2 時間発展を考えるための時空の分解

ここでは，Einstein 方程式

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad \text{where} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \quad \text{and} \quad \kappa = 8\pi G \tag{2.1}$$

を「時間発展を追う」形式に書き換える方法を説明する．

2.1 ADM 形式 (ADM formulation)

2.1.1 The 3+1 decomposition of space-time

The idea of space-time evolution was first formulated by Arnowitt, Deser, and Misner (ADM) [10]. The formulation was first motivated by a desire to construct a canonical framework in general relativity, but it also gave the community to the fundamental idea of time evolution of space and time: such as foliations of 3-dimensional hypersurface (Figure 2.1). This scheme is often called '3+1 formulation', 'the ADM formulation', or 'Cauchy approach'.

3-metric, lapse function, shift vectors

Let us denote the hypersurface $\Sigma(t)$ which is the three-dimensional spatial space with a parameter t. The evolution of spacetime is expressed as the dynamics of $\Sigma(t)$. The formulation begins by decomposing the metic as

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu, \quad (\mu,\nu = 0,1,2,3) \\ \text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij}\, dx^i\, dx^j, \quad (i,j = 1,2,3) \end{aligned}$$

Let the unit normal vector of the slices be n^μ, where

$$n_\mu = (-\alpha, 0, 0, 0), \qquad n^\mu = g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha).$$

We then have a 3+1 decomposed metric as

$$\begin{aligned} ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (-\alpha^2 + \beta_l \beta^l)\, dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i\, dx^j \end{aligned} \tag{2.2}$$

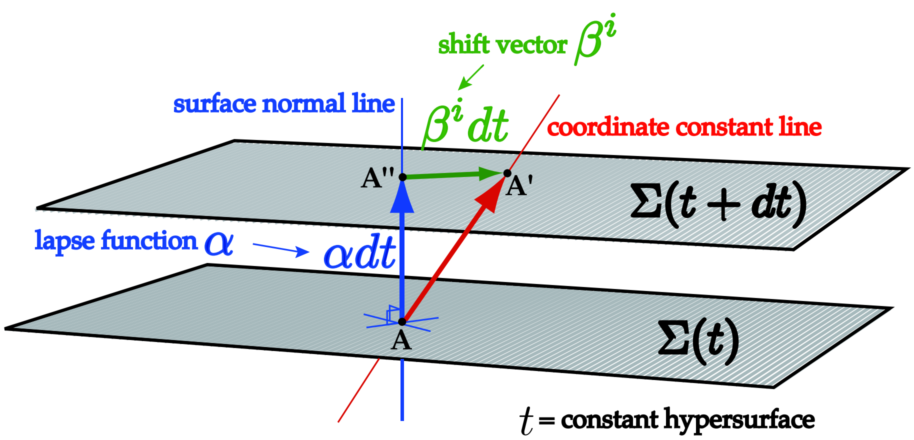

Figure 2.1: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted α or N, and β^i or N^i, respectively.

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l\beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

where α and β_j are defined as

$$\alpha \equiv 1/\sqrt{-g^{00}}, \qquad \beta_j \equiv g_{0j}. \tag{2.3}$$

and called the lapse function and shift vector, respectively.

Projection onto Σ

In order to decompose the Einstein equation into 3+1, we introduce the projection operator $\perp^\mu_\nu$ normal to n^μ,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad \gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu. \tag{2.4}$$

We also call the spatial components of γ_{ij} the intrinsic 3-metric g_{ij}.[2]

The projections of the Einstein equation can be the following three:

$$\begin{aligned} G_{\mu\nu}\, n^\mu\, n^\nu &= \kappa\, T_{\mu\nu}\, n^\mu\, n^\nu \equiv \kappa\rho_H & (2.5)\\ G_{\mu\nu}\, n^\mu \perp^\nu_i &= \kappa\, T_{\mu\nu}\, n^\mu \perp^\nu_i \equiv -\kappa J_i & (2.6)\\ G_{\mu\nu} \perp^\mu_i \perp^\nu_j &= \kappa\, T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv \kappa S_{ij}, & (2.7) \end{aligned}$$

where ρ_H, J_i and S_{ij} are energy density, momentum density and stress tensor, respectively, defined by an observer moving along $n_\mu = (-\alpha, 0, 0, 0)$. That is, the energy-momentum tensor, $T_{\mu\nu}$, is decomposed as

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}. \tag{2.8}$$

Extrinsic curvature

In order to express equations (2.5)-(2.7) tractable, we introduce the extrinsic curvature K_{ij} as

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = \cdots = \frac{1}{2\alpha}\left(-\partial_t\gamma_{ij} + \beta_{i|j} + \beta_{j|i}\right) = -\frac{1}{2}\pounds_n\gamma_{ij}. \tag{2.9}$$

Projection of the Einstein equation onto the 3-hypersurface Σ is given using the Gauss-Codacci relation: The Gauss equation,

$${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = {}^{(4)}R^\mu{}_{\nu\rho\sigma} \perp_\mu{}^\alpha \perp_\beta{}^\nu \perp_\gamma{}^\rho \perp_\delta{}^\sigma - K^\alpha{}_\gamma K_{\beta\delta} + K^\alpha{}_\delta K_{\beta\gamma}\,, \tag{2.10}$$

and the Codacci equation,

$$D_j K_i{}^j - D_i K = -{}^{(4)}R_{\rho\sigma} n^\sigma \perp_i{}^\rho\,, \tag{2.11}$$

where $K = K^i{}_i$, and D_μ is the covariant differentiation with respect to γ_{ij}.

[2] If n_μ is space-like, then $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$.

2.1.2 The Standard ADM formulation

The projections (2.5)-(2.7) can be derived as follows.

The Standard ADM formulation [63, 75]: **Box 2.1**

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{2.12}$$

$$\begin{aligned} \partial_t K_{ij} = \; & \alpha \, {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha + (D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} \\ & -\alpha\Lambda\gamma_{ij} - \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}, \end{aligned} \tag{2.13}$$

  where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\mathcal{H}^{ADM} := {}^{(3)}R + K^2 - K_{ij}K^{ij} - 2\kappa\rho - 2\Lambda \approx 0, \tag{2.14}$$

$$\mathcal{M}_i^{ADM} := D_jK^j{}_i - D_iK - \kappa J_i \approx 0, \tag{2.15}$$

  where ${}^{(3)}R = {}^{(3)}R^i{}_i$: these are called the Hamiltonian (or energy) and momentum constraint equations, respectively.

The formulation has 12 free first-order dynamical variables (γ_{ij}, K_{ij}), with 4 freedom of gauge choice (α, β_i) and with 4 constraint equations, (2.14) and (2.15). The rest freedom expresses 2 modes of gravitational waves.

What are constraints?

The ADM formulation is a kind of constrained system, like Maxwell equations.

	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$	Hamiltonian constraint (2.14)
	div $\mathbf{B} = 0$	Momentum constraints (2.15)
evolution eqs.	$\partial_t\mathbf{E} = \mathrm{rot}\ \mathbf{B} - 4\pi\mathbf{j}$	$\partial_t\gamma_{ij} = \cdots$ (2.12)
	$\partial_t\mathbf{B} = -\mathrm{rot}\ \mathbf{E}$	$\partial_t K_{ij} = \cdots$ (2.13)

Table 2.1: Maxwell equations and ADM equations.

