filename="dualnull.tex" 2003-0403 Hisaaki Shinkai hshinkai@postman.riken.go.jp

Dual null formulation (and its Quasi-Spherical version)

This note is for actual coding of the double null formulation by Hayward [1, 2], expecially its quasi-spherical approximated version [3, 4].

Contents

1 Dual-null formulation (without conformal scaling)

1.1 metric

$$
\begin{aligned}
ds^2 &= h_{ij}(dx^i + p^i dx^+ + m^i dx^-)(dx^j + p^j dx^+ + m^j dx^-) - 2e^{-f} dx^+ dx^- \qquad (1)\\
&= h_{ij} dx^i dx^j + 2p_l dx^l dx^+ + 2m_l dx^l dx^- + p_l p^l dx^{+2} + m_l m^l dx^{-2} + 2(p_l m^l - e^{-f}) dx^+ dx^- \qquad (2)
\end{aligned}
$$

In the matrix form,

$$
g_{ab} = \begin{pmatrix} p^2 & -e^{-f} + p_l m^l & p_j \\ -e^{-f} + p_l m^l & m^2 & m_j \\ p_i & m_i & h_{ij} \end{pmatrix} \tag{3}
$$

and

$$
g^{ab} = \begin{pmatrix} 0 & -e^f & e^f m^i \\ -e^f & 0 & e^f p^i \\ e^f m^j & e^f p^j & h^{ij} - e^f(p^i m^j + p^j m^i) \end{pmatrix} \tag{4}
$$

1.2 Lie derivatives

We use Lie derivatives $L_\pm$ along the null normal vectors

$$
l_\pm = u_\pm - s_\pm = e^{-f} g^{-1}(n^\mp), \tag{5}
$$

that is

$$
\begin{aligned}
{l_+}^a &= u_+^a - p^a = (1,0,0,0) - (0,0,p^1,p^2) = (1,0,-p^1,-p^2) \qquad (6a)\\
{l_-}^a &= u_-^a - m^a = (0,1,0,0) - (0,0,m^1,m^2) = (0,1,-m^1,-m^2). \qquad (6b)
\end{aligned}
$$

The Lie derivative is defined in general as

$$
\begin{aligned}
\pounds_\xi S &= \xi^\alpha \partial_\alpha S, \qquad (7a)\\
\pounds_\xi V^\mu &= \xi^\alpha \nabla_\alpha V^\mu - V^\alpha \nabla_\alpha \xi^\mu = \xi^\alpha \partial_\alpha V^\mu - V^\alpha \partial_\alpha \xi^\mu, \qquad (7b)\\
\pounds_\xi U_\mu &= \xi^\alpha \nabla_\alpha U_\mu + U_\alpha \nabla_\mu \xi^\alpha, = \xi^\alpha \partial_\alpha U_\mu + U_\alpha \partial_\mu \xi^\alpha, \qquad (7c)\\
\pounds_\xi T^{\mu\nu} &= \xi^\alpha \nabla_\alpha T^{\mu\nu} - T^{\alpha\nu} \nabla_\alpha \xi^\mu - T^{\mu\alpha} \nabla_\alpha \xi^\nu = \xi^\alpha \partial_\alpha T^{\mu\nu} - T^{\alpha\nu} \partial_\alpha \xi^\mu - T^{\mu\alpha} \partial_\alpha \xi^\nu, \qquad (7d)\\
\pounds_\xi W_{\mu\nu} &= \xi^\alpha \nabla_\alpha W_{\mu\nu} + W_{\alpha\nu} \nabla_\mu \xi^\alpha + W_{\mu\alpha} \nabla_\nu \xi^\alpha = \xi^\alpha \partial_\alpha W_{\mu\nu} + W_{\alpha\nu} \partial_\mu \xi^\alpha + W_{\mu\alpha} \partial_\nu \xi^\alpha \qquad (7e)
\end{aligned}
$$

therefore for scalar, vector and tensor quantities,

$$
\begin{aligned}
L_+ S &= {l_+}^a \partial_a S = \partial_{x^+} S - p^k \partial_k S \qquad (8a)\\
L_- S &= {l_-}^a \partial_a S = \partial_{x^-} S - m^k \partial_k S \qquad (8b)\\
L_+ V^a &= \partial_{x^+} V^a - p^k \partial_k V^a - V^k \partial_k p^a \qquad (8c)\\
L_- V^a &= \partial_{x^-} V^a - m^k \partial_k V^a - V^k \partial_k m^a \qquad (8d)\\
L_+ h_{ij} &= \partial_{x^+} h_{ij} - p^k \partial_k h_{ij} + 2h_{k(i} \partial_{j)} p^k \qquad (8e)\\
L_- h_{ij} &= \partial_{x^-} h_{ij} - m^k \partial_k h_{ij} + 2h_{k(i} \partial_{j)} m^k \qquad (8f)
\end{aligned}
$$

