APCTP Winter School, January 25-26, 2008

Formulation Problem in Numerical Relativity

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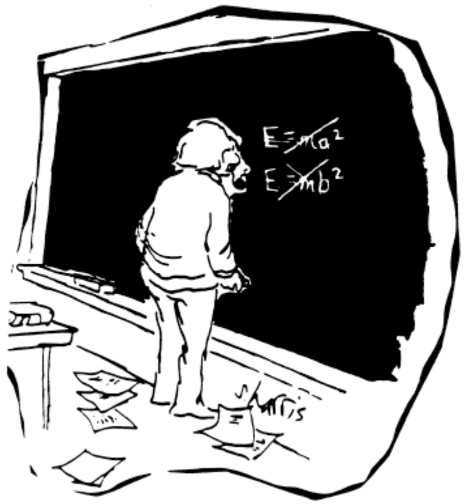
真貝寿明（しんかいひさあき） 신 카 이 히 사 아 키

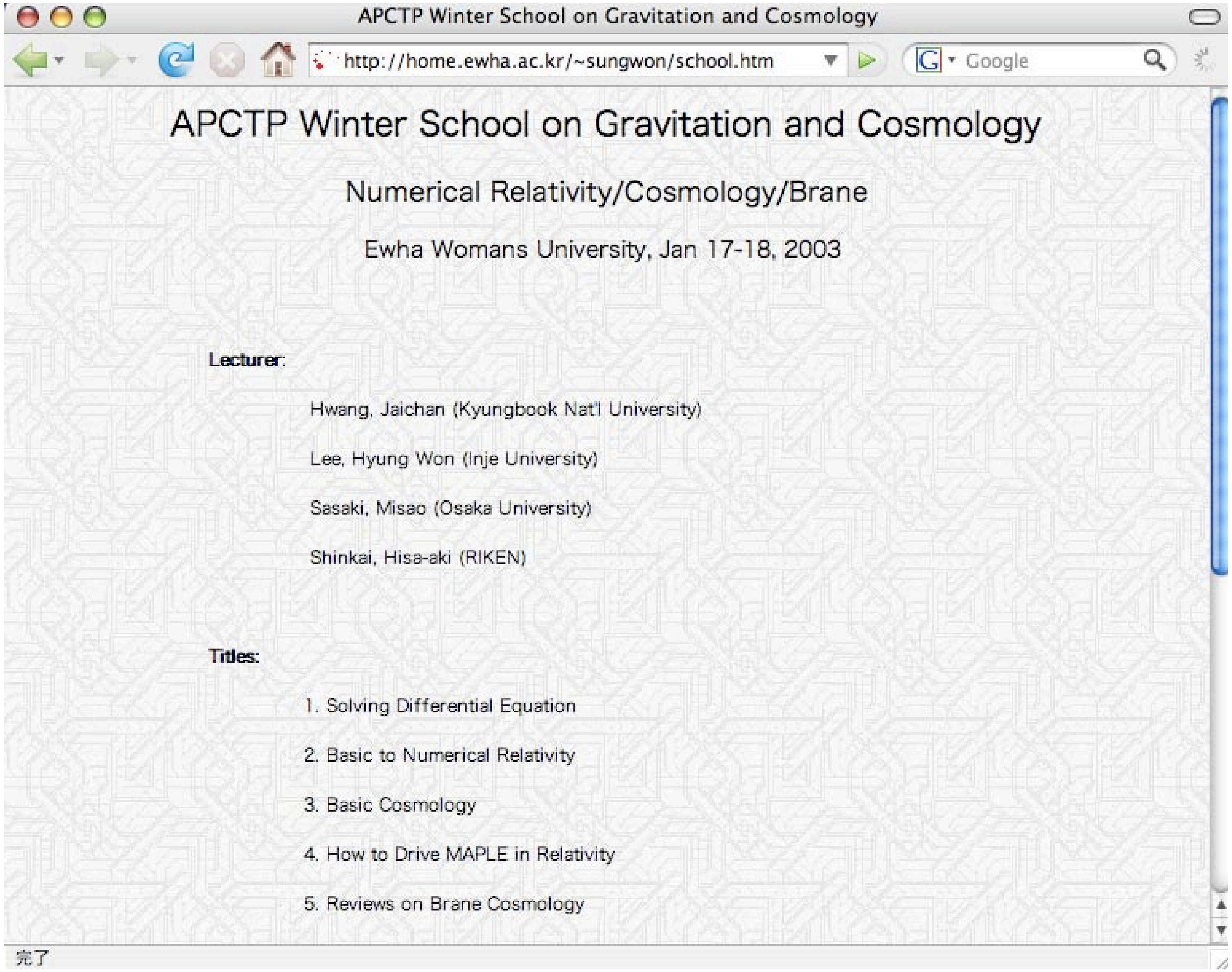
1. Introduction
   - What is the "Formulation Problem" ?
   - Historical Review
2. The Standard Approach to Numerical Relativity
   - The ADM formulation
   - The BSSN formulation
   - Hyperbolic formulations
3. Robust system for Constraint Violation
   - Adjusted systems
4. Outlook



http://www.is.oit.ac.jp/~shinkai/

APCTP Winter School, January 17-18, 2003

http://home.ewha.ac.kr/~sungwon/school.html

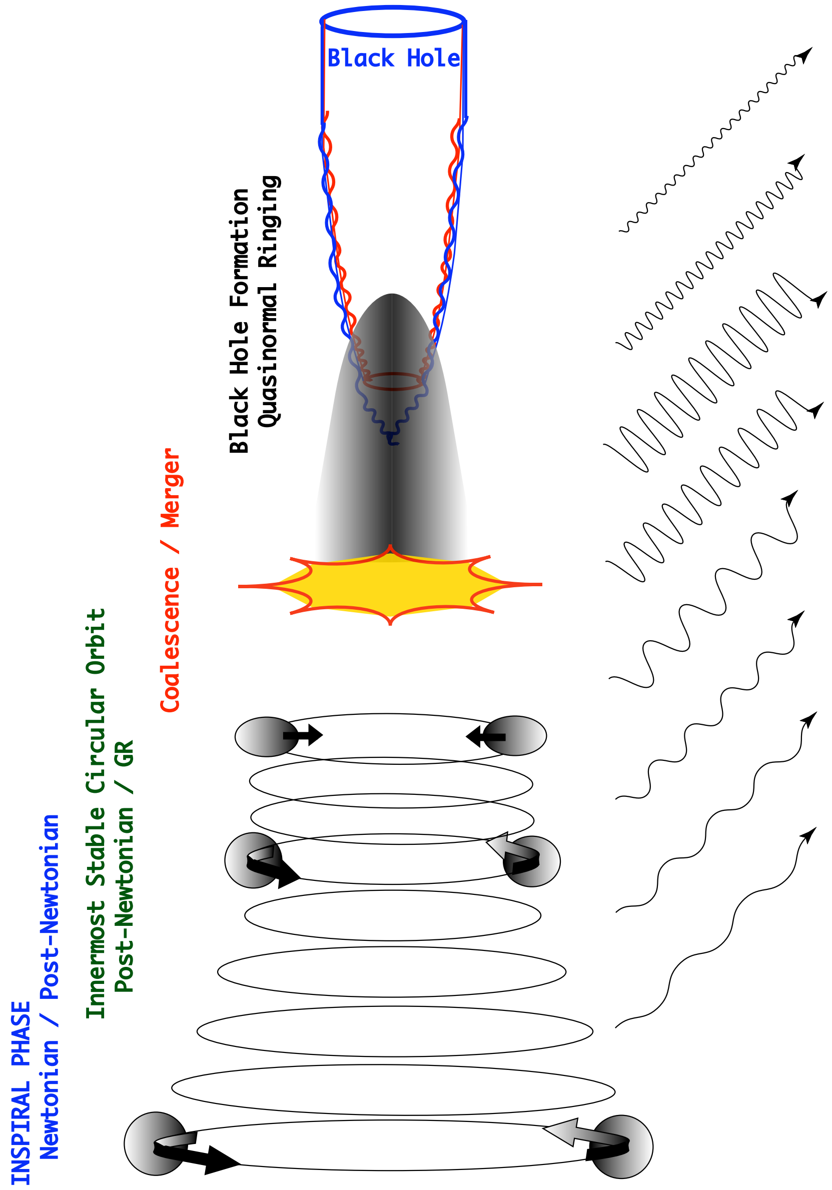


In the last 5 years, ...

Binary BH-BH coalescence simulations are available!!

Breakthrough suddenly occurs.

- Pretorius (2005)
- Univ. Texas Brownsville (2006)
- NASA-Goddard (2006)



In the last 5 years, ...

Binary BH-BH coalescence simulations are available!!

- Pretorius (2005) --> Princeton Univ.
- Univ. Texas Brownsville (2006) --> Rochester Univ.
- NASA-Goddard (2006)
  - Louisiana State Univ.
  - Jena Univ.
  - Pennsylvania State Univ.

"Gold-Rush of parameter searches" (B. Bruegmann, July 2007 @GRG)

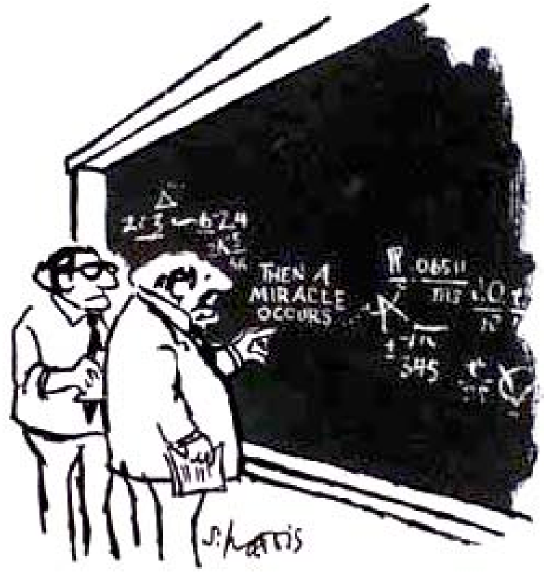
But ..... Why it works?

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

The Einstein equation

geodesics

spacetime curvature → matter distribution

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein tensor

Energy-Momentum tensor

cosmological constant

Solve for metric
$g_{\mu\nu}$(t,x,y,z)
(10 components)

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu := g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ & g_{xx} & g_{xy} & g_{xz} \\ & & g_{yy} & g_{yz} \\ sym. & & & g_{zz} \end{pmatrix}$$

flat spacetime (Minkowskii spacetime):

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

The Einstein equation:

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Chandrasekhar says ...

"Einstein equations are easy to solve. Look at the *Exact Solutions* book. There are more than 400 solutions. "

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Exact Solutions book says ...

1st Edition (1980): "... checked 2000 references, ..., there are now over 100 papers on exact solutions every year, ..."

2nd Edition (2003): "... we looked at 4000 new papers published during 1980-1999, ... "

D. Kramer, et al, Exact Solutions to Einstein's Field Equations, (Cambridge, 1980)
H. Stephani, et al, Exact Solutions to Einstein's Field Equations, (Cambridge, 2003)

Why don't we solve it using computers?

- dynamical behavior, no symmetry in space, ...
- strong gravitational field, gravitational wave! ...
- any dimension, any theories, ...

Numerical Relativity

= Solve the Einstein equations numerically.

= Necessary for unveiling the nature of strong gravity.

For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

The most robust way to study the strong gravitational field. Great.

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

What are the difficulties?

- for 10-component metric, highly nonlinear partial differential equations.
  mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- completely free to choose cooordinates, gauge conditions, and even for decomposition of the space-time.
- has singularity in its nature.

How to solve it?