Constraint propagations
In order to see the constraints are conserved during the evolution or not, we have to check how the constraints evolve. The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (2.14) and the momentum constraints (2.15), can be written as [33, 59]

The Constraint Propagations of the Standard ADM: **Box 2.2**

$$\begin{aligned}\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\ &\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \quad (2.16)\\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\ &\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \quad (2.17)\end{aligned}$$

From these equations, we know that **if the constraints are satisfied on the initial slice Σ, then the constraints are satisfied throughout evolution** (in principle).

Standard ADM vs Original ADM
We should remark here the 'original' ADM formulation. The evolution equations in Box 2.1 is the version by Smarr and York which is now the standard convention for numerical relativists. They adapted K_{ij} as a fundamental variable instead of the conjugate momentum π^{ij}, which was in the original Arnowitt-Deser-Misner's canonical formulation. Note that there is one replacement in (2.13) using (2.14) in the process of conversion from the original ADM to the standard ADM equations.

More detail description (vacuum case): The Hamiltonian density can be written as

$$\mathcal{H}_{GR} \quad = \quad \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \sqrt{-g}R = \alpha\sqrt{\gamma}[{}^{(3)}R - K^2 + K_{ij}K^{ij}],$$

where π^{ij} is the canonically conjugate momentum to γ_{ij},

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = -\sqrt{\gamma}(K^{ij} - K\gamma^{ij}),$$

omitting the boundary terms. The variation of $\mathcal{H}_{GR}$ with respect to α and β_i yields the constraints, and the dynamical equations are given by $\dot{\gamma}_{ij} = \dfrac{\delta \mathcal{H}_{GR}}{\delta \pi^{ij}}$ and $\dot{\pi}^{ij} = -\dfrac{\delta \mathcal{H}_{GR}}{\delta h_{ij}}$.

$$\begin{aligned}\partial_t \gamma_{ij} &= 2\frac{N}{\sqrt{\gamma}}(\pi_{ij} - (1/2)\gamma_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} &= -\sqrt{\gamma}N({}^{(3)}R^{ij} - (1/2){}^{(3)}R\gamma^{ij}) + (1/2)\frac{N}{\sqrt{\gamma}}h^{ij}(\pi_{mn}\pi^{mn} - (1/2)\pi^2) - 2\frac{N}{\sqrt{\gamma}}(\pi^{in}\pi_n{}^j - (1/2)\pi\pi^{ij}) \\ &\quad +\sqrt{\gamma}(D^iD^jN - \gamma^{ij}D^mD_mN) + \sqrt{\gamma}D_m(\gamma^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)}\end{aligned}$$

2.1.3 Matter equations

The energy-momentum tensor, $T_{\mu\nu}$, and its evolution equations are model dependent. Let us see two introductory cases briefly.

Scalar field
We start from the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2\kappa} - \epsilon\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)\right] \tag{2.18}$$

where $V(\Phi)$ is a potential of the scalar field. The parameter ϵ is the signature of the field ψ and takes the value $+1$ (normal field) or -1 (ghost field). From the variation of Lagrangian, we get [3]

$$\begin{aligned}\delta S_g &= \delta\int\sqrt{-g}\frac{R}{2\kappa}d^4x = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\delta g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] &(2.20)\\ \delta S_\phi &= \int d^4x\left(-\epsilon\frac{1}{2}\right)\left[-g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) + \phi_{,\mu}\phi_{,\nu}\right]\sqrt{-g}\delta g^{\mu\nu}, \\ &\quad + \int d^4x\epsilon\left[(\sqrt{-g}g^{\mu\nu}\phi_{,\mu})_{,\nu} - \sqrt{-g}\frac{\partial V}{\partial\phi}\right]\delta\phi. &(2.21)\end{aligned}$$

Therefore, we naturally set $T_{\mu\nu}$ as

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad T_{\mu\nu} = \epsilon\left[\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)\right]. \tag{2.22}$$

The field equation (Klein-Gordon equation) for the scalar field becomes

$$\Box\phi = \frac{\partial V}{\partial\phi}, \quad \text{that is} \quad \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\phi_{,\mu})_{,\nu} = \frac{\partial V}{\partial\phi}. \tag{2.23}$$

The equation (2.23) can be constructed also in a first-order form. For example, in a plane symmetric spacetime, $ds^2 = -\alpha^2dt^2 + 2\beta dtdx + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$, where all metric components are functions of x and t, we introduce the conjugate momentum

$$\Pi = \frac{\sqrt{\gamma}}{\alpha}(-\partial_t\phi + \frac{\beta}{\gamma_{11}}\partial_x\phi), \tag{2.24}$$

where $\gamma = \det\gamma_{ij}$, and write down eq.(2.23) into two first-order partial differential equations:

$$\begin{aligned}\partial_t\phi &= \frac{\beta}{\gamma_{11}}\partial_x\phi - \frac{\alpha}{\sqrt{\gamma}}\Pi, &(2.25)\\ \partial_t\Pi &= \alpha\sqrt{\gamma}\frac{dV}{d\phi} + \partial_x\frac{1}{\gamma_{11}}[\beta\Pi - \alpha\sqrt{\gamma}\partial_x\phi]. &(2.26)\end{aligned}$$

Consequently, the dynamical variables are γ_{ij} and K_{ij} (and ϕ and Π, when a scalar field exists).

[3] Note that from $\delta g = gg^{ab}\delta g_{ab} = -gg_{ab}\delta g^{ab}$,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}. \tag{2.19}$$

Perfect fluid [6]
We assume the perfect fluid stress-energy tensor,

$$T_{\mu\nu} = (\rho + \rho\varepsilon + p)u_\mu u_\nu + p g_{\mu\nu} \tag{2.27}$$

where ρ, ε and p are the proper mass density, the specific internal energy and the pressure, respectively, and u_μ is the 4-velocity of the fluid.

The evolution equation for the fluid is given by the Bianchi identity, $T^{\mu\nu}{}_{;\nu} = 0$. The projections $n^\mu T_\mu{}^\nu{}_{;\nu} = 0$ and $h_i{}^\mu T_{\mu;\nu}^{\ \nu} = 0$ give respectively,

$$\begin{aligned}
\partial_t(\sqrt{\gamma}\rho_H) + \partial_\ell(\sqrt{\gamma}\rho_H V^\ell) &= -\partial_\ell(\sqrt{\gamma}p(V^\ell + \beta^\ell)) + \alpha\sqrt{\gamma}pK \\
&\quad -(\partial_\ell\alpha)\sqrt{\gamma}J^\ell + \frac{\alpha\sqrt{\gamma}J^\ell J^m K_{\ell m}}{\rho_H + p}, \qquad (2.28)\\
\partial_t(\sqrt{\gamma}J_i) + \partial_\ell(\sqrt{\gamma}J_i V^\ell) &= -\alpha\sqrt{\gamma}\,\partial_i p - \sqrt{\gamma}(p + \rho_H)\,\partial_i\alpha \\
&\quad + \frac{1}{2}\alpha\sqrt{\gamma}\,(\partial_i\gamma_{kl})\frac{J^k J^l}{p + \rho_H} + \sqrt{\gamma}J_\ell(\partial_i\beta^\ell), \qquad (2.29)
\end{aligned}$$

where

$$\begin{aligned}
\rho_H &= \rho + \rho\varepsilon, \qquad (2.30)\\
V^i &= \frac{u^i}{u^0} = \frac{\alpha J^i}{p + \rho_H} - \beta^i. \qquad (2.31)
\end{aligned}$$

These represent the energy conservation and the Euler equation. The continuity equation, $(\rho u^\mu)_{;\mu} = 0$ (the GR version of $\partial_t\rho + \partial_i(\rho v_i) = 0$), gives

$$\partial_t(\sqrt{\gamma}\alpha u^0\rho) + \partial_\ell(\sqrt{\gamma}\alpha u^0\rho V^\ell) = 0. \tag{2.32}$$

The normalization of the 4-velocity, $u^\mu u_\mu = -1$, also gives us

$$\alpha u^0 = \frac{p + \rho_H}{\sqrt{(p + \rho_H)^2 - J^\ell J_\ell}}. \tag{2.33}$$

We also need the equation of state,

$$p = p(\varepsilon, \rho). \tag{2.34}$$

- For the perfect fluid, the variables are fluid components (ρ, ε, p), which are related by (2.34) so that the freedom is 2. We can say the combination (ρ, ρ_H), instead.
- The momentum J_i is also freely speciable. From (ρ, ρ_H, J_i),

$$S_{ij} = \frac{J_i J_j}{\rho + \rho_H} + p\gamma_{ij} \tag{2.35}$$

- For the total 5 variables, we have 5 equations (2.28), (2.29), and (2.32).