1.3 Geometrical quantities

From the original expressions The fields $(\theta_\pm, \sigma_\pm, \nu_\pm, \omega)$ encode the extrinsic curvature of the dual-null foliation. These extrinsic fields are unique up to duality $\pm \mapsto \mp$ and diffeomorphisms which relabel the null hypersurfaces, i.e. $dx^\pm \mapsto e^{\lambda_\pm} dx^\pm$ for functions $\lambda_\pm(x^\pm)$.

$$\begin{aligned}
\theta_\pm &= *L_\pm *1 && \text{(9a)}\\
\sigma_\pm &= \perp L_\pm h - \theta_\pm h && \text{(9b)}\\
\nu_\pm &= L_\pm f && \text{(9c)}\\
\omega &= \tfrac{1}{2} e^f h([l_-, l_+]) && \text{(9d)}
\end{aligned}$$

where $*$ is the Hodge operator of h_{ab}. The functions $\theta_\pm$ are the expansions, the traceless bilinear forms $\sigma_\pm$ are the shears, the 1-form ω is the twist, measuring the lack of integrability of the normal space, and the functions $\nu_\pm$ are the inaffinities, measuring the failure of the null normals to be affine.

HS notes For more friendly expressions, these are

$$\begin{aligned}
\theta_\pm &= \frac{1}{\sqrt{\det h_{ij}}} L_\pm \sqrt{\det h_{ij}} && \text{(10a)}\\
\sigma_{\pm ab} &= h_a^c h_b^d L_\pm h_{cd} - \theta_\pm h_{ab} && \text{(10b)}\\
\nu_\pm &= L_\pm f && \text{(10c)}\\
\omega_a &= \tfrac{1}{2} e^f h_{ab} [l_-, l_+]^b = \tfrac{1}{2} e^f h_{ab} (l_-^c \nabla_c l_+^b - l_+^c \nabla_c l_-^b) \\
&= \tfrac{1}{2} e^f h_{ab} (l_-^c \partial_c l_+^b - l_+^c \partial_c l_-^b) && \text{(10d)}
\end{aligned}$$

More concrete,

$$\begin{aligned}
\theta_+ &= \frac{1}{\sqrt{\det h_{ij}}} [\partial_{x^+} \sqrt{\det h_{ij}} - p^k \partial_k \sqrt{\det h_{ij}}] && \text{(11a)}\\
\theta_- &= \frac{1}{\sqrt{\det h_{ij}}} [\partial_{x^-} \sqrt{\det h_{ij}} - m^k \partial_k \sqrt{\det h_{ij}}] && \text{(11b)}\\
\sigma_{+ab} &= h_a^c h_b^d L_+ h_{cd} - \theta_+ h_{ab} \\
&= h_a^c h_b^d [\partial_{x^+} h_{cd} - p^k \partial_k h_{cd} + 2 h_{k(c} \partial_{d)} p^k] - \theta_+ h_{ab} && \text{(11c)}\\
\sigma_{-ab} &= h_a^c h_b^d L_- h_{cd} - \theta_- h_{ab} \\
&= h_a^c h_b^d [\partial_{x^-} h_{cd} - m^k \partial_k h_{cd} + 2 h_{k(c} \partial_{d)} m^k] - \theta_- h_{ab} && \text{(11d)}\\
\nu_+ &= L_+ f = \partial_{x^+} f - p^k \partial_k f && \text{(11e)}\\
\nu_- &= L_- f = \partial_{x^-} f - m^k \partial_k f && \text{(11f)}\\
\omega_a &= \tfrac{1}{2} e^f h_{ab} (\partial_{x^-} l_+^b - m^k \partial_k l_+^b - \partial_{x^+} l_-^b + p^k \partial_k l_-^b) && \text{(11g)}
\end{aligned}$$

1.4 Full version

1.4.1 Full set of Einstein equation 1

Inverting the definitions of the momentum fields yields the propagation equations

$$\begin{aligned}
\perp (L_+ s_- - L_- s_+) &= 2 e^{-f} h^{-1}(\omega) + [s_-, s_+] && \text{(12a)}\\
\perp L_\pm h &= \theta_\pm h + \sigma_\pm && \text{(12b)}\\
L_\pm f &= \nu_\pm . && \text{(12c)}
\end{aligned}$$

The full set of Einstein equations is obtained with the below.