Numerical Relativity – basic issues

0. How to foliate space-time

   Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

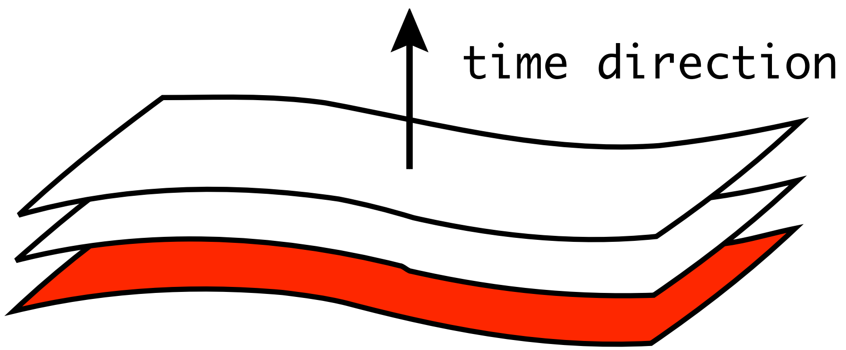
Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

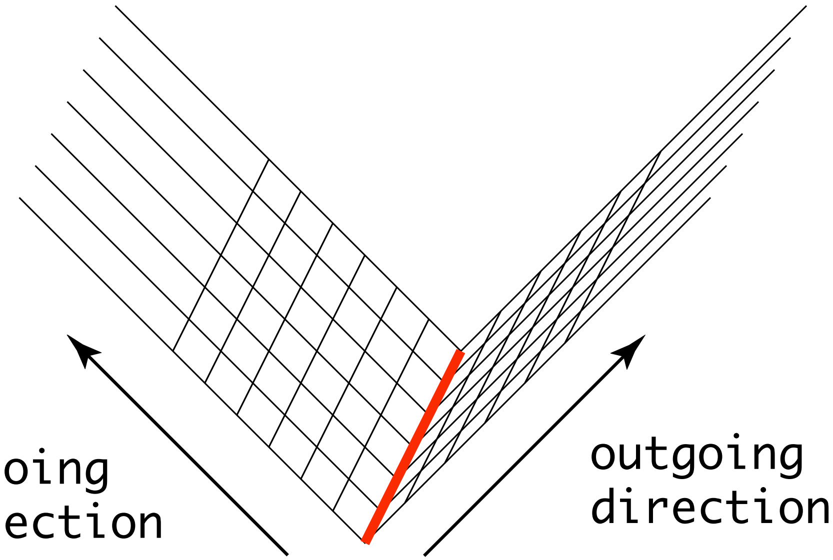
First Question: How to foliate space-time?

Cauchy approach
or ADM 3+1 formulation



Σ: Initial 3-dimensional Surface

Characteristic approach
(if null, dual-null 2+2 formulation)

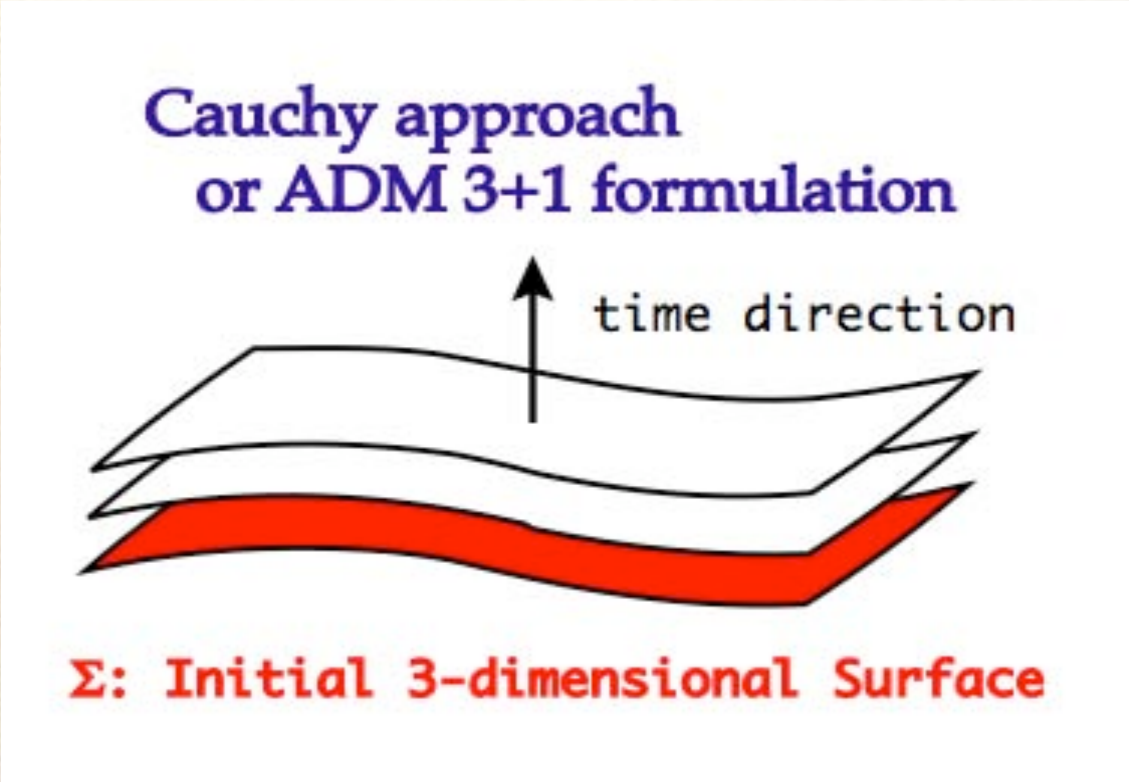


S: Initial 2-dimensional Surface

3+1 versus 2+2

	Cauchy (3+1) evolution	Characteristic (2+2) evolution
pioneers	ADM (1961), York-Smarr (1978)	Bondi *et al* (1962), Sachs (1962), Penrose (1963)
variables	easy to understand the concept of time evolution	has geometrical meanings 1 complex function related to 2 GW polarization modes
foliation	has Hamilton structure	allows implementation of Penrose's space-time compactification
initial data	need to solve constraints	no constraints
evolution	PDEs need to avoid constraint violation	ODEs with consistent conditions propagation eqs along the light rays
singularity	need to avoid by some method	can truncate the grid
disadvantages	can not cover space-time globally	difficulty in treating caustics hard to treat matter

"3+1" formulation



Numerical Relativity – basic issues

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Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
  - Solve the constraint equations

- Time Evolution

  do time=1, time_end
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  - Evolve the variables
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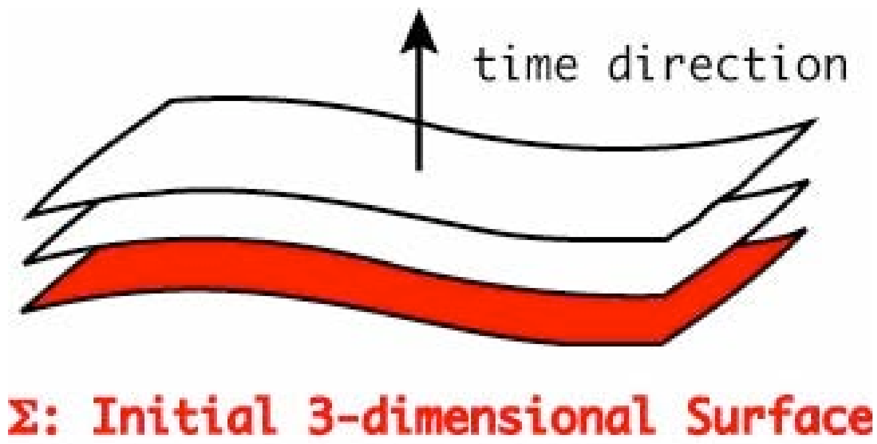
  end do



The 3+1 decomposition of space-time: The ADM formulation

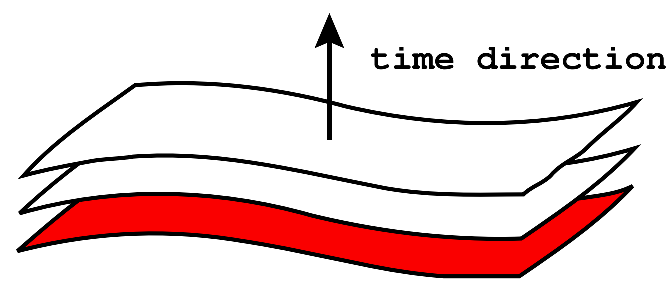
[1] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. by L.Witten, (Wiley, New York, 1962).

[2] J.W. York, Jr. in *Sources of Gravitational Radiation*, (Cambridge, 1979)

Dynamics of Space-time = Foliation of Hypersurface

- Evolution of t =const. hypersurface $\Sigma(t)$.

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu, \quad (\mu,\nu=0,1,2,3) \\ \text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij}dx^i dx^j, \quad (i,j=1,2,3) \end{aligned}$$



Σ: Initial 3-dimensional Surface

- The unit normal vector of the slices, n^μ.

$$\begin{aligned} n_\mu &= (-\alpha,0,0,0) \\ n^\mu &= g^{\mu\nu}n_\nu = (1/\alpha, -\beta^i/\alpha) \end{aligned}$$

- The lapse function, α. The shift vector, β^i.

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

The decomposed metric:

$$\begin{aligned} ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (-\alpha^2 + \beta_l \beta^l) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j \end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l \beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

where α and β_j are defined as $\alpha \equiv 1/\sqrt{-g^{00}}, \quad \beta_j \equiv g_{0j}$.

- The unit normal vector of the slices, n^μ.

$$\begin{aligned} n_\mu &= (-\alpha, 0, 0, 0) \\ n^\mu &= g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha) \end{aligned}$$

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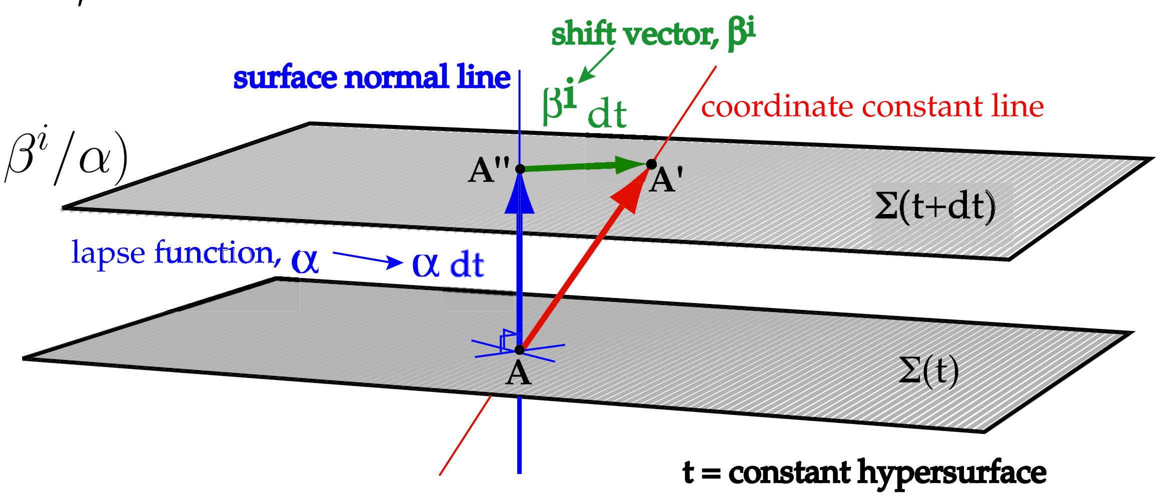


Projection of the Einstein equation:

- Projection operator (or intrinsic 3-metric) to $\Sigma(t)$,

$$\begin{aligned} \gamma_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu \\ \gamma^\mu_\nu &= \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu \end{aligned}$$

- Define the extrinsic curvature K_{ij},

$$\begin{aligned} K_{ij} &\equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} \\ &= -(\delta^\mu_i + n^\mu n_i)(\delta^\nu_j + n^\nu n_j) n_{\mu;\nu} \\ &= -n_{i;j} \\ &= \Gamma^\alpha_{ij} n_\alpha = \cdots = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + \beta_{i|j} + \beta_{j|i}\right). \end{aligned}$$

- Projection of the Einstein equation:

$$G_{\mu\nu}\, n^\mu\, n^\nu = 8\pi G\, T_{\mu\nu}\, n^\mu\, n^\nu \equiv 8\pi \rho_H \qquad \Rightarrow \text{ the Hamiltonian constraint eq.}$$

$$G_{\mu\nu}\, n^\mu \perp^\nu_i = 8\pi G\, T_{\mu\nu}\, n^\mu \perp^\nu_i \equiv -8\pi J_i \qquad \Rightarrow \text{ the momentum constraint eqs.}$$

$$G_{\mu\nu} \perp^\mu_i \perp^\nu_j = 8\pi G\, T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv 8\pi S_{ij} \qquad \Rightarrow \text{ the evolution eqs.}$$