2.1.4 Numerical Procedures

In numerical relativity, this free-evolution approach is also the standard. This is because solving the constraints (non-linear elliptic equations) is numerically expensive, and because free evolution allows us to monitor the accuracy of numerical evolution.

The normal numerical scheme (free evolution scheme):

1. preparation of the initial data
   solve the elliptic constraints for preparing the initial data (γ_{ij}, K_{ij}).
2. time evolution
   (a) specify the gauge conditions (slicing conditions) for the lapse α and shift β_i.
   (b) evolve (γ_{ij}, K_{ij}) by using the evolution equations.
   (c) monitor the accuracy of simulations by checking the constraints.
   (d) extract physical quantities.
3. step back to 2 and repeat.

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2.2 Ashtekar 形式 (Ashtekar formulation)

ここでは，Ashtekar による一般相対論の拡張を紹介する．重力場の方程式は電磁気学の理論と非常に似た形式をもっているが，ゲージ理論的な特徴からは完全に対応していない．Ashtekar は，重力場を電場に相当する E と，場の強さに相当する接続量 A という 2 つの基本変数に変更することにより，ゲージ理論と対応する形式を導いた．

重力場と電磁気場の比較 **Box 2.3**

- 一般相対論は，時空の各点で局所座標系を選択できる（等価原理）とする．
- ゲージ理論は，理論が局所対称性（不変性）をもつ，とする．
  $\Psi(x) \longrightarrow e^{i\alpha}\Psi(x)$：大域対称性（大局的ゲージ不変性）$\Longrightarrow$ 保存則
  $\Psi(x) \longrightarrow e^{i\alpha(x)}\Psi(x)$：局所対称性（局所的ゲージ不変性）$\Longrightarrow$ 相互作用

	重力場	電磁気場
自由場の方程式	$\dfrac{d^2}{dt^2}\boldsymbol{x} = 0$	$(i\gamma^\mu\partial_\mu - m)\Psi = 0$
対称性	一般座標変換 $(x^\mu \longrightarrow x^{\mu'})$	局所ゲージ変換 $(\Psi \longrightarrow e^{i\alpha(x)}\Psi)$
共変微分	$\nabla_\mu = \partial_\mu + \Gamma$	$D_\mu \equiv \partial_\mu + iqA_\mu$
接続係数	${\Gamma^\mu}_{\alpha\beta}$ (局所的に 0 とできる)	A_μ (直接観測できない，ゲージ依存)
相互作用	$\dfrac{d^2x^\alpha}{d\tau^2} + {\Gamma^\alpha}_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = 0$	$(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\,\Psi = 0$
共変微分の非可換性	曲率テンソル ${R^\mu}_{\alpha\beta\gamma}$ (観測可能量)	電磁場テンソル $F_{\mu\nu}$ (ゲージ不変量)

2.2.1 From Einstein to Ashtekar; transformation of Lagragians

Here we try to understand Ashtekar's new formulation of general relativity [1] as the steps of rewriting the Lagragian formalism [2, 3]. Note that Ashtekar himself introduced his new variables through a kind of canonical transformation in the Hamiltonian formalism. [4]

Einstein-Hilbert action (metric $g_{\mu\nu}$)
First let us start from the Einstein-Hilbert action

$$S_E[g] = \int d^4x\,\sqrt{-g}\,R(g) \quad \sim \quad g\,\partial^2 g + (\partial g)^2 \tag{2.36}$$

which can be put into a canonical theory by means of the ADM method. That is, the metric $g_{\mu\nu}$ is decomposed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad = \quad -N^2\,dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.37}$$

[4]This subsection refers much to H. Ikemori's note in the proceedings of the 1st JGRG workshop at Tokyo (1991).

Theory	action		order of ∂_μ	independent variables
Einstein	Einstein-Hilbert action	S_E	2nd order	metric ($g_{\mu\nu}$)
	Palatini action	S_P	1st order	metric ($g_{\mu\nu}$) & Affine connection ($\Gamma^\lambda_{\mu\nu}$)
	Tetrad Palatini action	S_T	1st order	tetrad (${e^a}_\mu$) & spin connection (ω^{ab}_μ)
AshtekarOriginal	Jacobson-Smolin action	${}^+S_T$	1st order	tetrad (${e^a}_\mu$) & self-dual connection (${}^+\omega^{ab}_\mu$)

Table 2.2: Steps to the Ashtekar theory via Lagrangian formalism.

S_P with the Christoffel condition for Γ	$\Longrightarrow$	S_E
S_T with the Levi-Civita condition for ω^{ab} (torsion free condition)	$\Longrightarrow$	S_P
${}^+S_T$ with the Bianchi identity for R_{ab} ($R_{\mu[\nu\alpha\beta]}=0$)	$\Longrightarrow$	S_T

Table 2.3: Steps to the Ashtekar theory and their extensions.

where N is the lapse function (the same with α) and N^i is the shift vector (β^i) [5], and q_{ij} is the three metric. That is,

$$g_{\mu\nu} \;=\; \begin{pmatrix} -N^2+N_kN^k & N_j \\ N_i & q_{ij} \end{pmatrix}. \tag{2.38}$$

The canonical action, then, is given by

$$S_E[q,p] = \int d^4x\,[\dot{q}_{ij}\,p^{ij} - N\,\mathcal{C}_\mathcal{H} - N_i\,{\mathcal{C}_\mathcal{M}}^i] \tag{2.39}$$

where

$$\mathcal{C}_\mathcal{H} \;:=\; G_{ijkl}p^{ij}p^{kl} - \sqrt{q}^{(3)}R \tag{2.40}$$

$${\mathcal{C}_\mathcal{M}}^i \;:=\; -2\nabla_j p^{ij} \tag{2.41}$$

where $G_{ijkl} = \dfrac{1}{2\sqrt{q}}(q_{ik}q_{jl}+q_{il}q_{jk}-q_{ij}q_{kl})$.

Palatini action (metric $g_{\mu\nu}$, Affine connection $\Gamma^\alpha_{\mu\nu}$)
The Einstein-Hilbert action (2.36) consists of the terms with the second-order derivative or the square of the first order derivative of metric $g_{\mu\nu}$. Palatini's idea is to introduce the Affine connection $\Gamma^\alpha_{\mu\nu}(=\Gamma^\alpha_{\nu\mu})$ to be independent to the metric $g_{\mu\nu}$. The Palatini action

$$S_P[g,\Gamma] = \int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,R_{\mu\nu}(\Gamma) \quad \sim \quad g(\partial\Gamma+\Gamma\,\Gamma) \tag{2.42}$$

which is equivalent to the Einstein-Hilbert action (2.36) , $S_P = S_E$, when a connection $\Gamma^\lambda_{\mu\nu}$ satisfies the definition of the Christoffel symbol, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(g) \sim \partial g$. This condition is derived from the variation with respect to $\Gamma^\alpha_{\mu\nu}$,

$$\frac{\delta}{\delta\Gamma^\alpha_{\mu\nu}} S_P[g,\Gamma] = 0. \tag{2.43}$$

The action (2.42) contains up to the first-order derivatives.