1.4.2 Full set of Einstein equation 2

From Appendix B in [1] (with the current convention):

$$\begin{aligned}
L_+\theta_+ &= -\nu_+\theta_+ - \tfrac{1}{2}\theta_+^2 - \tfrac{1}{4}h^{ac}h^{bd}\sigma_{+ab}\sigma_{+cd} && (13a)\\
L_+\theta_- &= -\theta_+\theta_- + e^{-f}\left(-\tfrac{1}{2}\mathcal{R} + h^{ab}(\omega_a\omega_b - \tfrac{1}{2}D_aD_bf + \tfrac{1}{4}D_afD_bf + \omega_aD_bf - D_a\omega_b)\right) && (13b)\\
L_+\nu_- &= -\tfrac{1}{2}\theta_+\theta_- + \tfrac{1}{4}h^{ac}h^{bd}\sigma_{+ab}\sigma_{-cd} + e^{-f}\left(-\tfrac{1}{2}\mathcal{R} + h^{ab}(3\omega_a\omega_b - \tfrac{1}{4}D_afD_bf - \omega_aD_bf)\right) && (13c)\\
\perp L_+\sigma_{-ab} &= \tfrac{1}{2}(\theta_+\sigma_{-ab} - \theta_-\sigma_{+ab}) + h^{cd}\sigma_{+c(a}\sigma_{-b)d}\\
&\quad +2e^{-f}\left(\omega_a\omega_b - \tfrac{1}{2}D_aD_bf + \tfrac{1}{4}D_afD_bf + \omega_{(a}D_{b)}f - D_{(a}\omega_{b)}\right)\\
&\quad -e^{-f}h^{cd}\left(\omega_c\omega_d - \tfrac{1}{2}D_cD_df + \tfrac{1}{4}D_cfD_df + \omega_cD_df - D_c\omega_d\right)h_{ab} && (13d)\\
\perp L_+\omega_a &= -\theta_+\omega_a + \tfrac{1}{2}(D\nu_+ - D\theta_+ - \theta_+Df) + \tfrac{1}{2}h^{bc}D_c\sigma_{+ab} && (13e)
\end{aligned}$$

and

$$\begin{aligned}
L_-\theta_- &= -\nu_-\theta_- - \tfrac{1}{2}\theta_-^2 - \tfrac{1}{4}h^{ac}h^{bd}\sigma_{-ab}\sigma_{-cd} && (14a)\\
L_-\theta_+ &= -\theta_+\theta_- + e^{-f}\left(-\tfrac{1}{2}\mathcal{R} + h^{ab}(\omega_a\omega_b - \tfrac{1}{2}D_aD_bf + \tfrac{1}{4}D_afD_bf - \omega_aD_bf + D_a\omega_b)\right) && (14b)\\
L_-\nu_+ &= -\tfrac{1}{2}\theta_+\theta_- + \tfrac{1}{4}h^{ac}h^{bd}\sigma_{+ab}\sigma_{-cd} + e^{-f}\left(-\tfrac{1}{2}\mathcal{R} + h^{ab}(3\omega_a\omega_b - \tfrac{1}{4}D_afD_bf + \omega_aD_bf)\right) && (14c)\\
\perp L_-\sigma_{+ab} &= -\tfrac{1}{2}(\theta_+\sigma_{-ab} - \theta_-\sigma_{+ab}) + h^{cd}\sigma_{-c(a}\sigma_{+b)d}\\
&\quad +2e^{-f}\left(\omega_a\omega_b - \tfrac{1}{2}D_aD_bf + \tfrac{1}{4}D_afD_bf - \omega_{(a}D_{b)}f + D_{(a}\omega_{b)}\right)\\
&\quad -e^{-f}h^{cd}\left(\omega_c\omega_d - \tfrac{1}{2}D_cD_df + \tfrac{1}{4}D_cfD_df - \omega_cD_df + D_c\omega_d\right)h_{ab} && (14d)\\
\perp L_-\omega_a &= -\theta_-\omega_a - \tfrac{1}{2}(D\nu_- - D\theta_- - \theta_-Df) - \tfrac{1}{2}h^{bc}D_c\sigma_{-ab} && (14e)
\end{aligned}$$

where D is the covariant derivative of h_{ab}.

There is no $L_+\sigma_{+ab}$, $L_-\sigma_{-ab}$, $L_+\nu_+$, and $L_-\nu_-$.

In spherical symmetry, $(\sigma_\pm, \omega, D)$ vanish, while $(\theta_\pm, \nu_\pm, D_\pm)$ generally do not.