The Standard ADM formulation (aka York 1978):

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
\partial_t K_{ij} &= \alpha\ {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha \\
&\quad +(D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} \\
&\quad -8\pi G\alpha\{S_{ij} + (1/2)\gamma_{ij}(\rho_H - \mathrm{tr}S)\},
\end{aligned}$$

where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

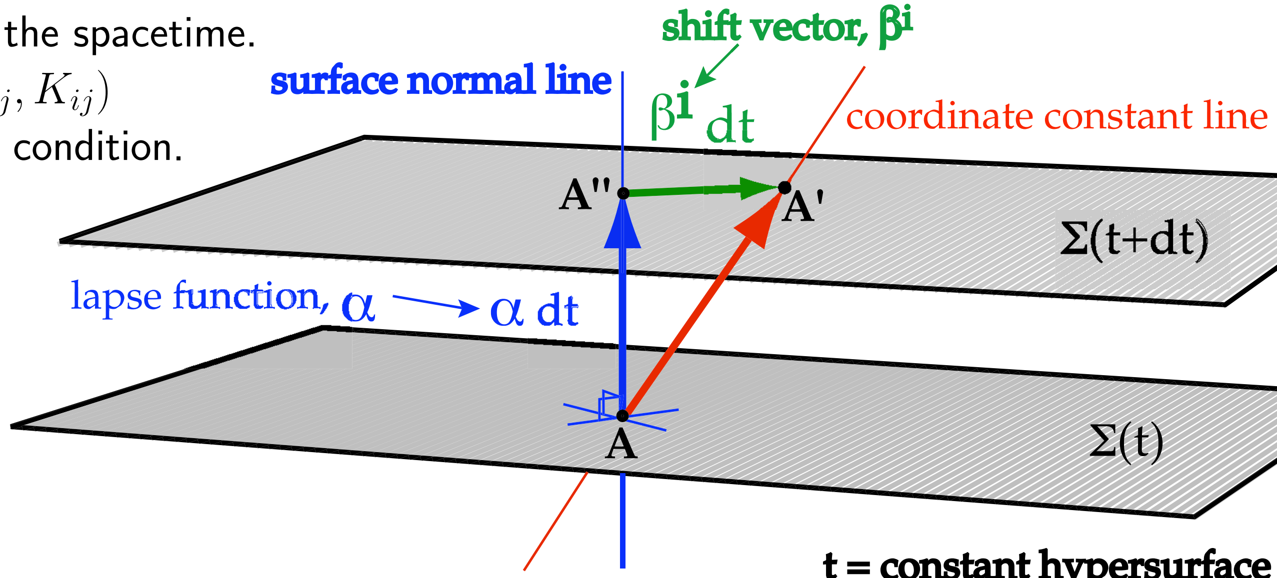
- Constraint equations:

$$\begin{aligned}
&\text{Hamiltonian constr.} \quad & \mathcal{H}^{ADM} &:= {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, \\
&\text{momentum constr.} \quad & \mathcal{M}_i^{ADM} &:= D_jK^j{}_i - D_iK \approx 0,
\end{aligned}$$

where ${}^{(3)}R = {}^{(3)}R^i{}_i$.

strategy 0 **The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)**

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} =$ rot $\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -$rot $\mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
    - Assume the background metric
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Need to solve elliptic PDEs
-- Conformal approach
-- Thin-Sandwich approach

- Time Evolution

    do time=1, time_end
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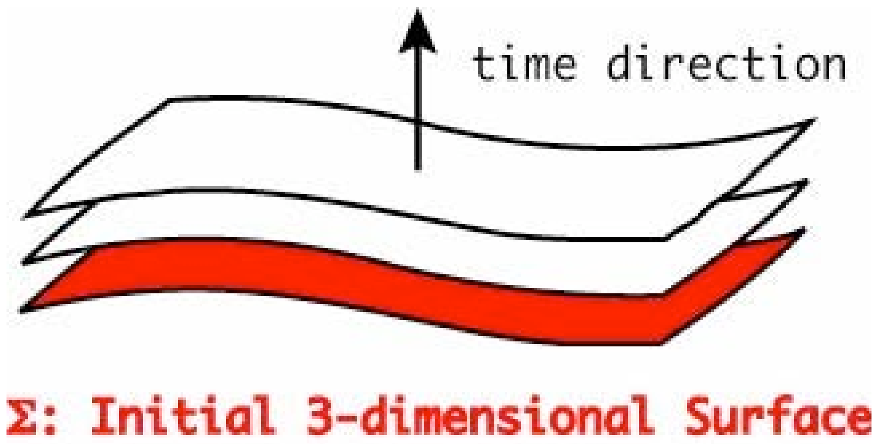
    end do



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end do

singularity avoidance,
simplify the system,
GW extraction, ...

Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - ◆ Assume the background metric
  - ◆ Solve the constraint equations ← Need to solve elliptic PDEs
    - -- Conformal approach
    - -- Thin-Sandwich approach

- Time Evolution

```
do time=1, time_end
   ◆ Specify the slicing condition
   ◆ Evolve the variables
   ◆ Check the accuracy
   ◆ Extract physical quantities
end do
```

Specify the slicing condition ← singularity avoidance, simplify the system, GW extraction, ...

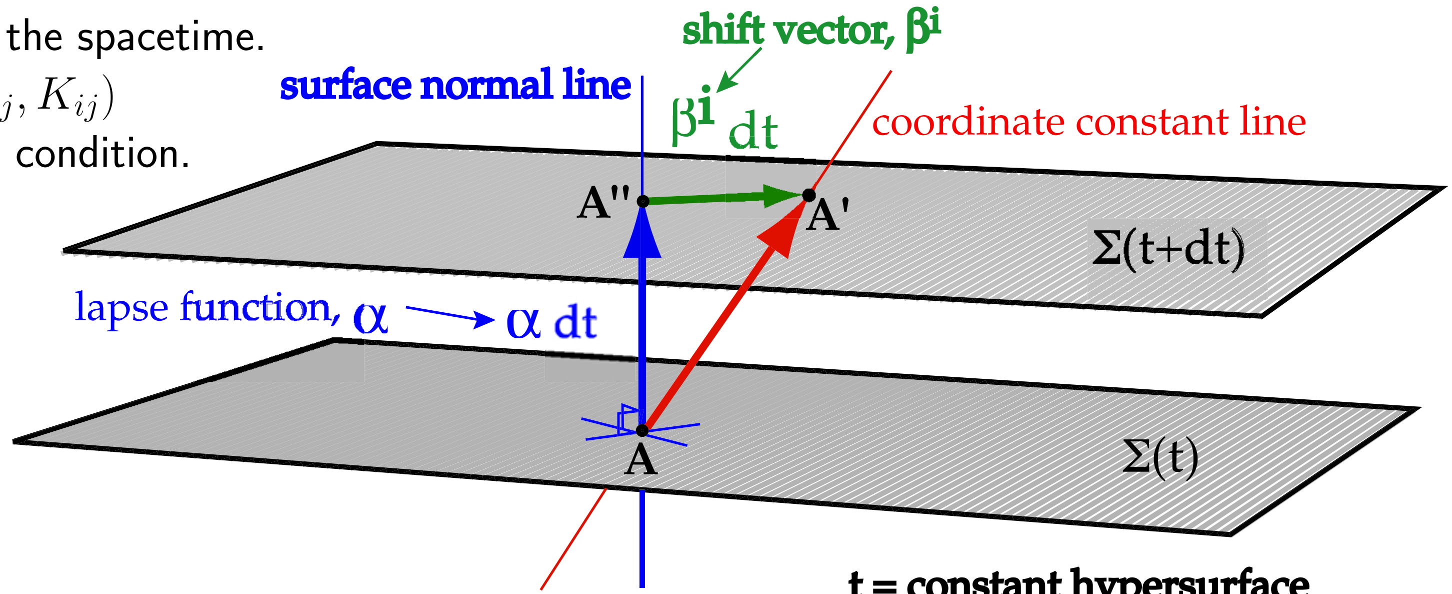
Evolve the variables / Check the accuracy ← Robust formulation ?
- -- modified ADM / BSSN
- -- hyperbolization
- -- asymptotically constrained

Formulation Problem

strategy 0

The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



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evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \text{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\text{rot } \mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

The Constraint Propagations of the Standard ADM:

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}$$

From these equations, we know that

if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

Primary / Secondary constraint
First-class / Second-class constraint

- Primary Constraints

$$\text{constraint } C_1(q,p) \approx 0$$
$$\text{constraint } C_2(q,p) \approx 0$$

- Secondary Constraints
  = when propagation of constraints require additional constraints

$$\begin{aligned} \dot{C}_i &= \{C_1, H\}_P = \{C_i, H'(q,p) + \lambda^k C_k\}_P \\ &= \{C_i, H'\}_P + \lambda^k \{C_i, C_k\}_P \approx 0 \end{aligned}$$

- First-Class Constraints
  = $\text{set of constraints } C_i \text{ satisfy } \{C_i, C_k\}_P \approx 0$

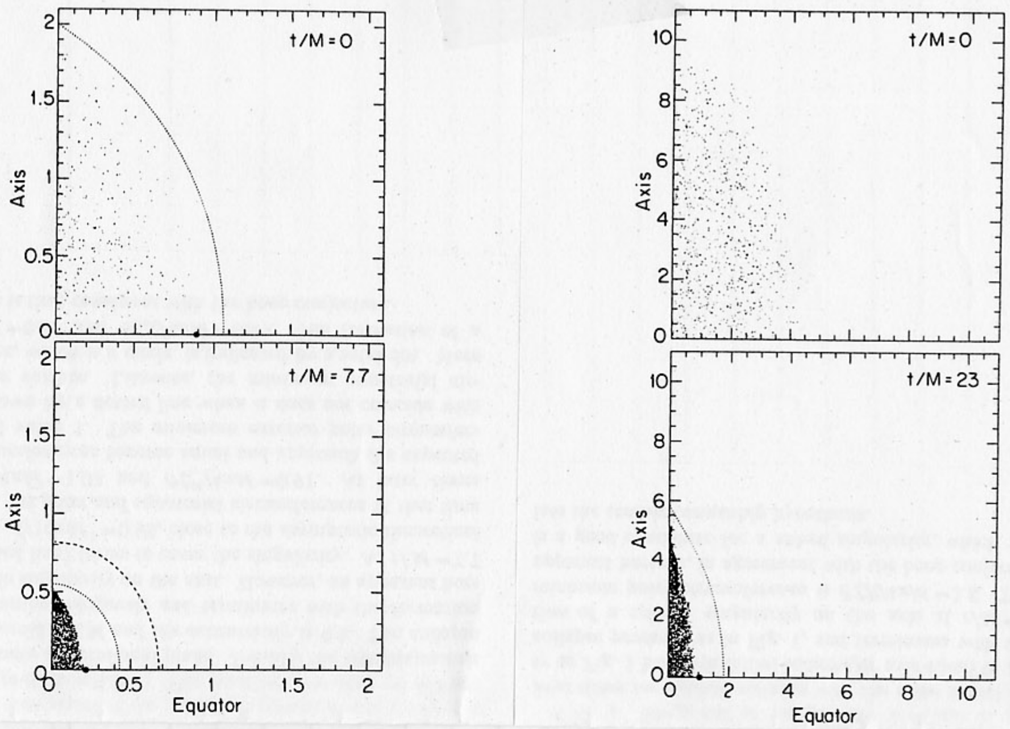
Numerical Relativity in the 20th century