[5]We use N and N^i instead of α and β, according to the convensions throughout this section.

テトラド (tetrad), トライアド (triad), スピン接続 (spin connection) **Box 2.4**

- 各時空点ごとに局所的な 4 次元直交座標系を定義する．直交座標の基底ベクトルを E^I として，これを任意の座標系で表したもの E^I_μ をテトラド（4 脚場）と呼ぶ．

$$g_{\mu\nu} = E^I_\mu \, E^J_\nu \, \eta_{IJ}, \qquad \eta_{IJ} = \text{diag}\,(-1,1,1,1)$$

- 同様に，3 次元空間で局所的に直交座標を導入した基底ベクトルをトライアド（3 脚場）と呼ぶ．

$$g_{ij} = E^a_i \, E^b_j \, \delta_{ab}$$

- 局所直交座標系の成分を持つベクトルに対する共変微分を

$$\nabla_\mu V^I = \partial_\mu V^I + \omega^I_{\mu J} V^J$$

と表すとき，$\omega^I_{\mu J}$ をスピン接続と呼ぶ．具体的には，

$$\omega^{IJ}_\mu \;=\; E^{I\nu}\nabla_\mu E^J_\nu = E^{\nu I}\partial_{[\mu}E^J_{\nu]} - E_{\mu K}E^{\rho I}E^{\nu J}\partial_{[\rho}E^K_{\nu]} + E^{\rho J}\partial_{[\rho}E^I_{\mu]}$$

Tetrad Palatini action (tetrad e^a_μ, spin connection ω^{ab}_μ)
The next step is the introduction of the internal symmetry, that is, to introduce the local Lorentz transformation as a gauge symmetry. We employ the orthonormal tetrad e^a_μ in stead of the metric $g_{\mu\nu}$, which acts as a basis of the local Lorentz frame. We also employ the spin connection $\omega^{ab}_\mu (= -\omega^{ba}_\mu)$ instead of the Affine connection $\Gamma^\alpha_{\mu\nu}$, which acts as a gauge field of the local Lorentz albebra so(3,1). The internal indices $a, b, \cdots$ are lowering and raising by the metric $\eta_{ab} = \text{diag}(-1,1,1,1)$. The tetrad plays a role of a square root of the metric,

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu \, e^b_\nu. \tag{2.44}$$

The Palatini action in the tetrad form is written as

$$S_T[e,\omega] \;=\; \int d^4x \, e \, E^\mu_a \, E^\nu_b \, R^{ab}{}_{\mu\nu}(\omega) \tag{2.45}$$

where e is the determinant of e^a_μ, and the E^μ_a is the inverse tetrad,

$$e := \det\, e^a_\mu = \sqrt{-g}, \qquad E^\mu_a = e^b_\nu \, g^{\mu\nu}\, \eta_{ab}. \tag{2.46}$$

Now that the internal symmetry is taken into account, the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$ will be replaced by the curvature $R^{ab}_{\mu\nu}(\omega)$ of the spin connection ω^{ab}_μ defined by

$$R^{ab}_{\mu\nu}(\omega) := \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu, \tag{2.47}$$

that is to say, the curvature 2-form R^{ab} is defined from the spin connection 1-form ω^{ab} by

$$R^{ab}(\omega) := d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{2.48}$$

in the language of the differential forms. The action (2.45), then, can be expressed also as

$$S_T[e,\omega] \;=\; \int \frac{1}{2}\,\varepsilon_{abcd} R^{ab}(\omega) \wedge e^c \wedge e^d. \tag{2.49}$$

The tetrad Palatini action (2.45) is equivalent to S_P only when the spin connection equals to the Levi-Civita connection $\omega^{ab} = \omega^{ab}(e)$, that is the torsion free condition,

$$De^a := de^a + \omega^a{}_b \wedge e^b = 0 \tag{2.50}$$

which is derived from the variation respect to ω^{ab},

$$\frac{\delta}{\delta\omega^{ab}} S_T[e,\omega] = 0. \tag{2.51}$$

Self-dual action (tetrad e^a_μ, self-dual connection ${}^+\omega^{ab}_\mu$)

The last step to the Ashtekar's formulation is the introduction of the self-dual connection ${}^+\omega^{ab}_\mu$. Note that the self-duality here is with respect to the internal indices and not with the space-time indices.

Self-duality, anti-self-duality: **Box 2.5**

Suppose F_{ab} is an anti-symmetric tensor, then the duality transformation is defined as

$${}^*F_{ab} := \frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}, \tag{2.52}$$

making use of the totally anti-symmetric symbol, ε^{abcd}. Note that the dual of dual is equal to the minus of the original,

$${}^*({}^*F_{ab}) = -F_{ab} \tag{2.53}$$

when we choose the Lorentzian signature and use the metric η_{ab} for lowering and raising the internal indices. Thus, the duality transformation (2.52) corresponds to $\pm i$ operation. If we suppose the complex combinations

$${}^\pm F_{ab} = \frac{1}{2}(F_{ab} \mp i\,{}^*F_{ab}), \tag{2.54}$$

then this satisfies the eigen-equations

$${}^*({}^\pm F_{ab}) = \pm i\,{}^\pm F_{ab}. \tag{2.55}$$

The notion of self-duality means an eigen-state of the duality transform operation and we call ${}^+F_{ab}$ self-dual part of F_{ab} (and ${}^-F_{ab}$ anti-self-dual part of F_{ab}).

The spin connection 1-form ω^{ab} which has a pair of anti-symmetric internal indices can be uniquely decomposed into the self-dual and anti-self-dual part,

$$\omega^{ab} = {}^+\omega^{ab} + {}^-\omega^{ab}. \tag{2.56}$$

The substitution of this relation into the definition of the curvature 2-form R^{ab} results in

$$R^{ab}(\omega) = R^{ab}({}^+\omega^{ab} + {}^-\omega^{ab}) = R^{ab}({}^+\omega^{ab}) + R^{ab}({}^-\omega^{ab}) := {}^+R^{ab} + {}^-R^{ab}, \tag{2.57}$$

which means that the R^{ab} can also be decomposed additively according to the decomposition with respect to the self-duality.

The previously mentioned tetrad-Palatini action (2.49)

$$S_T[e,\omega] = \int \frac{1}{2}\varepsilon_{abcd}R^{ab}(\omega)\wedge e^c \wedge e^d = \int {}^*R_{cd}(\omega)\wedge e^c\wedge e^d \tag{2.58}$$

is decomposed as

$$\begin{aligned} S_T[e,\omega] &= \int {}^*R_{ab}({}^+\omega)\wedge e^a\wedge e^b + \int {}^*R_{ab}({}^-\omega)\wedge e^a\wedge e^b \\ &:= {}^+S_T[e,{}^+\omega] + {}^-S_T[e,{}^-\omega] \end{aligned}$$

with regard to the contributions of self-dual and anti-self-dual connections.

Ashtekar's idea is to consider just a self-dual part of the action. The equivalence to the Einstein-Hilbert action is still preserved with regard to just a half of duality components.