1.4.3 HS notes: friendly expressions

For the equations in §1.4.1

$$\begin{aligned}
h^a_b\left(L_+m^b - L_-p^b\right) &= h^a_b\left(\partial_{x^+}m^b - p^k\partial_km^b - m^k\partial_kp^b - \partial_{x^-}p^b + m^k\partial_kp^b + p^k\partial_km^b\right)\\
&= h^a_b\left(\partial_{x^+}m^b - \partial_{x^-}p^b\right) = 2e^{-f}h^{ab}\omega_b + m^c\partial_cp^a - p^c\partial_cm^a && (15a)\\
h^c_ah^d_bL_+h_{cd} &= h^c_ah^d_b(\partial_{x^+}h_{cd} - p^k\partial_kh_{cd} + 2h_{k(c}\partial_{d)}p^k) = \theta_+h_{ab} + \sigma_{+ab} && (15b)\\
h^c_ah^d_bL_-h_{cd} &= h^c_ah^d_b(\partial_{x^-}h_{cd} - m^k\partial_kh_{cd} + 2h_{k(c}\partial_{d)}m^k) = \theta_-h_{ab} + \sigma_{-ab} && (15c)\\
L_+f &= \partial_{x^+}f - p^k\partial_kf = \nu_+ && (15d)\\
L_-f &= \partial_{x^-}f - m^k\partial_kf = \nu_- && (15e)
\end{aligned}$$

1.5 Quasi-spherical version

1.5.1 Full set of Einstein equation 1

Inverting the definitions of the momentum fields yields the propagation equations

$$\begin{aligned} \perp (L_+ s_- - L_- s_+) &= 2e^{-f} h^{-1}(\omega) + [s_-, s_+] & \text{(16a)} \\ \perp L_\pm h &= \theta_\pm h + \sigma_\pm & \text{(16b)} \\ L_\pm f &= \nu_\pm . & \text{(16c)} \end{aligned}$$

The full set of Einstein equations is obtained with the below (17a)-(17e). There is no $L_+\sigma_{+ab}$, $L_-\sigma_{-ab}$, $L_+\nu_+$, and $L_-\nu_-$.

1.5.2 Full set of Einstein equation 2 (quasi-spherical approximation)

After the truncations for quasi-spherical approximations,

$$\begin{aligned} L_\pm \theta_\pm &= -\nu_\pm \theta_\pm - \tfrac{1}{2}\theta_\pm^2 & \text{(17a)} \\ L_\pm \theta_\mp &= -\theta_+ \theta_- - e^{-f} r^{-2} & \text{(17b)} \\ L_\pm \nu_\mp &= -\tfrac{1}{2}\theta_+ \theta_- - e^{-f} r^{-2} & \text{(17c)} \\ \perp L_\pm \sigma_\mp &= \pm\tfrac{1}{2}(\theta_+ \sigma_- - \theta_- \sigma_+) & \text{(17d)} \\ \perp L_\pm \omega &= -\theta_\pm \omega \pm \tfrac{1}{2}(D\nu_\pm - D\theta_\pm - \theta_\pm D f) & \text{(17e)} \end{aligned}$$

where D is the covariant derivative of h_{ab}. In spherical symmetry, $(\sigma_\pm, \omega, D)$ vanish, while $(\theta_\pm, \nu_\pm, D_\pm)$ generally do not.

1.5.3 HS notes: friendly expressions

For the equations in §1.5.2

$$\begin{aligned} L_\pm \theta_\pm &= \partial_\pm \theta_\pm - s_\pm^k \partial_k \theta_\pm = -\nu_\pm \theta_\pm - \tfrac{1}{2}\theta_\pm^2 \qquad \text{that is} \\ L_+ \theta_+ &= \partial_+ \theta_+ - p^k \partial_k \theta_+ = -\nu_+ \theta_+ - \tfrac{1}{2}\theta_+^2 & \text{(18a)} \\ L_- \theta_- &= \partial_- \theta_- - m^k \partial_k \theta_- = -\nu_- \theta_- - \tfrac{1}{2}\theta_-^2 & \text{(18b)} \\ L_\pm \theta_\mp &= \partial_\pm \theta_\mp - s_\pm^k \partial_k \theta_\mp = -\theta_+ \theta_- - e^{-f} r^{-2} \qquad \text{that is} \\ L_+ \theta_- &= \partial_+ \theta_- - p^k \partial_k \theta_- = -\theta_+ \theta_- - e^{-f} r^{-2} & \text{(18c)} \\ L_- \theta_+ &= \partial_- \theta_+ - m^k \partial_k \theta_+ = -\theta_+ \theta_- - e^{-f} r^{-2} & \text{(18d)} \\ L_\pm \nu_\mp &= \partial_\pm \nu_\mp - s_\pm^k \partial_k \nu_\mp = -\tfrac{1}{2}\theta_+ \theta_- - e^{-f} r^{-2} \qquad \text{that is} \\ L_+ \nu_- &= \partial_+ \nu_- - p^k \partial_k \nu_- = -\tfrac{1}{2}\theta_+ \theta_- - e^{-f} r^{-2} & \text{(18e)} \\ L_- \nu_+ &= \partial_- \nu_+ - m^k \partial_k \nu_+ = -\tfrac{1}{2}\theta_+ \theta_- - e^{-f} r^{-2} & \text{(18f)} \\ h_a^c h_b^d L_\pm \sigma_{\mp cd} &= h_a^c h_b^d (\partial_{x^-} \sigma_{\mp cd} - s_\pm^k \partial_k \sigma_{\mp cd} + 2\sigma_{\mp k(c} \partial_{d)} s_\pm^k) \\ &= \pm\tfrac{1}{2}(\theta_+ \sigma_{-ab} - \theta_- \sigma_{+ab}) & \text{(18g)} \\ h_a^b L_\pm \omega_b &= h_a^b (\partial_\pm \omega_b - s_\pm^k \partial_k \omega_b + \omega_k \partial_b s_\pm^k) \\ &= -\theta_\pm \omega_a \pm \tfrac{1}{2}(D_a \nu_\pm - D_a \theta_\pm - \theta_\pm D_a f) \\ &= -\theta_\pm \omega_a \pm \tfrac{1}{2}(\partial_a \nu_\pm - \partial_a \theta_\pm - \theta_\pm \partial_a f) & \text{(18h)} \end{aligned}$$