1960s	Hahn-Lindquist	2 BH head-on collision	AnaPhys29(1964)304
	May-White	spherical grav. collapse	PR141(1966)1232
1970s	ÓMurchadha-York	conformal approach to initial data	PRD10(1974)428
	Smarr	3+1 formulation	PhD thesis (1975)
	Smarr-Cades-DeWitt-Eppley	2 BH head-on collision	PRD14(1976)2443
	Smarr-York	gauge conditions	PRD17(1978)2529
	ed. by L.Smarr	"Sources of Grav. Radiation"	Cambridge(1979)
1980s	Nakamura-Maeda-Miyama-Sasaki	axisym. grav. collapse	PTP63(1980)1229
	Miyama	axisym. GW collapse	PTP65(1981)894
	Bardeen-Piran	axisym. grav. collapse	PhysRep96(1983)205
	Stark-Piran	axisym. grav. collapse	unpublished
1990	Shapiro-Teukolsky	naked singularity formation	PRL66(1991)994
	Oohara-Nakamura	3D post-Newtonian NS coalesence	PTP88(1992)307
	Seidel-Suen	BH excision technique	PRL69(1992)1845
	Choptuik	critical behaviour	PRL70(1993)9
	NCSA group	axisym. 2 BH head-on collision	PRL71(1993)2851
	Cook et al	2 BH initial data	PRD47(1993)1471
	Shibata-Nakao-Nakamura	BransDicke GW collapse	PRD50(1994)7304
	Price-Pullin	close limit approach	PRL72(1994)3297
1995	NCSA group	event horizon finder	PRL74(1995)630
	NCSA group	hyperbolic formulation	PRL75(1995)600
	Anninos *et al*	close limit vs full numerical	PRD52(1995)4462
	Scheel-Shapiro-Teukolsky	BransDicke grav. collapse	PRD51(1995)4208
	Shibata-Nakamura	3D grav. wave collapse	PRD52(1995)5428
	Gunnersen-Shinkai-Maeda	ADM to NP	CQG12(1995)133
	Wilson-Mathews	NS binary inspiral, prior collapse?	PRL75(1995)4161
	Pittsburgh group	Cauchy-characteristic approach	PRD54(1996)6153
	Brandt-Brügmann	BH puncture data	PRL78(1997)3606
	Illinois group	synchronized NS binary initial data	PRL79(1997)1182
	Shibata-Baumgarte-Shapiro	2 NS inspiral, PN to GR	PRD58(1998)023002
	BH Grand Challenge Alliance	characteristic matching	PRL80(1998)3915
	Baumgarte-Shapiro	Shibata-Nakamura formulation	PRD59(1998)024007
	Brady-Creighton-Thorne	intermediate binary BH	PRD58(1998)061501
	Meudon group	irrotational NS binary initial data	PRL82(1999)892
	Shibata	2 NS inspiral coalesence	PRD60(1999)104052

Formation of Naked Singularities: The Violation of Cosmic Censorship

Stuart L. Shapiro and Saul A. Teukolsky
Center for Radiophysics and Space Research and Departments of Astronomy and Physics, Cornell University, Ithaca, New York 14853
(Received 7 September 1990)

We use a new numerical code to evolve collisionless gas spheroids in full general relativity. In all cases the spheroids collapse to singularities. When the spheroids are sufficiently compact, the singularities are hidden inside black holes. However, when the spheroids are sufficiently large, there are no apparent horizons. These results lend support to the hoop conjecture and appear to demonstrate that naked singularities can form in asymptotically flat spacetimes.



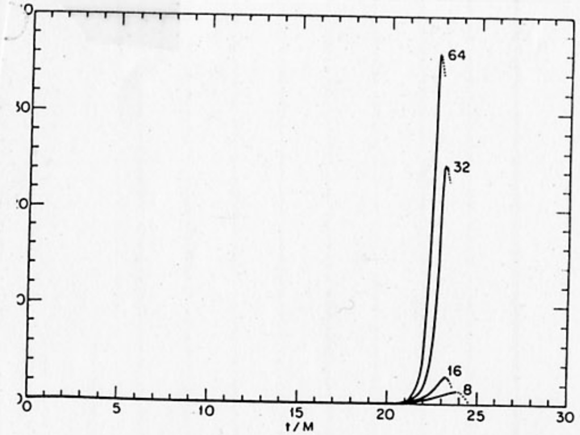


FIG. 3. Growth of the Riemann invariant I (in units of M^{-4}) vs time for the collapse shown in Fig. 2. The simulation was repeated with various angular grid resolutions. Each curve is labeled by the number of angular zones used. We use dots to show where the singularity has caused the code to become inaccurate.

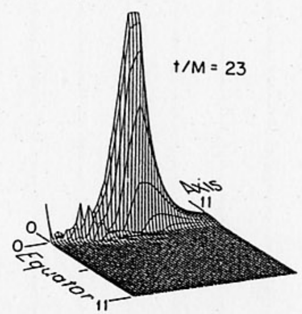


FIG. 4. Profile of I in a meridional plane for the collapse shown in Fig. 2. For the case of 32 angular zones shown here, the peak value of I is $24/M^4$ and occurs on the axis just outside the matter.

Critical Phenomena in Gravitational Collapse

Choptuik, Phys. Rev. Lett. 70 (1993) 9

TABLE I. Initial data specification for various one-parameter families discussed in text. For families (a)–(c), I specified the initial pulses to be purely in-going. For family (d), the functions $X_>(r)$, $Y_<(r)$ and $X_>(r)$, $Y_>(r)$ are late-time fits to subcritical and supercritical evolutions, respectively, of the pulse shape shown in Fig. 1(d).

Family	Form of initial data	p
(a)	$\phi(r) = \phi_0 r^3 \exp(-[(r-r_0)/\delta]^q)$	ϕ_0, r_0, δ, q
(b)	$\phi(r) = \phi_0 \tanh[(r-r_0)/\delta]$	ϕ_0
(c)	$\phi(r+r_0) = \phi_0 r^{-5}[\exp(1/r)-1]^{-1}$	ϕ_0
(d)	$X(r) = (1-\eta)X_<(r) + \eta X_>(r)$ $Y(r) = (1-\eta)Y_<(r) + \eta Y_>(r)$	η

TABLE II. Numerically determined values of the scaling exponent γ in the conjectured relationship $M_{BH} \simeq c_f |p-p^*|^\gamma$. μ_{min} and μ_{max} are the minimum and maximum mass fractions ($\mu \equiv M_{BH}/M$) of the black holes computed in the simulation and γ is the least-squares estimate of the scaling exponent.

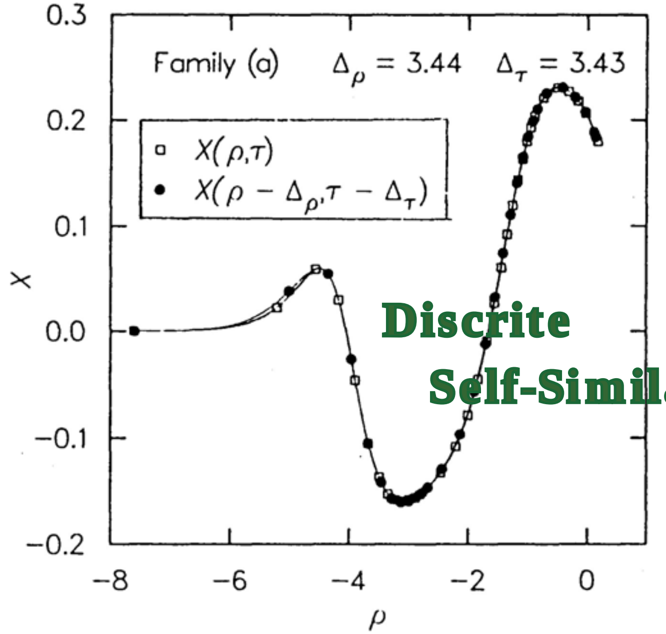
Family	Parameter	μ_{min}	μ_{max}	γ
(a)	ϕ_0	7.9×10^{-3}	8.9×10^{-1}	0.376
(a)	δ	1.3×10^{-3}	9.4×10^{-1}	0.372
(a)	q	3.1×10^{-3}	9.8×10^{-1}	0.372
(a)	r_0	1.3×10^{-2}	9.2×10^{-1}	0.379
(b)	ϕ_0	2.8×10^{-3}	4.0×10^{-1}	0.372
(c)	ϕ_0	4.9×10^{-3}	9.9×10^{-1}	0.366
(d)	η	2.2×10^{-5}	1.7×10^{-2}	0.380

Spherical Sym., Massless Scalar Field

(1) scaling

(2) echoing

(3) universality



Discrete Self-Similarity

FIG. 2. Illustration of the rescaling or echoing property observed in near-critical evolution of the scalar field. The curve marked with open squares shows the profile of the scalar field variable, X, at some proper central time T_0. The curve marked with solid circles is the profile at a later time $T_0 + e^{\Delta_\tau}$ but on a scale $e^{\Delta_\rho} \approx 30$ times smaller.

Head-on Collision of 2 Black-Holes (Misner initial data)

NCSA group 1995

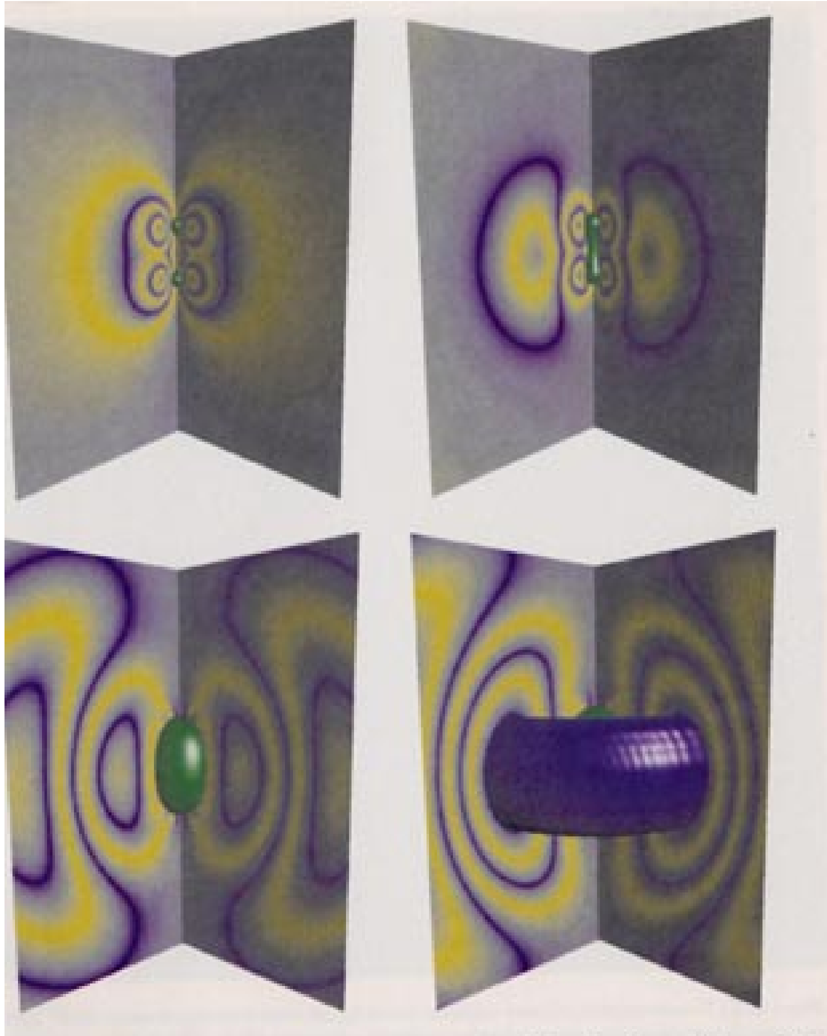


Fig. C.6. 3D evolution of the radiation field Ψ_4 of the head-on collision of two equal-mass black holes shown as a blue and yellow color-map

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

The Constraint Propagations of the Standard ADM:

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}$$

From these equations, we know that

if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

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&\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}$$