When the self-dual connection is equal to the self-dual part of the Levi-Civita connection

$${}^+\omega^{ab} = {}^+\omega^{ab}(e), \tag{2.59}$$

the variation $\frac{\delta}{\delta {}^+\omega}{}^+S_T[e,{}^+\omega] = 0$ is satisfied. It reduces the self-dual action equals to the Einstein-Hilbert action with a factor half, ${}^+S_T[e,{}^+\omega] = \frac{1}{2}S_T[e,\omega(e)] = \frac{1}{2}S_E[g]$.

The equivalence to the Einstein theory requires additional condition. Since the curvature of self-dual connection is given by its complex combination

$$R^{ab}({}^+\omega) = {}^+R^{ab}(\omega) = \frac{1}{2}\left(R^{ab}(\omega) - i\,{}^*R^{ab}(\omega)\right), \tag{2.60}$$

the action turns out to be

$$\begin{aligned} {}^+S_T[e,{}^+\omega(e)] &= \frac{1}{2}\int {}^*\left(R_{ab}(\omega(e)) - i\,{}^*R_{ab}(\omega(e))\right)\wedge e^a\wedge e^b \\ &= \frac{1}{2}\int \left({}^*R_{ab}(\omega(e)) + iR_{ab}(\omega(e))\right)\wedge e^a\wedge e^b \\ &= \frac{1}{2}S_T[e,\omega(e)] + i\frac{1}{2}\int R_{ab}(\omega(e))\wedge e^a\wedge e^b = \frac{1}{2}S_E[g] + 0, \end{aligned} \tag{2.61}$$

where the last imaginary term is vanished by virtue of the 1st Bianchi identity

$$R^a{}_b(\omega(e))\wedge e^b \equiv 0 \tag{2.62}$$

which is the cyclic identity $R_{\mu[\nu\alpha\beta]} = 0$ in the tensor form.

This means that the self-dual action would lead to the same equation of motion as the Einstein equation so far as the tetrad or equivalently the metric is concerned. The anti-self-dual action can also play the same role with the above discussion.

2.2.2 New Variables

The Ashtekar formalism can be regarded as a canonical theory starting from the self-dual action,

$${}^+S_T[e,{}^+\omega] = \int d^4x\, e\, E^\mu_a\, E^\nu_b\, R^{ab}{}_{\mu\nu}({}^+\omega). \tag{2.63}$$

where E^μ_a is the inverse tetrad, defined as $E^\mu_a := E^b_\nu g^{\mu\nu}\eta_{ab}$, which makes the inverse space-time metric as $q^{\mu\nu} = \eta^{ab}E^\mu_a E^\nu_b$ as we mentioned before. See notations in Table 2.4.

4-spacetime indices	$\mu, \nu, \rho, \cdots$	$0, \cdots, 3;$	raise and lower indices by	$g_{\mu\nu}$
SO(1,3) indices	$I, J, K, \cdots$	$(1), \cdots, (3)$		$\eta^{IJ} = \mathrm{diag}(-1,1,1,1)$
3-spacetime indices	$i, j, k, \cdots$	$1, \cdots, 3$		γ_{ij}
SO(3) indices	$a, b, c, \cdots$	$(1), \cdots, (3)$		δ_{ab}
volume forms	ϵ_{abc}	$\epsilon_{abc}\epsilon^{abc} = 3!$		
density	e	$\epsilon_{ijk} = e,$	$\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$	$\underset{\sim}{\epsilon}_{123} = 1, \tilde{\epsilon}^{123} = 1$
tetrad (inverse tetrad)	E^I_μ (E^μ_I)	$g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$	$E^\mu_I := e^J_\nu g^{\mu\nu}\eta_{IJ}$	
spin connection	ω^{IJ}_μ	$\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu.$		
curvature 2-form	$F^a_{\mu\nu}$	$F^a_{\mu\nu} := \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu - i\epsilon^a{}_{bc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$		

Table 2.4: Notations in §2.2.

Let us consider the $3+1$ decomposition of the self-dual theory in the tetrad form after the ADM decomposition. The spatial component of the tetrad, E^i_I acts as an inverse triad since it produces the inverse 3-metric, $q^{ij} = E^i_I E^j_I$. We further impose the gauge condition

$$E^0_I = 0 \tag{2.64}$$

then, the inverse tetrad is expressed as

$$E^\mu_a = \begin{pmatrix} E^0_0 & E^i_0 \\ E^0_I & E^i_I \end{pmatrix} = \begin{pmatrix} 1/N & -N^i/N \\ 0 & E^i_I \end{pmatrix} \tag{2.65}$$

Note that (2.64) allows $E^\mu_0 = (1/N, -N^i/N)$ as a normal vector field to the space like hypersurface spanned by the condition of $t =$const. This gauge choice is not a restriction on the general coordinate transformation but on the local Lorentz transformation.

New variables (densitized inverse triad $\tilde{E}^i_a$, self-dual connection ${}^+\mathcal{A}^a_\mu$)
The key feature of Ashtekar's formulation of general relativity [1] is the introduction of a self-dual connection as one of the basic dynamical variables.

Ashtekar variables (New variables) [1]: **Box 2.6**

The geometry in the Ashtekar formulation is expressed by the pair of new variables, $(\tilde{E}^i_a, \mathcal{A}^a_i)$.

- self-dual connection (Ashtekar connection)
  We define $so(3,\mathrm{C})$ connections

$$ {}^{\pm}\mathcal{A}^I_\mu := \omega^{0I}_\mu \mp \frac{i}{2}\epsilon^I{}_{JK}\,\omega^{JK}_\mu, \tag{2.66} $$

  where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only the self-dual part of the connection ${}^{+}\mathcal{A}^a_\mu$ and to use its spatial part ${}^{+}\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^{+}\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

- densitized inverse triad $\tilde{E}^i_a$

$$ \tilde{E}^i_a := eE^i_a, \tag{2.67} $$

  where $e := \det E^a_i$ is a density.

This pair forms a canonical set.

For later convenience, we denote the relation,

$$ e^2 = \det g_{ij} = \det \tilde{E}^i_a = (\det E^a_i)^2 = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a\tilde{E}^j_b\tilde{E}^k_c, \tag{2.68} $$

where $\epsilon_{ijk} := \epsilon_{abc}E^a_iE^b_jE^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$. [6]

In the case of pure gravitational spacetime with cosmological constant Λ, the Hilbert action takes the form

$$ {}^{+}S_A[\tilde{E}, {}^{+}\mathcal{A}] \;=\; \int \mathrm{d}^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a + \underset{\sim}{N}\,\mathcal{C}_H + N^i\,\mathcal{C}_{Mi} + \mathcal{A}^a_0\,\mathcal{C}_{Ga}], \tag{2.69} $$

where $\underset{\sim}{N} := e^{-1}N$. The latter terms are understood as Lagrange multipliers ($\mathcal{A}^a_0, N^i$, and $\underset{\sim}{N}$) and their accompanied constraints, $\mathcal{C}_H \approx 0, \mathcal{C}_{Mi} \approx 0$ and $\mathcal{C}_{Ga} \approx 0$, which are

$$ \mathcal{C}_H \;:=\; (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_bF^c_{ij} - 2\Lambda\,\det\tilde{E} \tag{2.70} $$

$$ \mathcal{C}_{Mi} \;:=\; F^a_{ij}\tilde{E}^j_a \tag{2.71} $$

$$ \mathcal{C}_{Ga} \;:=\; \mathcal{D}_i\tilde{E}^i_a \tag{2.72} $$

where $F^a_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}^a_{\nu]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form, and $\mathcal{D}_i\tilde{E}^j_a := \partial_i\tilde{E}^j_a - i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$.