2 Dual-null formulation (with conformal scaling)

2.1 Introducing the conformal decomposition

It is also possible to integrate all the way from $\Im^-$ to $\Im^+$ by a conformal transformation.

We take the conformal decomposition of

$$h_{ab} = r^2 k_{ab}, \tag{19}$$

such that

$$D_\pm \tilde{*}1 = 0 \qquad (D_\pm \det k_{ab} = 0) \tag{20}$$

where $\tilde{*}$ is the Hodge operator of k, satisfying $*1 = \tilde{*}r^2$.

The shear equations, composed into a second-order equation for k, become

$$\Delta k_{ab} = 0 \tag{21}$$

where Δ is the quasi-spherical wave operator:

$$\begin{aligned} \Delta\phi &= -2e^f \left(L_{(+}L_{-)}\phi + 2r^{-1}L_{(+}rL_{-)}\phi\right) \\ &= -e^f \left(L_+L_-\phi + L_-L_+\phi + 2r^{-1}L_+rL_-\phi + 2r^{-1}L_-rL_+\phi\right) \end{aligned} \tag{22}$$

In practice, one may use the conformal factor

$$\Omega = r^{-1} \tag{23}$$

and the rescaled expansions and shears

$$\vartheta_\pm = r\theta_\pm \tag{24a}$$

$$\varsigma_\pm = r^{-1}\sigma_\pm \tag{24b}$$

which are finite and generally non-zero at $\Im^\mp$.

2.2 Full version

Not available here.

2.3 Quasi-spherical version

Of the dynamical fields and operators introduced above, $(s_\pm, \sigma_\pm, \omega, D)$ vanish in spherical symmetry, while $(h, f, \theta_\pm, \nu_\pm, \Delta_\pm)$ generally do not. The quasi-spherical approximation consists of linearizing in $(s_\pm, \sigma_\pm, \omega, D)$, i.e. setting to zero any second-order terms in these quantities. This yields a greatly simplified truncation [3] of the full field equations, the first-order dual-null form of the vacuum Einstein system[2]. In particular, the truncated equations decouple into a three-level hierarchy, the last level being irrelevant to determining the gravitational waveforms. The remaining equations are the quasi-spherical equations

$$\Delta_\pm\Omega = -\tfrac{1}{2}\Omega^2\vartheta_\pm, \tag{25}$$

$$\Delta_\pm f = \nu_\pm, \tag{26}$$

$$\Delta_\pm\vartheta_\pm = -\nu_\pm\vartheta_\pm \underbrace{-\tfrac{1}{4}\Omega||\varsigma_\pm||^2}_{\textit{2nd order}}, \tag{27}$$

$$\Delta_\pm\vartheta_\mp = -\Omega(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}), \tag{28}$$

$$\Delta_\pm\nu_\mp = -\Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f} \underbrace{-\tfrac{1}{4}\langle\varsigma_+, \varsigma_-\rangle}_{\textit{2nd order}}), \tag{29}$$

and the linearized equations

$$\Delta_{\pm}k = \Omega\varsigma_{\pm}, \tag{30}$$
$$\Delta_{\pm}\varsigma_{\mp} = \Omega(\underbrace{\varsigma_{+}\cdot k^{-1}\cdot\varsigma_{-}}_{\textit{1st but missing in below}} - \tfrac{1}{2}\vartheta_{\mp}\varsigma_{\pm}). \tag{31}$$

These are all ordinary differential equations; no transverse D derivatives occur. Thus we have an effectively two-dimensional system to be integrated independently at each angle of the sphere. The "2nd order" terms are pointed out in [5].