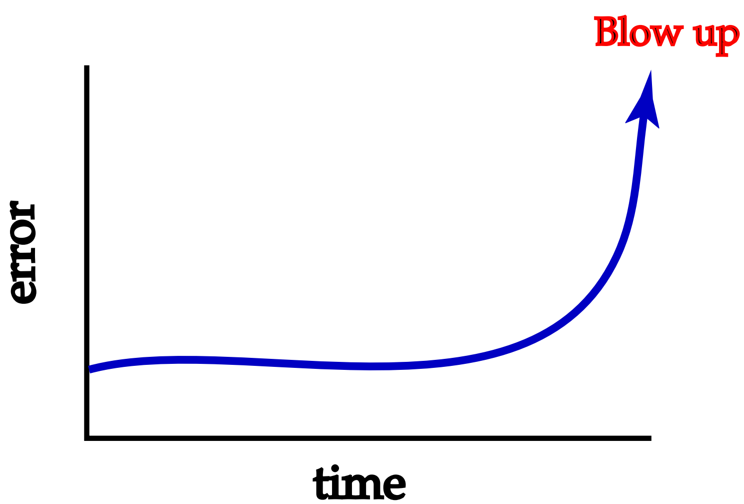
From these equations, we know that

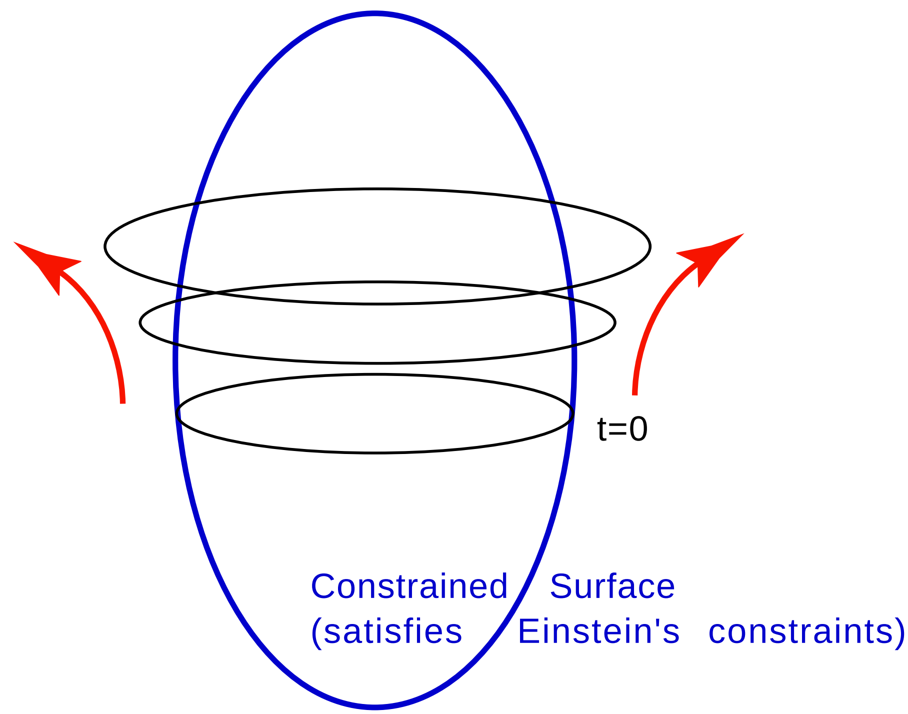
if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

But this is NOT TRUE in NUMERICS....

- By the period of 1990s, NR had provided a lot of physics:
  Gravitational Collapse, Critical Behavior, Naked Singularity, Event Horizons,
  Head-on Collision of BH-BH and Gravitational Wavve, Cosmology, $\cdots$
- However, for the BH-BH/NS-NS inspiral coalescence problem, $\cdots$ why ???

  Many (too many) trials and errors, hard to find a definit recipe.



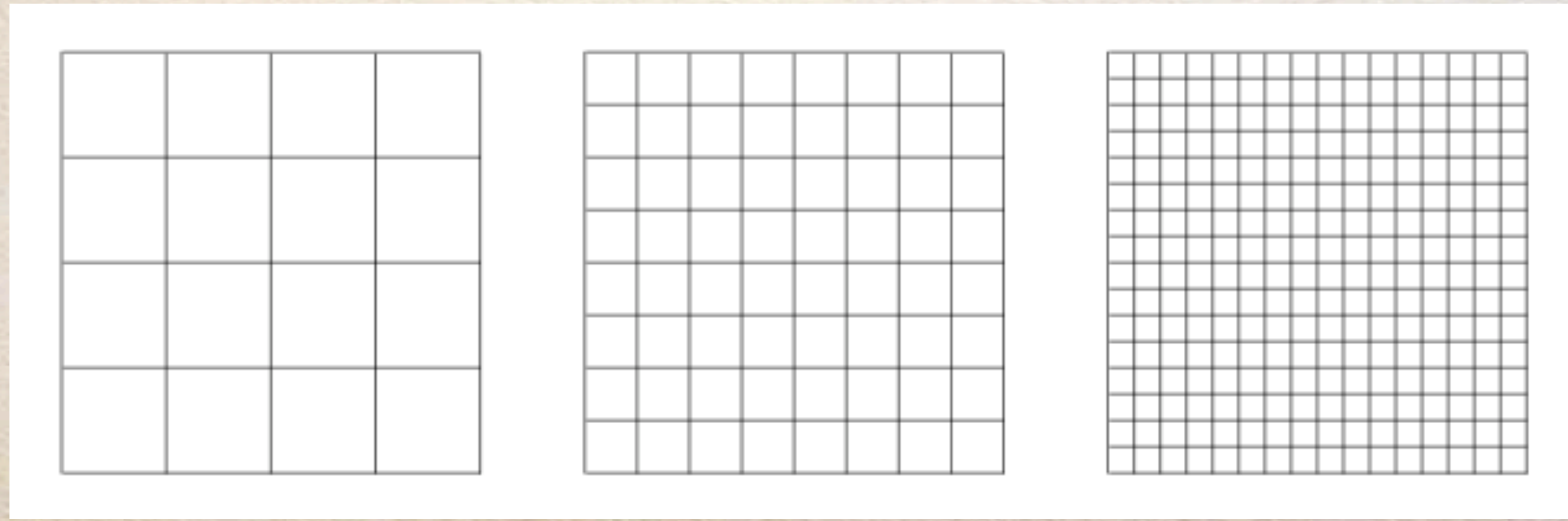


Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

"Convergence"

= higher resolution runs approach to the continuum limit.
(All numerical codes must have this property.)

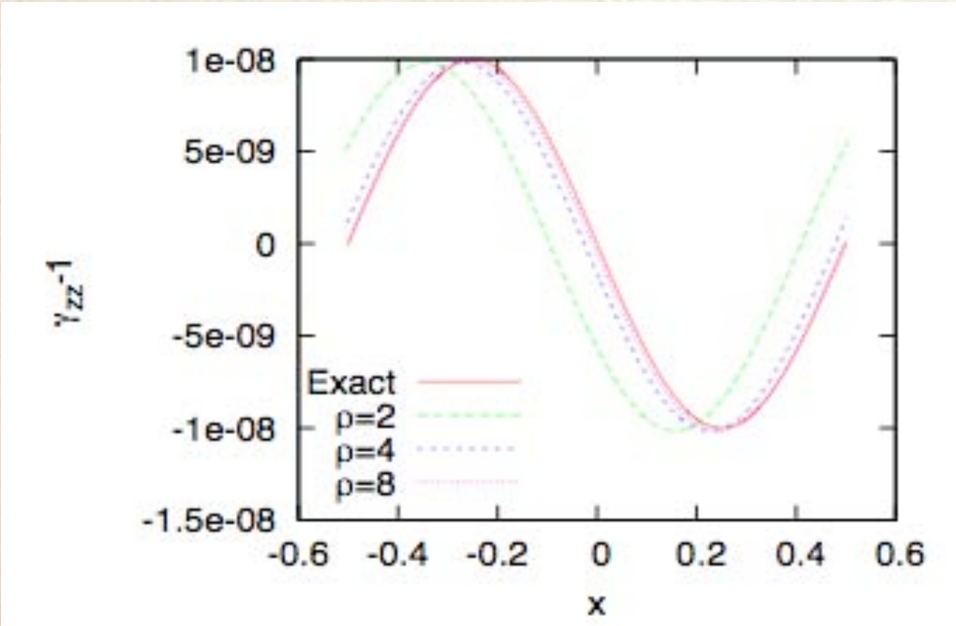
- When the code has 2nd order finite difference scheme, $O\left((\Delta x)^2\right)$ then the error should be scaled with $O\left((\Delta x)^2\right)$
- "Consistency", Choptuik, PRD 44 (1991) 3124



“Accuracy”

= The numerical results represent the actual solutions.
(All numerical codes must have this property.)

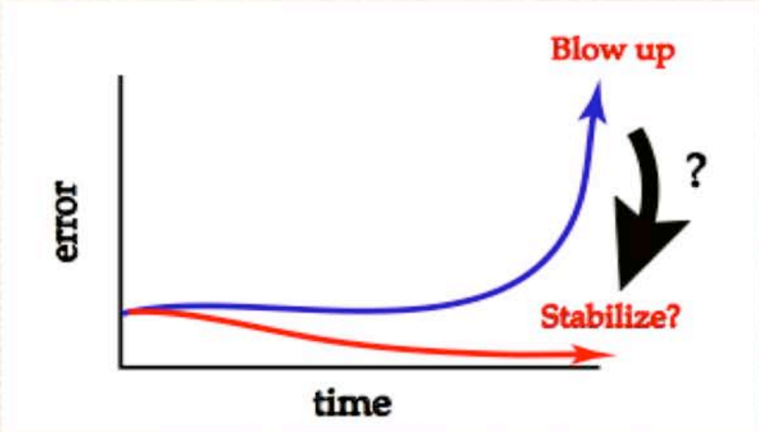
- Check the code with known results.



Gauge wave test in BSSN;
Kiuchi, HS, PRD (2008)

"Stability"

- **We** mean that a numerical simulation continues without any blow-ups and data remains on the constrained surface.



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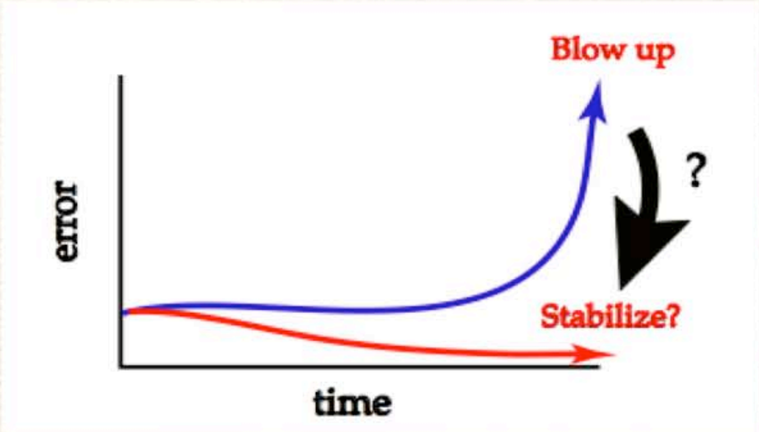
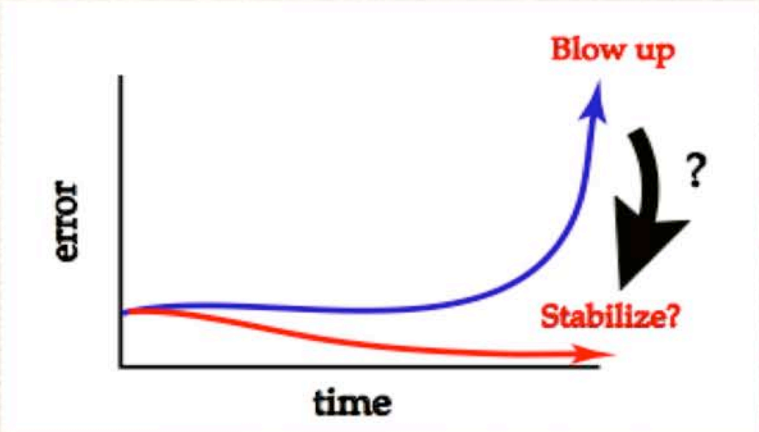


- **Mathematicians** define in terms of the PDE well-posedness.

$$||u(t)|| \le e^{\kappa t}||u(0)||$$

“Stability”

- **We** mean that a numerical simulation continues without any blow-ups and data remains on the constrained surface.