[6] When $(i,j,k) = (1,2,3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.

The Ashtekar formulation [1]: **Box 2.7**

The dynamical variables are $(\tilde{E}^i_a, \mathcal{A}^a_i)$.

$$\mathcal{A}^a_i := \omega^{0a}_i - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i \quad (2.73)$$

$$\tilde{E}^i_a := eE^i_a \quad (2.74)$$

- The evolution equations for a set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ are

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c, \quad (2.75)$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + 2\Lambda \underset{\sim}{N} \tilde{e}^a_i, \quad (2.76)$$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c \mathcal{A}^b_j X^{ji}_c$, and $F^a_{ij} := 2\partial_{[i}\mathcal{A}^a_{j]} - i\epsilon^a{}_{bc}\mathcal{A}^b_i \mathcal{A}^c_j$.

- Constraint equations: (Hamiltonian, momentum and Gauss constraints)

$$\mathcal{C}^{\rm ASH}_H := (i/2)\epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - 2\Lambda \det \tilde{E} \approx 0, \quad (2.77)$$

$$\mathcal{C}^{\rm ASH}_{Mi} := F^a_{ij}\tilde{E}^j_a \approx 0, \quad (2.78)$$

$$\mathcal{C}^{\rm ASH}_{Ga} := \mathcal{D}_i \tilde{E}^i_a \approx 0. \quad (2.79)$$

- Gauge variables are the lapse function $\underset{\sim}{N}$, the shift vector N^i, and the triad lapse $\mathcal{A}^a_0$.

The set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ forms a canonical relation,

$$\{\tilde{E}^i_a(x), \tilde{E}^j_b(y)\} = 0, \quad (2.80)$$

$$\{\mathcal{A}^a{}_i(x), \tilde{E}^j_b(y)\} = i\delta^j{}_i \delta^a{}_b \delta(x-y), \quad (2.81)$$

$$\{\mathcal{A}^a{}_i(x), \mathcal{A}^b{}_j(y)\} = 0. \quad (2.82)$$

The dynamical degrees of freedom are summarized in Table 2.5.

covariant vars.		canonical vars.	gauge conditions	gauge vars.
E^μ_a (16)	$\Longrightarrow$	$\tilde{E}^i_a$ (9)	$E^0_a = 0$ (3)	N^i (3) + $\underset{\sim}{N}$ (1)
$+\omega^{ab}_\mu$ (12)	$\Longrightarrow$	$\mathcal{A}^a_i$ (9)		$\mathcal{A}^a_0$ (3)

Table 2.5: Dynamical degrees of freedom.

2.2.3 Einstein vs. Ashtekar

Let us compare the features of Ashtekar's formulation of general relativity with the conventional one. See Table 2.6 for brief summary.

From the viewpoint of classical dynamics

If we apply this formulation to the time evolution of Lorenzian space-time, the bottleneck is the additional constraint $\mathcal{C}_G$ and the reality conditions.

Einstein theory	Ashtekar theory
purely geometrical theory 2nd order derivative theory dynamical eqs are non-polynomial does contain the inverse of variables does not admit degenerate metric	gauge theoretical features 1st order derivative theory dynamical els are polynomial dynamical eqs are (weakly) hyperbolic does not contain the inverse of variables does admit degenerate metric
constraints are $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{M}}$	additional constraint, $\mathcal{C}_{\mathcal{G}}$
	additional "reality condition" to recover real geometry

Table 2.6: Einstein vs Ashtekar theories

- Additional gauge variables ($\mathcal{A}_0^a$)
  When we consider the space-time evolution as foliations of space-like hypersurfaces, the ADM formulation says that we have gauge freedoms which are expressed with the lapse function, α or N, and with the shift vector, β^i or N^i. In Ashtekar's theory, there is additional gauge variable, $\mathcal{A}_0^a$, which we named "triad lapse" [7]. This freedom appears due to the introduction of the internal indices. We somehow have to spacify $\mathcal{A}_0^a$ in a proper manner. See Fig. 2.2.

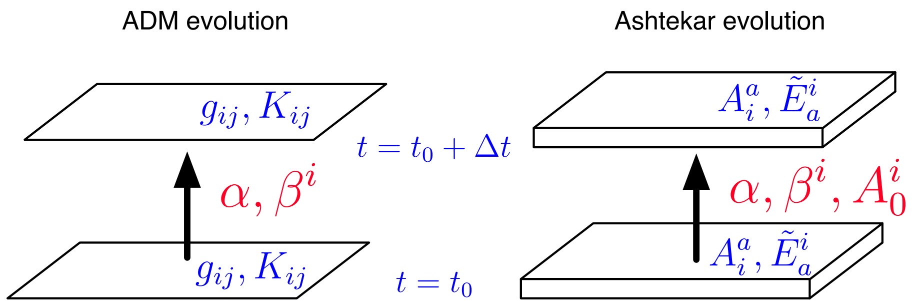

Figure 2.2: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted α or N, and β^i or N^i, respectively.

- Additional "Gauss constraint" ($\mathcal{C}_{\mathcal{G}}$)
  In ADM formulation, we have Hamiltonian (scalar) and momentum (vector) constraint equations. These are the first-class, and we have to solve these 4-equations when we prepare the initial data for time evolutions.

  In Ashtekar's theory, we have additional Gauss constraint ($\mathcal{C}_{\mathcal{G}}$), which has 3 components. The set of constraints forms the first-class, therefore we have to solve them when we prepare the initial data.

- Reality conditions to recover classical GR
  We have to consider the reality conditions when we use this formalism to describe the classical Lorentzian spacetime. The reality conditions are, so far, posed on the metric or the triad.

  Fortunately, the metric will remain on its real-valued constraint surface during time evolution automatically if we prepare initial data which satisfies the reality condition[6].

  More practically, we further can require that triad is real-valued. But again this reality condition appears as a gauge restriction on $\mathcal{A}_0^a$[9], which can be imposed at every time step.

[7] Actually, HS asked Ashtekar to name this variable, and he named it after a minute.

From the fact that the reality of the spacetime is conserved if we solve reality conditions initially, so we propose to prepare ADM initial data for evolution in Ashtekar's variables by transforming variables and introducing internal variables as they satisfy $\mathcal{C}_G$.

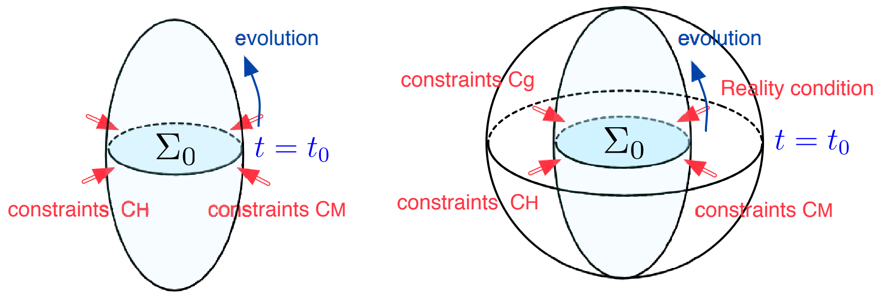

Figure 2.3: Images of constraints, as a solution space in the Einstein manifold. (Left) The ADM approach has two constraints, $\mathcal{C}_\mathcal{H}$ and $\mathcal{C}_{\mathcal{M}i}$, which specify a solution so as it satisfies the Einstein equations. (Right) The Ashtekar formulation has another constraint, $\mathcal{C}_{\mathcal{G}_a}$, and reality condition.