The initial-data formulation is based on a spatial surface S orthogonal to $l_{\pm}$ and the null hypersurfaces $\Sigma_{\pm}$ generated from S by $l_{\pm}$, assumed future-pointing. The initial data for the above equations are $(\Omega, f, k, \vartheta_{\pm})$ on S and $(\varsigma_{\pm}, \nu_{\pm})$ on $\Sigma_{\pm}$. We will take l_{+} and l_{-} to be outgoing and ingoing respectively.

For the quasi-spherical approximation to be valid near $\Im^{\pm}$, a modification of the integration scheme is suggested, such that initial data is given at spatial infinity i^0 rather than Σ. Specifically, one may fix $(\Omega, f, \vartheta_{\pm}, k) = (0, 0, \pm\sqrt{2}, \epsilon)$ at i^0, where $\epsilon = d\vartheta\otimes d\vartheta + \sin^2\vartheta d\varphi\otimes d\varphi$ is the standard metric of a unit sphere. The first step would then be to integrate backwards along $\Im^{-}$ as far as Σ, fixing $\nu_{+} = 0$ on Σ_{+}. The remaining coordinate data is given by ν_{-} on the ingoing null hypersurface Σ_{-}, which is left free so that one may adapt the foliation of Σ_{-} to the surfaces which are most spherical.

2.4 HS notes

For scalar and tensor quantities:

$$\Delta_{\pm}S = \perp L_{\pm}S = \perp(\partial_{\pm}S - s^k_{\pm}\partial_k S) = \partial_{\pm}S - s^k_{\pm}\partial_k S$$
$$\Delta_{\pm}W_{ij} = \perp L_{\pm}W_{ij} = h^a_i h^b_j L_{\pm}W_{ab} = h^a_i h^b_j(\partial_{\pm}W_{ab} - s^k_{\pm}\partial_k W_{ab} + 2W_{k(a}\partial_{b)}s^k_{\pm})$$

Therefore we can write the evolution equations as the following:

$$\Delta_{\pm}\Omega = \partial_{\pm}\Omega - s^k_{\pm}\partial_k\Omega = -\tfrac{1}{2}\Omega^2\vartheta_{\pm} \quad \text{that is}$$
$$\partial_{+}\Omega = p^k\partial_k\Omega - \tfrac{1}{2}\Omega^2\vartheta_{+} \tag{32a}$$
$$\partial_{-}\Omega = m^k\partial_k\Omega - \tfrac{1}{2}\Omega^2\vartheta_{-} \tag{32b}$$
$$\Delta_{\pm}f = \partial_{\pm}f - s^k_{\pm}\partial_k f = \nu_{\pm} \quad \text{that is}$$
$$\partial_{+}f = p^k\partial_k f + \nu_{+} \tag{32c}$$
$$\partial_{-}f = m^k\partial_k f + \nu_{-} \tag{32d}$$
$$\Delta_{\pm}\vartheta_{\pm} = \partial_{\pm}\vartheta_{\pm} - s^k_{\pm}\partial_k\vartheta_{\pm} = -\nu_{\pm}\vartheta_{\pm} \quad \text{that is}$$
$$\partial_{+}\vartheta_{+} = p^k\partial_k\vartheta_{+} - \nu_{+}\vartheta_{+} \tag{32e}$$
$$\partial_{-}\vartheta_{-} = m^k\partial_k\vartheta_{-} - \nu_{-}\vartheta_{-} \tag{32f}$$
$$\Delta_{\pm}\vartheta_{\mp} = \partial_{\pm}\vartheta_{\mp} - s^k_{\pm}\partial_k\vartheta_{\mp} - \Omega(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \quad \text{that is}$$
$$\partial_{+}\vartheta_{-} = p^k\partial_k\vartheta_{-} - \Omega(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \tag{32g}$$
$$\partial_{-}\vartheta_{+} = m^k\partial_k\vartheta_{+} - \Omega(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \tag{32h}$$
$$\Delta_{\pm}\nu_{\mp} = \partial_{\pm}\nu_{\mp} - s^k_{\pm}\partial_k\nu_{\mp} - \Omega^2(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \quad \text{that is}$$
$$\partial_{+}\nu_{-} = p^k\partial_k\nu_{-} - \Omega^2(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \tag{32i}$$
$$\partial_{-}\nu_{+} = m^k\partial_k\nu_{+} - \Omega^2(\tfrac{1}{2}\vartheta_{+}\vartheta_{-} + e^{-f}) \tag{32j}$$