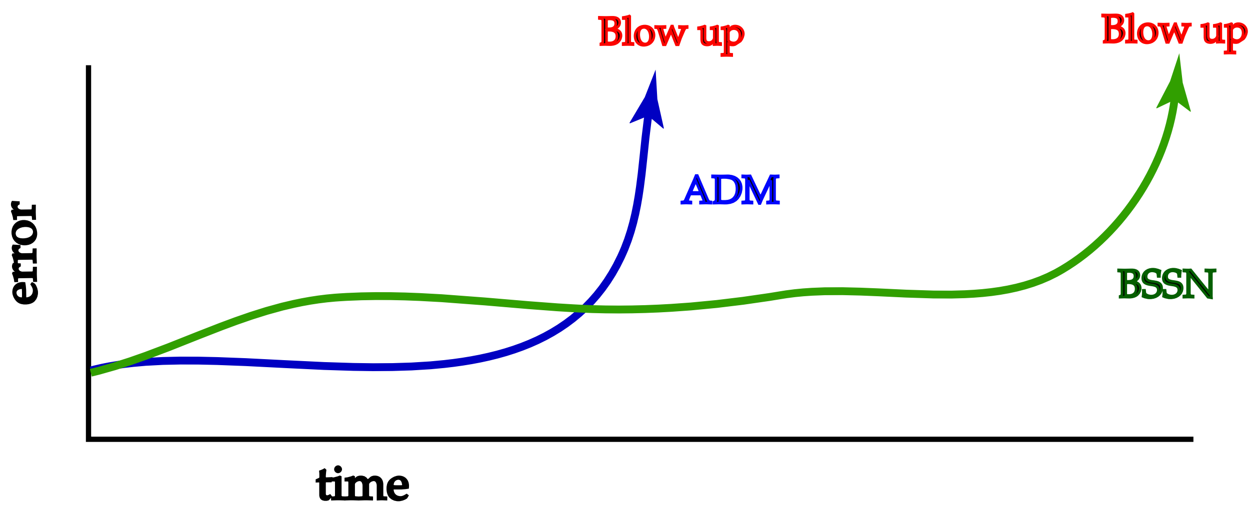
- **Mathematicians** define in terms of the PDE well-posedness.

$$||u(t)|| \leq e^{\kappa t}||u(0)||$$

- **Programmers** define for selecting a finite differencing scheme (judged by von Neumann's analysis).
  Lax's equivalence theorem says that if a numerical scheme is consistent (converging) and stable, then the simulation represents the right (converging) solution.

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

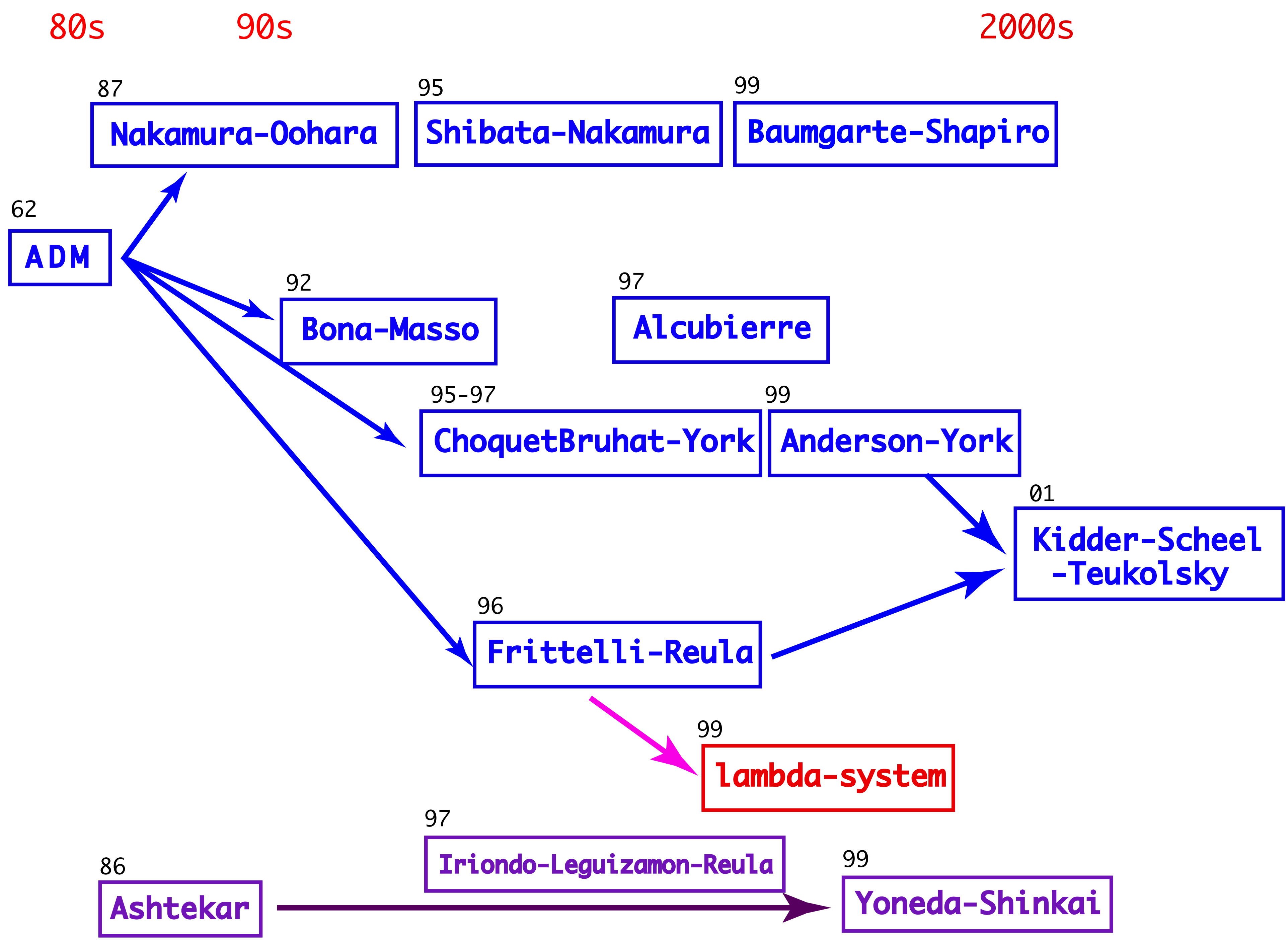
- Many (too many) trials and errors, hard to find a definit recipe.



Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner (ADM) formulation
strategy 1: Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation
strategy 2: Hyperbolic formulations
strategy 3: "Asymptotically constrained" against a violation of constraints

By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?



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Ashtekar
99
Yoneda-Shinkai

strategy 1 Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation

T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. **90**, 1 (1987)
M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995)
T.W. Baumgarte and S.L. Shapiro, Phys. Rev. D **59**, 024007 (1999)

The popular approach. Nakamura's idea in 1980s.
BSSN is a tricky nickname. BS (1999) introduced a paper of SN (1995).

- define new set of variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

  and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

- The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

Momentum constraint was used in Γ^i-eq.

- Calculate Riemann tensor as

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^{\phi}_{ij}$$
$$R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi)$$
$$\tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}$$

- Constraints are $\mathcal{H}, \mathcal{M}_i$.
  But thre are additional ones, $\mathcal{G}^i, \mathcal{A}, \mathcal{S}$.

Hamiltonian and the momentum constraint equations

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \quad (1)$$
$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \quad (2)$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \quad (3)$$
$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \quad (4)$$
$$\mathcal{S} = \tilde{\gamma} - 1, \quad (5)$$

Why BSSN better than ADM?
Is the BSSN best? Are there any alternatives?

Some known fact (technical):

- Trace-out A_{ij} at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

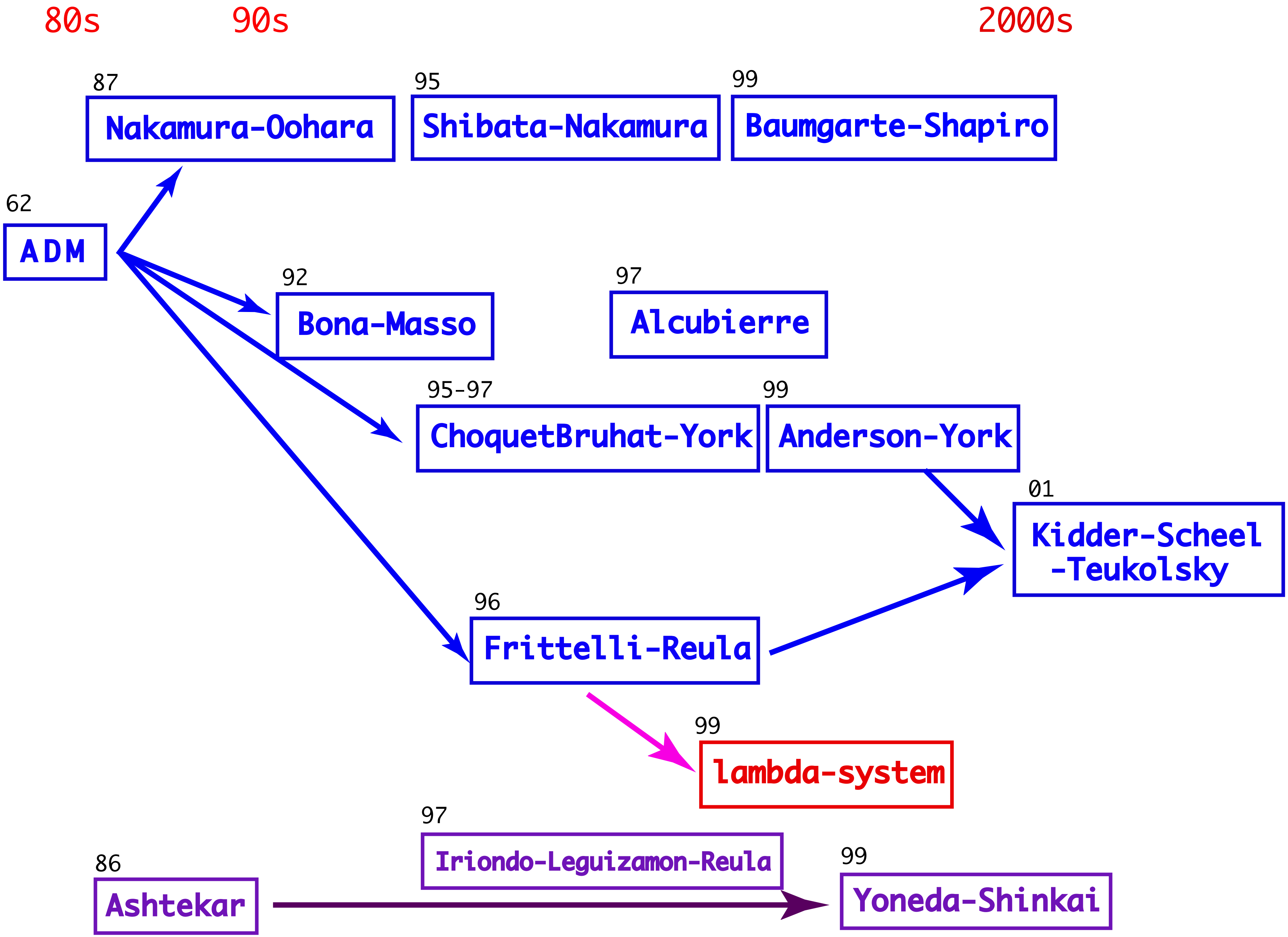
  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

Some guesses:

- BSSN has a wider range of parameters that give us stable evolutions in von Neumann's stability analysis. Miller, [gr-qc/0008017]

- The eigenvalues of BSSN *evolution equations* has fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability can be caused by "zero eigenvalues" that violate "gauge mode".