In our actual simulation, we prepare our initial data using the standard ADM approach, so that we have no difficulties in maintaining these reality conditions.

	ADM formulation		connection formulation			
			Re(metric)		Re(triad)	
						Σ_0 (Σ_t)
variables	γ_{ij}	6	$\tilde{E}^i_a$	18	$\tilde{E}^i_a$	18 (9)
	K_{ij}	6	$\mathcal{A}^a_i$	18	$\mathcal{A}^a_i$	18 (9)
gauge	N	1	N	1	N	1 (1)
	N^i	3	N^i	3	N^i	3 (3)
			$\mathcal{A}^a_0$	6	$\mathcal{A}^a_0$	3 (3)
constraints	$\mathcal{C}_H$	1	$\mathcal{C}_H$	1	$\mathcal{C}_H$	1 (1)
	$\mathcal{C}_{Mi}$	3	$\mathcal{C}_{Mi}$	3	$\mathcal{C}_{Mi}$	3 (3)
			$\mathcal{C}_{Ga}$	6	$\mathcal{C}_{Ga}$	6 (3)
reality condition			primary	6 (Σ_0)	primary	9 (0)
			secondary	6 (Σ_0)	secondary	6 (0)
GW freedom		2×2		2×2		2×2

Table 2.7: Number of components in actual simulations. We here count the numbers of freedom in components, i.e. one complex number has two components.

2.2.4 Reality conditions

Notice that the metric in Ashtekar's formulation is not necessary to be real. In order to recover the real metric, we must impose the reality conditions.

To ensure the metric is real-valued, we need to impose two conditions; the primary is that the doubly densitized contravariant metric $\tilde{\tilde{\gamma}}^{ij} := e^2\gamma^{ij}$ is real,

$$\Im(\tilde{E}^i_a \tilde{E}^{ja}) = 0, \tag{2.83}$$

and the secondary condition is that the time derivative of $\tilde{\tilde{\gamma}}^{ij}$ is real,

$$\Im\{\partial_t(\tilde{E}^i_a\tilde{E}^{ja})\} = 0. \quad (2.84)$$

Using the equations of motion for $\tilde{E}^i_a$ (2.75), the Gauss constraint (2.79) and the primary reality condition (2.83), we can replace the secondary condition (2.84) with a different constraint

$$W^{ij} := \Re(\epsilon^{abc}\tilde{E}^k_a\tilde{E}^{(i}_b\mathcal{D}_k\tilde{E}^{j)}_c) \approx 0, \quad (2.85)$$

which fixes six components of $\mathcal{A}^a_i$ and $\tilde{E}^i_a$. Moreover, in order to recover the original lapse function $N := \underset{\sim}{N}e$, we demand $\Im(N/e) = 0$, i.e. the density e be real and positive. This requires that e^2 be positive, i.e.

$$\det\tilde{E} > 0. \quad (2.86)$$

The secondary condition of (2.86),

$$\Im[\partial_t(\det\tilde{E})] = 0, \quad (2.87)$$

is automatically satisfied (see [9]). Therefore, in order to ensure that e is real, we only require (2.86).

Rather stronger reality conditions are sometimes useful in Ashtekar's formalism for recovering the real 3-metric and extrinsic curvature. These conditions are

$$\begin{aligned} \Im(\tilde{E}^i_a) &= 0 \quad (2.88)\\ \text{and}\ \Im(\dot{\tilde{E}}^i_a) &= 0, \quad (2.89)\end{aligned}$$

and we call them the "primary triad reality condition" and the "secondary triad reality condition", respectively. Using the equations of motion of $\tilde{E}^i_a$, the Gauss constraint (2.79), the metric reality conditions (2.83), (2.84) and the primary condition (2.88), we see that (2.89) is equivalent to [9]

$$\Re(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2}e^{-1}e^b_i\underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}^i_b + N^i\Re(\mathcal{A}^a_i). \quad (2.90)$$

From this expression we see that the second triad reality condition restricts the three components of "triad lapse" vector $\mathcal{A}^a_0$. Therefore (2.90) is not a restriction on the dynamical variables ($\mathcal{A}^a_i$ and $\tilde{E}^i_a$) but on the slicing, which we should impose on each hypersurface. Thus the second triad reality condition does not restrict the dynamical variables any further than the second metric condition does.

2.2.5 Trick for passing a degenerate point

Next, we examine the possibilities of passing a degenerate point. A 'degenerate point', we use here, is defined as the point in the spacetime where the density e of 3-space vanishes. In the Ashtekar formulation, all the equations do not include any inverse of e apparently, so that we expect we can 'pass' such a degenerate point.

In order to say 'pass' degenerate points, we start from requiring the finiteness of the fundmental variables (and their derivatives), $\tilde{E}^i_a, \mathcal{A}^a_i, N/e, N^i, \mathcal{A}^a_0$, and the condition that the calculation must be finished in finite coordinate time. Although these are natural conditions for pursuiting the evolutions of spacetime, we concluded that continuing evolutions including a degenerate point in its foliation of 3-space is generally break one of above conditions. The difficulties are that the term ω^{bc}_i in $\mathcal{A}^a_i$ diverges generally and a requirement of finite coordinate time fails when we pass a degenerate point. This means generally we face a trouble when we pass a degenerate point directly in Lorentzian spacetime even if we use Ashtekar's variables.

However, since the variables are originally defined as complex numbers, if we are allowed to break the reality condition locally in the neibour of a degenerate point, which we also assume its degeneracy exists only on the real section of spacetime, then we can 'pass' a degenerate point by such a 'deformed slice approach'. Note that, in our proposal, the foliation maintains $3+1$ dimensions $\boldsymbol{R}^3 \times \boldsymbol{R}$ in $\boldsymbol{C}^4$.

In order to recover a real metric spacetime again later, we have to impose 'reality recovering condition' on the foliation, which requires us to determine shooting parameters in complex part of gauge variables. We showed this technique actually works, by demonstrating a numerical evolution for an analytic solution of degenerate point in flat spacetime[8]. We see that the time evolution does work properly in the sense that the real part of evolution recovers the analytic evolutions and the imaginary part of metric vanishes asymptotically.

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2.3 高次元の場合 (Higher-dimensional ADM formulation)

2.3.1 Application to $N+1$-dimensional space-time

Let us describe how the ADM equations turns to be in higher-dimensional cases. The set of equations are shown in [1] in the context of constraint propagation equations.

The Standard ADM formulation in $N+1$-dim. [1] **Box 2.6**

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the N-metric and extrinsic curvature. The N-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \tag{2.91}$$

$$\partial_t K_{ij} = \alpha^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell{}_j K_{i\ell} - D_i D_j \alpha + \beta^k(D_k K_{ij}) + (D_j\beta^k)K_{ik} + (D_i\beta^k)K_{kj} - \frac{2\alpha}{N-1}\gamma_{ij}\Lambda, -\kappa\alpha\left(S_{ij} - \frac{1}{N-1}\gamma_{ij}T\right) \tag{2.92}$$

  where $K = K^i{}_i$, and ${}^{(N)}R_{ij}$ and D_i denote N-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\text{Hamiltonian constr.} \quad \mathcal{H}^{ADM} := {}^{(N)}R + K^2 - K_{ij}K^{ij} - 2\Lambda - 2\kappa\rho \approx 0,$$
$$\text{momentum constr.} \quad \mathcal{M}_i^{ADM} := D_j K^j{}_i - D_i K - \kappa J_i \approx 0,$$

  where ${}^{(N)}R = {}^{(N)}R^i{}_i$.