$$\Delta_\pm k_{ij} = h_i^a h_j^b(\partial_\pm k_{ab} - s_\pm^k \partial_k k_{ab} + 2k_{k(a}\partial_{b)} s_\pm^k) = \Omega\varsigma_{\pm ij} \qquad \text{that is}$$

$$h_i^a h_j^b \partial_+ k_{ab} = h_i^a h_j^b(p^k \partial_k k_{ab} - 2k_{k(a}\partial_{b)} p^k) + \Omega\varsigma_{+ij} \tag{32k}$$

$$h_i^a h_j^b \partial_- k_{ab} = h_i^a h_j^b(m^k \partial_k k_{ab} - 2k_{k(a}\partial_{b)} m^k) + \Omega\varsigma_{-ij} \tag{32l}$$

$$\Delta_\pm \varsigma_{\mp ij} = h_i^a h_j^b(\partial_\pm \varsigma_{\mp ab} - s_\pm^k \partial_k \varsigma_{\mp ab} + 2\varsigma_{\mp k(a}\partial_{b)} s_\pm^k) = -\tfrac{1}{2}\Omega\vartheta_\mp\varsigma_{\pm ij} \qquad \text{that is}$$

$$h_i^a h_j^b \partial_+ \varsigma_{-ab} = h_i^a h_j^b(p^k \partial_k \varsigma_{-ab} - 2\varsigma_{-k(a}\partial_{b)} p^k) - \tfrac{1}{2}\Omega\vartheta_-\varsigma_{+ij} \tag{32m}$$

$$h_i^a h_j^b \partial_- \varsigma_{+ab} = h_i^a h_j^b(m^k \partial_k \varsigma_{+ab} - 2\varsigma_{+k(a}\partial_{b)} m^k) - \tfrac{1}{2}\Omega\vartheta_+\varsigma_{-ij} \tag{32n}$$

3 Numerical Scheme for Quasi-Spherical approx. space-time

The variables are $(\Omega, f, \vartheta_+, \vartheta_-, \nu_+, \nu_-, k_{ab}, \varsigma_{+ab}, \varsigma_{-ab})$

1. Prepare initial data on Σ
   - Set metric components (f, k_{ab}) on Σ.
   - Set extrinsic curvature components $(\vartheta_\pm)$ on Σ.
2. Prepare data on Σ_-
   - Assume $m^a = 0$ on Σ_-
   - Set (ς_-, ν_-) on Σ_-.
   - Integrate $(\Omega, f, \vartheta_\pm, \nu_+, k_{ab}, \varsigma_{+ab})$ by Δ_- equations.

$$\partial_-\Omega = m^k\partial_k\Omega - \tfrac{1}{2}\Omega^2\vartheta_- \tag{33a}$$

$$\partial_- f = m^k\partial_k f + \nu_- \tag{33b}$$

$$\partial_-\vartheta_- = m^k\partial_k\vartheta_- - \nu_-\vartheta_- \tag{33c}$$

$$\partial_-\vartheta_+ = m^k\partial_k\vartheta_+ - \Omega(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{33d}$$

$$\partial_-\nu_+ = m^k\partial_k\nu_+ - \Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{33e}$$

$$h_i^a h_j^b \partial_- k_{ab} = h_i^a h_j^b(m^k \partial_k k_{ab} - 2k_{k(a}\partial_{b)} m^k) + \Omega\varsigma_{-ij} \tag{33f}$$

$$h_i^a h_j^b \partial_- \varsigma_{+ab} = h_i^a h_j^b(m^k \partial_k \varsigma_{+ab} - 2\varsigma_{+k(a}\partial_{b)} m^k) - \tfrac{1}{2}\Omega\vartheta_+\varsigma_{-ij} \tag{33g}$$

3. One step to go in U (from 0 to Δx^+ first time)
   - Assume $p^a = 0$ in U.
   - Set (ς_+, ν_+) on $\Sigma_+(\Delta x^+)$.
   - Integrate $(\Omega, f, \vartheta_\pm, \nu_-, k_{ab}, \varsigma_{-ab})$ by Δ_+ equations.

$$\partial_+\Omega = p^k\partial_k\Omega - \tfrac{1}{2}\Omega^2\vartheta_+ \tag{34a}$$

$$\partial_+ f = p^k\partial_k f + \nu_+ \tag{34b}$$

$$\partial_+\vartheta_+ = p^k\partial_k\vartheta_+ - \nu_+\vartheta_+ \tag{34c}$$

$$\partial_+\vartheta_- = p^k\partial_k\vartheta_- - \Omega(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{34d}$$

$$\partial_+\nu_- = p^k\partial_k\nu_- - \Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{34e}$$

$$h_i^a h_j^b \partial_+ k_{ab} = h_i^a h_j^b(p^k \partial_k k_{ab} - 2k_{k(a}\partial_{b)} p^k) + \Omega\varsigma_{+ij} \tag{34f}$$

$$h_i^a h_j^b \partial_+ \varsigma_{-ab} = h_i^a h_j^b(p^k \partial_k \varsigma_{-ab} - 2\varsigma_{-k(a}\partial_{b)} p^k) - \tfrac{1}{2}\Omega\vartheta_-\varsigma_{+ij} \tag{34g}$$

- Note that we do not integrate ν_+ and ς_+, that is we do not have them except on $x^+ = 0$ surface and on the Σ_+ surface. Therefore we have to use $\nu_+(x^+ = 0)$ and $\varsigma_+(x^+ = 0)$ in the above RHSs.

