l^a n^b \bar{m}^c n^d, \quad l^a\text{-directed longitudinal component, } \{-2, 0\} \tag{9}$$
$$\Psi_4 \equiv \psi_{ABCD} \iota^A \iota^B \iota^C \iota^D = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \quad l^a\text{-directed transverse component, } \{-4, 0\} \tag{10}$$

where $\{p, q\}$ indicates *spin- and boost-weighted type* and prime-operation will be defined later.

Gunnarsen-Shinkai-Maeda [1] derived a transformation formula of Weyl scalar Ψ_i from ADM variables (γ_{ij}, K_{ij}), motivated by an application to interpret numerically generated space-time. Here, we consider vacuum space-time. Let $(\mathcal{M}, \eta_{ab})$ be real, 4-dimensional Lorentz vector space with volume form ε_{abcd}; $\varepsilon_{abcd}\varepsilon^{abcd} = -4!$. Let (t^a, x^a, y^a, z^a) be orthonormal basis of $(\mathcal{M}, \eta_{ab})$, and define

$$t^a t_a = +s \quad (s = \pm 1), \quad \varepsilon_{abc} = \varepsilon^{abcd} t_d, \tag{11}$$

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$ We formulate our equations in the signatures both $(+,-,-,-)$ and $(-,+,+,+)$ by choosing $s = 1$ or -1, respectively[1], because the former notation is common in working with the spinors.

First, we define the Weyl curvature C_{abcd} by (5) and decompose those into its electric and a magnetic components,

$$E_{ab} \equiv - \ C_{ambn} t^m t^n, \quad B_{ab} \equiv - \ {}^*C_{ambn} t^n t^m, \tag{12}$$

where ${}^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn} C_{mncd}$ is a dual of the Weyl tensor. These decomposed elements E_{ab} and B_{ab} are also presented by the 3-metric γ_{ab} and the extrinsic curvature K_{ab} as [6]

$$E_{ab} = R_{ab} - K_a{}^m K_{bm} + K K_{ab} - \frac{2}{3}\Lambda\gamma_{ab}, \tag{13}$$

$$B_{ab} = \varepsilon_a{}^{mn} D_m K_{nb}. \tag{14}$$

This is why we emphasize that our inputs are '3+1' elements. It follows from two constraint equations that the fields E_{ab}, B_{ab} are both trace-free and symmetric. We can reconstruct the Weyl curvature completely from E_{ab} and B_{ab} by

$$C_{abcd} = 4t_{[a}E_{b][c}t_{d]} + 2\varepsilon_{ab}{}^m B_{m[c}t_{d]} + 2\varepsilon_{cd}{}^m B_{m[a}t_{b]} + \varepsilon_{ab}{}^m \varepsilon_{cd}{}^n E_{mn}. \tag{15}$$

The next step is to choose a unit vector field $\hat{z}^a$ on Σ, and to decompose E_{ab}, B_{ab} into components along and perpendicular to $\hat{z}^a$. We set

$$e = E_{ab}\hat{z}^a\hat{z}^b, \tag{16}$$

$$e_a = E_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), \tag{17}$$

$$e_{ab} = E_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}e s_{ab}, \tag{18}$$

$$b = B_{ab}\hat{z}^a\hat{z}^b, \tag{19}$$

$$b_a = B_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), \tag{20}$$

$$b_{ab} = B_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}b s_{ab}, \tag{21}$$

where $s_{ab} = \gamma_{ab} - \hat{z}_a\hat{z}_b$. We note that E_{ab}, B_{ab} is again reconstructed from (16)-(21)

$$E_{ab} = e\hat{z}_a\hat{z}_b + 2e_{(a}\hat{z}_{b)} + e_{ab} - (1/2)s_{ab}e. \tag{22}$$

$$B_{ab} = b\hat{z}_a\hat{z}_b + 2b_{(a}\hat{z}_{b)} + b_{ab} - (1/2)s_{ab}b. \tag{23}$$

Such decompositions will be useful to discuss the effects of curvatures on the transversal plane to the $\hat{z}^a$ direction.

We put a rotation operator on the plane spanned by $\hat{x}_a$ and $\hat{y}_a$ as,

$$J_a{}^b \equiv \varepsilon_a{}^{bcd}\hat{z}_c t_d. \tag{24}$$

It is easy to check this mapping preserves s_{ab}, and is also easy to check $J_a{}^c J_c{}^b = -(\delta_a{}^b + s\hat{z}_a\hat{z}^b)$, which shows us $J_a{}^b$ has a complex structure, i.e., $J_a{}^b$ lets us define complex multiples of vectors

[1]That is, the metric is $\eta_{ab} = s(t_a t_b - x_a x_b - y_a y_b - z_a z_b)$.

$x^a \in P_z$, according to the formula $(m+in)x^a = mx^a + nJ_b{}^a x^b$. In short, $J_a{}^b$ expresses a rotation by 90 degrees in the plane orthogonal to $\hat{z}^a$.

By substituting (15) and (2, 3) into (6)-(10), we get Ψ_i using (**??**) and (24):

$$\Psi_0 = -(e_{ab} + sJ_a{}^c b_{bc})m^a m^b, \tag{25}$$
$$\Psi_1 = -(s/\sqrt{2})\,(e_a + sJ_a{}^c b_c)m^a, \tag{26}$$
$$\Psi_2 = -(1/2)\,(e + ib), \tag{27}$$
$$\Psi_3 = -(s/\sqrt{2})(e_a - sJ_a{}^c b_c)\bar{m}^a, \tag{28}$$
$$\Psi_4 = -(e_{ab} - sJ_a{}^c b_{bc})\bar{m}^a \bar{m}^b. \tag{29}$$

This relation has been applied to many groups' numerical codes, and helps their simulation's physical understandings. Weyl scalars are also useful for evaluating Riemann (Kretchman) invariant as

$$C_{abcd}C^{abcd} = \Psi_4\Psi_0 - 4\Psi_1\Psi_3 + 3\Psi_2^2. \tag{30}$$

Note that

$$R_{abcd}R^{abcd} = C_{abcd}C^{abcd} + 2R_{ab}R^{ab} - (1/3)R^2. \tag{31}$$

I hope this note helps you.

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