2.3.2 $N+1$-formalism in Einstein-Gauss-Bonnet gravity

As one of the application to an alternative gravity model, Gauss-Bonnet gravity is extensively studied. Since dynamical studies have not yet been done, we first set up the ADM-type decomposition of the equations[2].

Gauss-Bonnet action

Einstein-Gauss-Bonnet action is given by

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}) + \mathcal{L}_{\text{matter}}\right] \tag{2.93}$$

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

where κ^2 is the $(N+1)$-dimensional gravitational constant, $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\text{matter}}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature and the matter Lagrangian, respectively. This action will reproduce the standard $(N+1)$-dimensional Einstein gravity, if we set the coupling constant $\alpha_{GB}(\geq 0)$ equals to zero. [8]

[8] The Greek indices $(\mu, \nu, \cdots)$ move $1, \cdots, N+1$, while the Latin indices $(i, j, \cdots)$ move $1, \cdots, N$.

The action gives the gravitational equation

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu} \tag{2.94}$$

where

$$\begin{aligned}
\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu}, \\
\mathcal{H}_{\mu\nu} &= 2\left[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\ \nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}^{\ \alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\right] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}, \\
\mathcal{T}_{\mu\nu} &\equiv -2\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{matter}}.
\end{aligned}$$

Projections to Hypersurface Σ_N (spacelike or timelike)

The projection operator,

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad n_\mu n^\mu = \varepsilon \tag{2.95}$$

where n_μ is the unit-normal vector to Σ with n_μ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Σ is spacelike (timelike) if n_μ is timelike (spacelike).

The induced N-dimensional metric γ_{ij} is defined by $\gamma_{ij} = \perp_{ij}$.

The projections of the gravitational equation:

$$\begin{aligned}
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\, n^\mu\, n^\nu &= \kappa^2\, T_{\mu\nu}\, n^\mu\, n^\nu =: \kappa^2 \rho_H, && (2.96)\\
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\, n^\mu \perp^\nu_{\ \rho} &= \kappa^2\, T_{\mu\nu}\, n^\mu \perp^\nu_{\ \rho} =: -\kappa^2 J_\rho, && (2.97)\\
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}) \perp^\mu_{\ \rho} \perp^\nu_{\ \sigma} &= \kappa^2\, T_{\mu\nu} \perp^\mu_{\ \rho} \perp^\nu_{\ \sigma} =: \kappa^2 S_{\rho\sigma}, && (2.98)
\end{aligned}$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \;\; T = -\rho_H + S^\ell_{\ \ell}$$

Introduce the extrinsic curvature K_{ij}

$$K_{ij} \;:=\; -\frac{1}{2}\pounds_n h_{ij} = -\perp^\alpha_{\ i}\perp^\beta_{\ j}\nabla_\alpha n_\beta, \tag{2.99}$$

where $\pounds_n$ denotes the Lie derivative in the n-direction and ∇ and D_i is the covariant differentiation with respect to $g_{\mu\nu}$ and γ_{ij}, respectively.

- Projection of the $(N+1)$-dimensional Riemann tensor onto Σ_N

$$\text{Gauss eq.} \quad \mathcal{R}_{\alpha\beta\gamma\delta} \perp^\alpha_{\ i}\perp^\beta_{\ j}\perp^\gamma_{\ k}\perp^\delta_{\ l} = R_{ijkl} - \varepsilon K_{ik}K_{jl} + \varepsilon K_{il}K_{jk}, \tag{2.100}$$

$$\text{Codacci eq.} \quad \mathcal{R}_{\alpha\beta\gamma\delta} \perp^\alpha_{\ i}\perp^\beta_{\ j}\perp^\gamma_{\ k}\, n^\delta = -2D_{[i}K_{j]k}, \tag{2.101}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta} \perp^\alpha_{\ i}\perp^\gamma_{\ k}\, n^\beta\, n^\delta = \pounds_n K_{ik} + K_{i\ell}\, K^\ell_{\ k}\,, \tag{2.102}$$

- Curvature relations

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\nu\sigma} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu} \\
&\quad - n_\rho D_\mu K_{\nu\sigma} + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho}) \\
&\quad + n_\mu n_\rho K_{\nu\alpha}K^\alpha_{\ \sigma} - n_\mu n_\sigma K_{\nu\alpha}K^\alpha_{\ \rho} - n_\nu n_\rho K_{\mu\alpha}K^\alpha_{\ \sigma} + n_\nu n_\sigma K_{\mu\alpha}K^\alpha_{\ \rho} \\
&\quad + n_\mu n_\rho \pounds_n K_{\nu\sigma} - n_\mu n_\sigma \pounds_n K_{\nu\rho} - n_\nu n_\rho \pounds_n K_{\mu\sigma} + n_\nu n_\sigma \pounds_n K_{\mu\rho}, && (2.103)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &= R_{\mu\nu} - \varepsilon\Big[KK_{\mu\nu} - 2K_{\mu\alpha}K^\alpha_{\ \nu} + n_\mu\left(D_\alpha K^\alpha_{\ \nu} - D_\nu K\right) + n_\nu\left(D_\alpha K^\alpha_{\ \mu} - D_\mu K\right)\Big] \\
&\quad + n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta} + \varepsilon\pounds_n K_{\mu\nu} + n_\mu n_\nu \gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}, && (2.104)
\end{aligned}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \tag{2.105}$$

$N+1$ **Einstein-Gauss-Bonnet equations [2]** **Box 2.7**

Substituting (2.103)-(2.105) into (2.95) or (2.5)-(2.7), we find:

(a) dynamical equations for γ_{ij}:

$$M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon(-K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab})$$
$$+2\alpha_{GB}\Big[H_{ij} + \varepsilon(M\pounds_n K_{ij} - 2M_i{}^a\pounds_n K_{aj} - 2M_j{}^a\pounds_n K_{ai} - W_{ij}{}^{ab}\pounds_n K_{ab})\Big] = \kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j$$

(b) Hamiltonian constraint equation:

$$M + \alpha_{GB}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) = -2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu$$

(c) momentum constraint equation:

$$N_i + 2\alpha_{GB}\left(MN_i - 2M_i{}^aN_a + 2M^{ab}N_{iab} - M_i{}^{cab}N_{abc}\right) = -\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu{}_i$$

$$\begin{aligned}
M_{ijkl} &= R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk}) \\
M_{ij} &= \gamma^{ab}M_{iajb} = R_{ij} - \varepsilon(KK_{ij} - K_{ia}K^a{}_j) \\
M &= \gamma^{ab}M_{ab} = R - \varepsilon(K^2 - K_{ab}K^{ab}) \\
N_{ijk} &= D_iK_{jk} - D_jK_{ik} \\
N_i &= \gamma^{ab}N_{aib} = D_aK_i{}^a - D_iK \\
W_{ij}{}^{kl} &= M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl} \\
H_{ij} &= MM_{ij} - 2(M_{ia}M^a{}_j + M^{ab}M_{iajb}) + M_{iabc}M_j{}^{abc} \\
&\quad -2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a{}_j + K_{ia}K^a{}_bM^b{}_j + K_{ja}K^a{}_bM^b{}_i + K^{ac}K_c{}^bM_{iajb} \\
&\qquad +N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}{}_j - N_{iab}N_j{}^{ab}\Big] \\
&\quad -\frac{1}{4}\gamma_{ij}[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}] \\
&\quad -\varepsilon\gamma_{ij}[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c{}^b - 2N_aN^a + N_{abc}N^{abc}]
\end{aligned}$$

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