  M. Alcubierre, et al, [PRD 62 (2000) 124011]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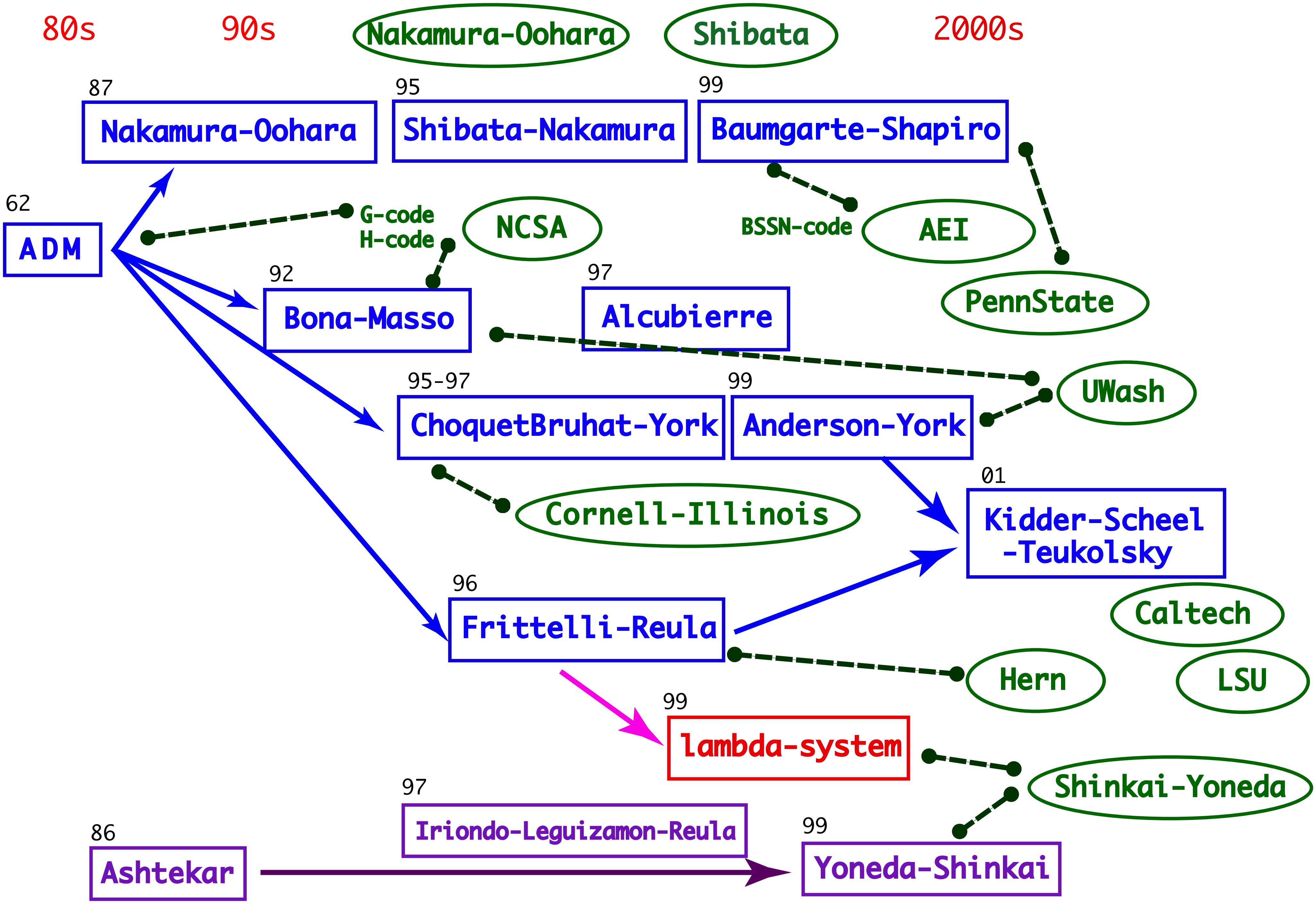
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strategy 2 **Hyperbolic formulation**

Construct a formulation which reveals a hyperbolicity explicitly.
For a first order partial differential equations on a vector u,

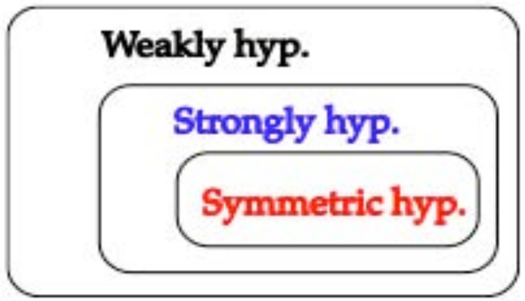
$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & & \\ & A & \\ & & \quad \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

Hyperbolic Formulation
(1) Definition

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & A & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvalues.
symmetric hyperbolic	if A is real and symmetric (Hermitian).

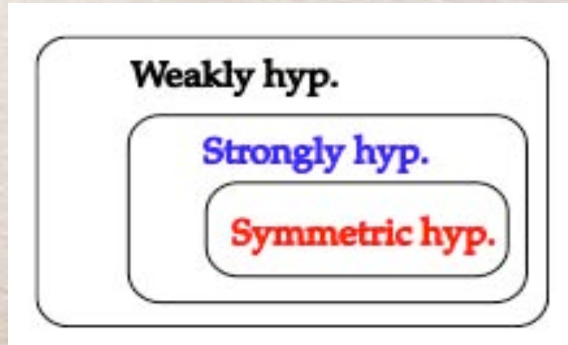


Hyperbolic Formulation (2) Expectations

- if strongly/symmetric hyperbolic ==> well-posed system
  - Given initial data + source terms -> a unique solution exists
  - The solution depends continuously on the data
  - Exists an upper bound on (unphysical) energy norm

$$||u(t)|| \leq e^{\kappa t}||u(0)||$$

- Better boundary treatments
  <== existence of characteristic field
- Known numerical techniques in Newtonian hydro-dynamics



strategy 2 Hyperbolic formulation

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However,

- ADM is not hyperbolic.
- BSSN is not hyperbolic.
- Many many hyperbolic formulations are presented. Why many? $\Rightarrow$ Exercise.

One might ask ...

Are they actually helpful?

Which level of hyperbolicity is necessary?

Exercise 1 of hyperbolic formulation

Wave equation $$(\partial_t \partial_t - c^2 \partial_x \partial_x) u = 0$$

Exercise 1 of hyperbolic formulation

Wave equation $(\partial_t \partial_t - c^2 \partial_x \partial_x) u = 0$

[1a] use u as one of the fundamental variables.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} \tag{6}$$

Eigenvalues $= \pm c$. Not a symmetric hyperbolic, but a kind of strongly hyperbolic.

[1b]

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic.

[2a] Let $U = \dot{u}, V = u'$,

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix} \tag{8}$$

Eigenvalues $= \pm c$. Not a symmetric hyperbolic, but a kind of strongly hyperbolic.

[2b] Let $U = \dot{u}, V = cu'$,

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix} \tag{9}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic.

Exercise 1 of hyperbolic formulation

Wave equation $(\partial_t\partial_t - c^2\partial_x\partial_x)u = 0$

[3a] Let $v = \dot{u}, w = v'$,

$$\partial_t \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Eigenvalues $= 0, \pm c$. Not a symmetric hyperbolic, nor a strongly hyperbolic.

[3b] Let $v = \dot{u}, w = cv'$,

$$\partial_t \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

Eigenvalues $= 0, \pm c$. Not a symmetric hyperbolic, nor a strongly hyperbolic.

[4] Let $f = \dot{u} - cu', g = \dot{u} + cu'$,

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \partial_x \begin{pmatrix} f \\ g \end{pmatrix} \tag{12}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic, de-coupled.

Exercise 2 of hyperbolic formulation | Maxwell equations

Consider the Maxwell equations in the vacuum space,

$$\operatorname{div}\mathbf{E} = 0, \tag{1a}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{1b}$$

$$\operatorname{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0, \tag{1c}$$

$$\operatorname{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{1d}$$

Exercise 2 of hyperbolic formulation | Maxwell equations | (cont.)

- Take a pair of variables as $u^i = (E_1, E_2, E_3, B_1, B_2, B_3)^T$, and write (1c) and (1d) in the matrix form

$$\partial_t \begin{bmatrix} E_i \\ B_i \end{bmatrix} \cong \underbrace{\begin{bmatrix} A^{l\ j}_{\ i} & B^{l\ j}_{\ i} \\ C^{l\ j}_{\ i} & D^{l\ j}_{\ i} \end{bmatrix}}_{\text{Hermitian?}} \partial_l \begin{bmatrix} E_j \\ B_j \end{bmatrix}. \tag{2}$$

- In the Maxwell case, we see immediately

$$\partial_t u_i = c \begin{pmatrix} 0 & \epsilon_i^{\ lm} \\ -\epsilon_i^{\ lm} & 0 \end{pmatrix} \partial_l u_m$$

or with the actual components

$$\partial_t \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = c \begin{pmatrix} 0 & \begin{pmatrix} 0 & -\delta^l_3 & \delta^l_2 \\ \delta^l_3 & 0 & -\delta^l_1 \\ -\delta^l_2 & \delta^l_1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \delta^l_3 & -\delta^l_2 \\ -\delta^l_3 & 0 & \delta^l_1 \\ \delta^l_2 & -\delta^l_1 & 0 \end{pmatrix} & 0 \end{pmatrix} \partial_l \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

That is, symmetric hyperbolic system.

Exercise 2 of hyperbolic formulation | Maxwell equations | (cont.)

- The eigen-equation of the characteristic matrix becomes

$$\det\begin{pmatrix} A^{l}{}_{i}{}^{j} - \lambda^{l}\delta_i^j & B^{l}{}_{i}{}^{j} \\ C^{l}{}_{i}{}^{j} & D^{l}{}_{i}{}^{j} - \lambda^{l}\delta_i^j \end{pmatrix} = \det\begin{pmatrix} \begin{pmatrix} -\lambda^l & 0 & 0 \\ 0 & -\lambda^l & 0 \\ 0 & 0 & -\lambda^l \end{pmatrix} & c\begin{pmatrix} 0 & -\delta_3^l & \delta_2^l \\ \delta_3^l & 0 & -\delta_1^l \\ -\delta_2^l & \delta_1^l & 0 \end{pmatrix} \\ c\begin{pmatrix} 0 & \delta_3^l & -\delta_2^l \\ -\delta_3^l & 0 & \delta_1^l \\ \delta_2^l & -\delta_1^l & 0 \end{pmatrix} & \begin{pmatrix} -\lambda^l & 0 & 0 \\ 0 & -\lambda^l & 0 \\ 0 & 0 & -\lambda^l \end{pmatrix} \end{pmatrix} = 0$$

We therefore obtain the eigenvalues as

$$0 \text{ (2 multi)}, \qquad \pm c\sqrt{(\delta_1^l)^2 + (\delta_2^l)^2 + (\delta_3^l)^2} \equiv \pm c \text{ (2 each)}$$

By adding constraints (1a) and (1b) in the RHS of equations, and see what will be happend.

$$\partial_t u_i = c \begin{pmatrix} 0 & -\epsilon_i{}^{lm} \\ \epsilon_i{}^{lm} & 0 \end{pmatrix} \partial_l u_m + c \begin{pmatrix} x \\ y \end{pmatrix} \partial_k E_k + c \begin{pmatrix} z \\ w \end{pmatrix} \partial_k B_k, \tag{3}$$

where x, y, z, w are parameters.

Exercise 3 of hyperbolic formulation | Adjusted Maxwell equations (cont.)

By adding constraints (1a) and (1b) in the RHS of equations, and see what will be happend.

$$\partial_t u_i = c \begin{pmatrix} 0 & -\epsilon_i{}^{lm} \\ \epsilon_i{}^{lm} & 0 \end{pmatrix} \partial_l u_m + c \begin{pmatrix} x \\ y \end{pmatrix} \partial_k E_k + c \begin{pmatrix} z \\ w \end{pmatrix} \partial_k B_k, \qquad (3)$$

where x, y, z, w are parameters.