4. Integrate along l^- direction

   - Assume $m^a = 0$ in U.
   - Set (ς_+, ν_+) on $\Sigma_+(\Delta x^+)$.
   - Integrate (ν_+, ς_{+ab}) by Δ_- equations.

$$\partial_-\nu_+ = m^k\partial_k\nu_+ - \Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{35a}$$
$$h_i^a h_j^b \partial_-\varsigma_{+ab} = h_i^a h_j^b(m^k\partial_k\varsigma_{+ab} - 2\varsigma_{+k(a}\partial_{b)}m^k) - \tfrac{1}{2}\Omega\vartheta_+\varsigma_{-ij} \tag{35b}$$

   where we need interpolate $\vartheta_\pm, \Omega, f, \varsigma_{-ij}$ in ODE solver, since they are only given on the grid points. To get the accurate solution, we have better to integrate these variables (except ς_-).

$$\partial_-\Omega = m^k\partial_k\Omega - \tfrac{1}{2}\Omega^2\vartheta_- \tag{36a}$$
$$\partial_- f = m^k\partial_k f + \nu_- \tag{36b}$$
$$\partial_-\vartheta_- = m^k\partial_k\vartheta_- - \nu_-\vartheta_- \tag{36c}$$
$$\partial_-\vartheta_+ = m^k\partial_k\vartheta_+ - \Omega(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{36d}$$

   As for ς_- and ν_-, we have to interpolate them from the grid points.

   - In order to check the accuracy, we also integrate (k_{ab}) and compare it with the results of step 2.

$$h_i^a h_j^b \partial_- k_{ab} = h_i^a h_j^b(m^k\partial_k k_{ab} - 2k_{k(a}\partial_{b)}m^k) + \Omega\varsigma_{-ij} \tag{37}$$

5. check the accuracy

   - If bad, then go back to the step 2. Integrate equations using the obtained $\nu_+(\Delta x^+)$ and $\varsigma_+(\Delta x^+)$ together with $\nu_+(0)$ and $\varsigma_+(0)$ with their linear interpolations.
   - If good, then fix the values $\nu_+(\Delta x^+)$ and $\varsigma_+(\Delta x^+)$ and go to the step 2 for the integration from Δx^+ to $2\Delta x^+$.

4 Schwarzschild metric as an example

We are using the metric with the form

$$ds^2 = \Omega^{-2}k_{ij}(dx^i + p^i dx^+ + m^i dx^-)(dx^j + p^j dx^+ + m^j dx^-) - 2e^{-f}dx^+dx^- \tag{38}$$

Schwarzschild metric is

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{39a}$$
$$= (1 - \frac{2m}{r})[-dt^2 + dr_*^2] + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{39b}$$
$$= -(1 - \frac{2m}{r})2dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{39c}$$

where

$$r_* = r + 2m\ln(\frac{r}{2m} - 1), \qquad \frac{dr}{dr_*} = 1 - \frac{2m}{r} \tag{40}$$

and

$$u = \frac{1}{\sqrt{2}}(t - r_*), \qquad v = \frac{1}{\sqrt{2}}(t + r_*). \tag{41}$$

Immediately we obtain

$$\Omega = r^{-1} \tag{42a}$$
$$k_{ij} = diag(1, \sin^2\theta) \tag{42b}$$
$$f = -\ln\left(1 - \frac{2m}{r}\right) \tag{42c}$$
$$m^i = p^i = 0 \tag{42d}$$

Take $\partial_+ = \partial_v, \partial_- = \partial_u$, then

$$\theta_\pm = \pm\frac{\sqrt{2}}{r}(1 - \frac{2m}{r}) \qquad \text{that is}$$
$$\vartheta_\pm = \pm\sqrt{2}(1 - \frac{2m}{r}) \tag{43a}$$
$$\nu_\pm = \mp\sqrt{2}\frac{m}{r^2} \tag{43b}$$
$$\sigma_{\pm ab} = -\theta_\pm \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \text{that is}$$
$$\varsigma_{\pm ab} = -\Omega\theta_\pm \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{43c}$$

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