- The actual components are

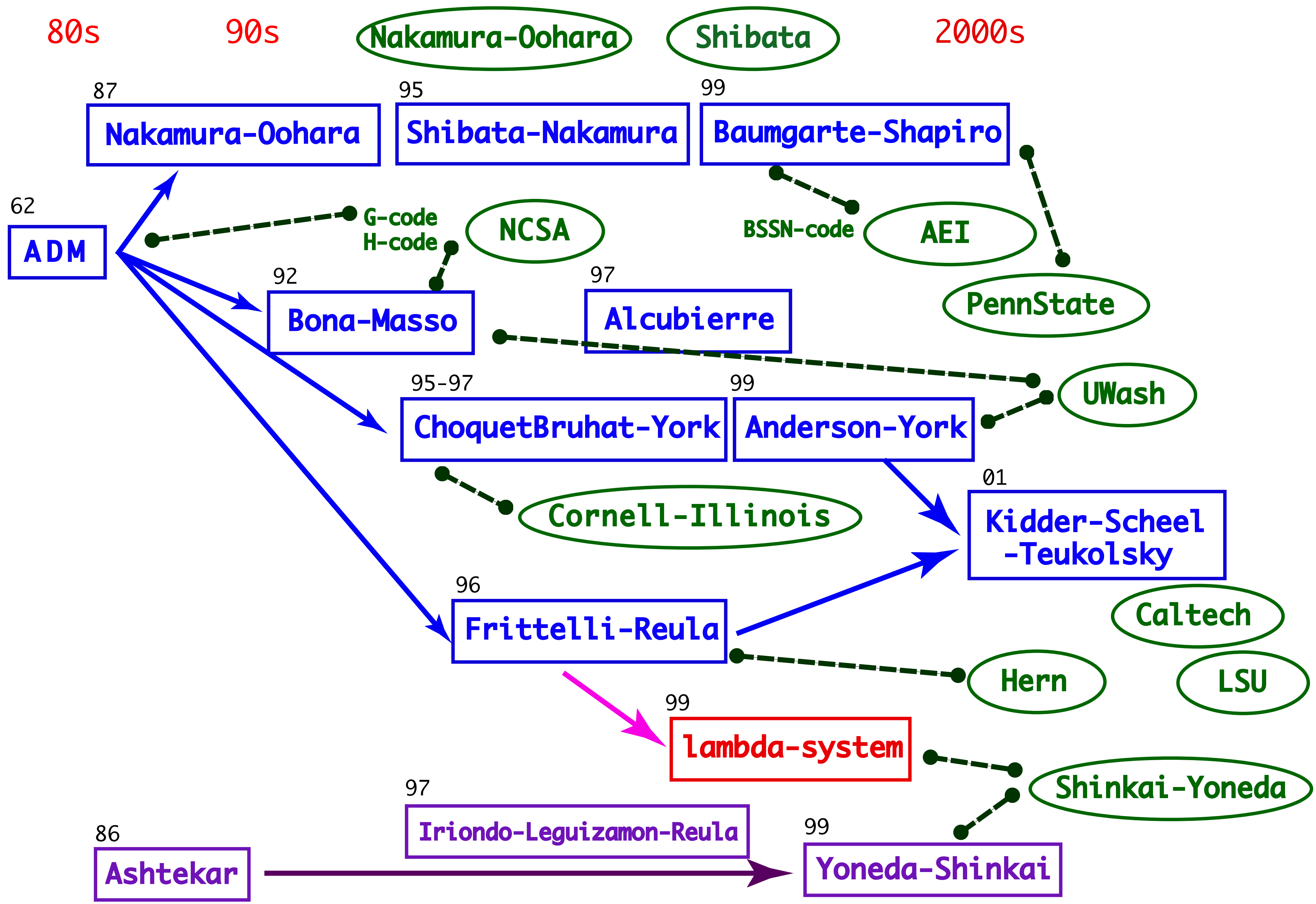
$$\partial_t \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = c \begin{pmatrix} x \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} & z \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} + \begin{pmatrix} 0 & -\delta_3^l & \delta_2^l \\ \delta_3^l & 0 & -\delta_1^l \\ -\delta_2^l & \delta_1^l & 0 \end{pmatrix} \\ y \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} + \begin{pmatrix} 0 & \delta_3^l & -\delta_2^l \\ -\delta_3^l & 0 & \delta_1^l \\ \delta_2^l & -\delta_1^l & 0 \end{pmatrix} & w \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} \end{pmatrix} \partial_l \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

  We see that adding constraint terms break the symmetricity of the characteristic matrix.

- The eigenvalues will be changed as

  $$\frac{c}{2}\left(x + w \pm \sqrt{x^2 - 2xw + w^2 + 4yz}\right)(\delta_1^l + \delta_2^l + \delta_3^l) \text{ (1 each)}, \qquad \pm c \text{ (2 each)}.$$

  The zero eigenvalues disappear by adding constraints, and they can be also $|c|$ if the parameters have the relation $(yz - xw - 1)^2 = (x + w)^2$.



80s
90s
Nakamura-Oohara
Shibata
2000s
87
Nakamura-Oohara
95
Shibata-Nakamura
99
Baumgarte-Shapiro
62
ADM
G-code
H-code
NCSA
BSSN-code
AEI
92
Bona-Masso
97
Alcubierre
PennState
95-97
ChoquetBruhat-York
99
Anderson-York
UWash
Cornell-Illinois
01
Kidder-Scheel
-Teukolsky
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Frittelli-Reula
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lambda-system
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Ashtekar
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Yoneda-Shinkai

Kidder-Scheel-Teukolsky hyperbolic formulation (Anderson-York + Frittelli-Reula)

Phys. Rev. D. 64 (2001) 064017

- Construct a First-order form using variables $(K_{ij}, g_{ij}, d_{kij})$ where $d_{kij} \equiv \partial_k g_{ij}$
  Constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{C}_{kij}, \mathcal{C}_{klij})$ where $\mathcal{C}_{kij} \equiv d_{kij} - \partial_k g_{ij}$, and $\mathcal{C}_{klij} \equiv \partial_{[k} d_{l]ij}$
- Densitize the lapse, $Q = \log(N g^{-\sigma})$
- Adjust equations with constraints

$$\begin{aligned}
\hat{\partial}_0 g_{ij} &= -2NK_{ij} \\
\hat{\partial}_0 K_{ij} &= (\cdots) + \gamma N g_{ij}\mathcal{H} + \zeta N g^{ab}\mathcal{C}_{a(ij)b} \\
\hat{\partial}_0 d_{kij} &= (\cdots) + \eta N g_{k(i}\mathcal{M}_{j)} + \chi N g_{ij}\mathcal{M}_k
\end{aligned}$$

- Re-deining the variables $(P_{ij}, g_{ij}, M_{kij})$

$$\begin{aligned}
P_{ij} &\equiv K_{ij} + \hat{z} g_{ij} K, \\
M_{kij} &\equiv (1/2)[\hat{k} d_{kij} + \hat{e} d_{(ij)k} + g_{ij}(\hat{a} d_k + \hat{b} b_k) + g_{k(i}(\hat{c} d_{j)} + \hat{d} b_{j)})], \quad d_k = g^{ab} d_{kab}, b_k = g^{ab} d_{abk}
\end{aligned}$$

The redefinition parameters

- do not change the eigenvalues of evolution eqs.
- do not effect on the principal part of the constraint evolution eqs.
- do affect the eigenvectors of evolution system.
- do affect nonlinear terms of evolution eqs/constraint evolution eqs.

Numerical experiments of KST hyperbolic formulation

Weak wave on flat spacetime.
-> No non-principal part.

-> We can observe the features of hyperbolicity.

-> Using constraints in RHS may improve the blow-up.

PHYSICAL REVIEW D **66**, 064011 (2002)

Stability properties of a formulation of Einstein's equations

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(Received 27 May 2002; published 19 September 2002)

We study the stability properties of the Kidder-Scheel-Teukolsky (KST) many-parameter formulation of Einstein's equations for weak gravitational waves on flat space-time from a continuum and numerical point of view. At the continuum, performing a linearized analysis of the equations around flat space-time, it turns out that they have, essentially, no non-principal terms. As a consequence, in the weak field limit the stability properties of this formulation depend only on the level of hyperbolicity of the system. At the discrete level we present some simple one-dimensional simulations using the KST family. The goal is to analyze the type of instabilities that appear as one changes parameter values in the formulation. Lessons learned in this analysis can be applied in other formulations with similar properties.

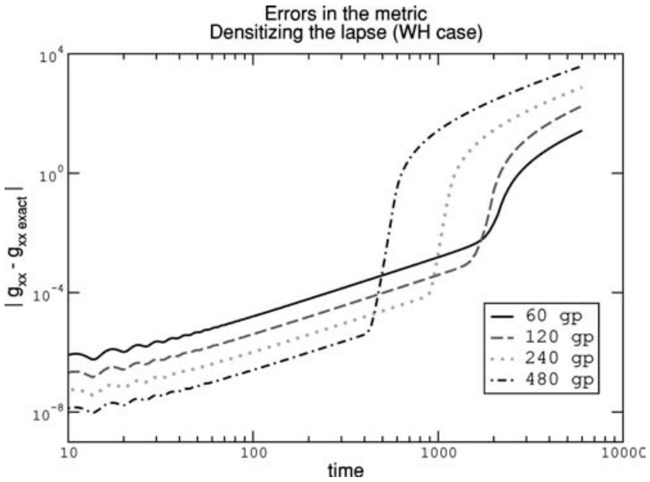

FIG. 7. L_2 norms of the errors for the metric.

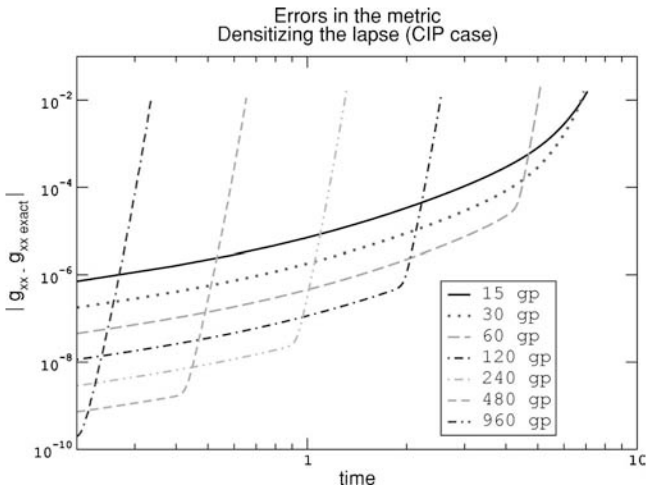

FIG. 9. L_2 norm of the errors for the metric.

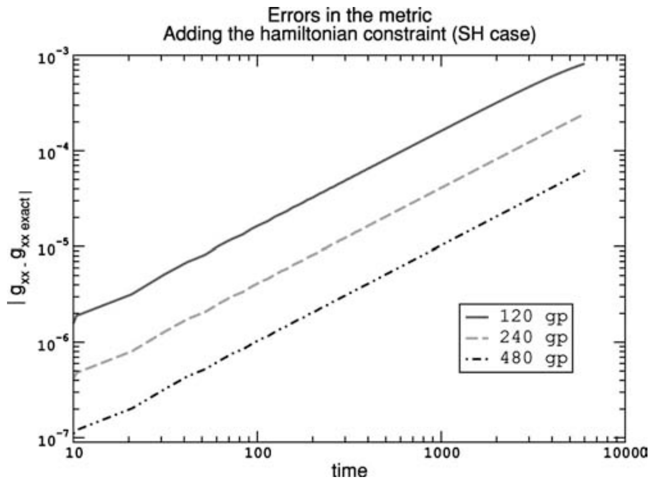

FIG. 12. L_2 norm of the errors for the metric.

Class. Quantum Grav. **17** (2000) 4799–4822. Printed in the UK PII: S0264-9381(00)50711-4

Hyperbolic formulations and numerical relativity: experiments using Ashtekar's connection variables

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Received 3 May 2000, in final form 13 September 2000

Abstract. In order to perform accurate and stable long-time numerical integration of the Einstein equation, several hyperbolic systems have been proposed. Here we present a numerical comparison between weakly hyperbolic, strongly hyperbolic and symmetric hyperbolic systems based on Ashtekar's connection variables. The primary advantage for using this connection formulation in this experiment is that we can keep using the same dynamical variables for all levels of hyperbolicity. Our numerical code demonstrates gravitational wave propagation in plane-symmetric spacetimes, and we compare the accuracy of the simulation by monitoring the violation of the constraints. By comparing with results obtained from the weakly hyperbolic system, we observe that the strongly and symmetric hyperbolic system show better numerical performance (yield less constraint violation), but not so much difference between the latter two. Rather, we find that the symmetric hyperbolic system is not always the best in terms of numerical performance.

This study is the first to present full numerical simulations using Ashtekar's variables. We also describe our procedures in detail.

$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c\, \tilde{E}^i_c,$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0,$$

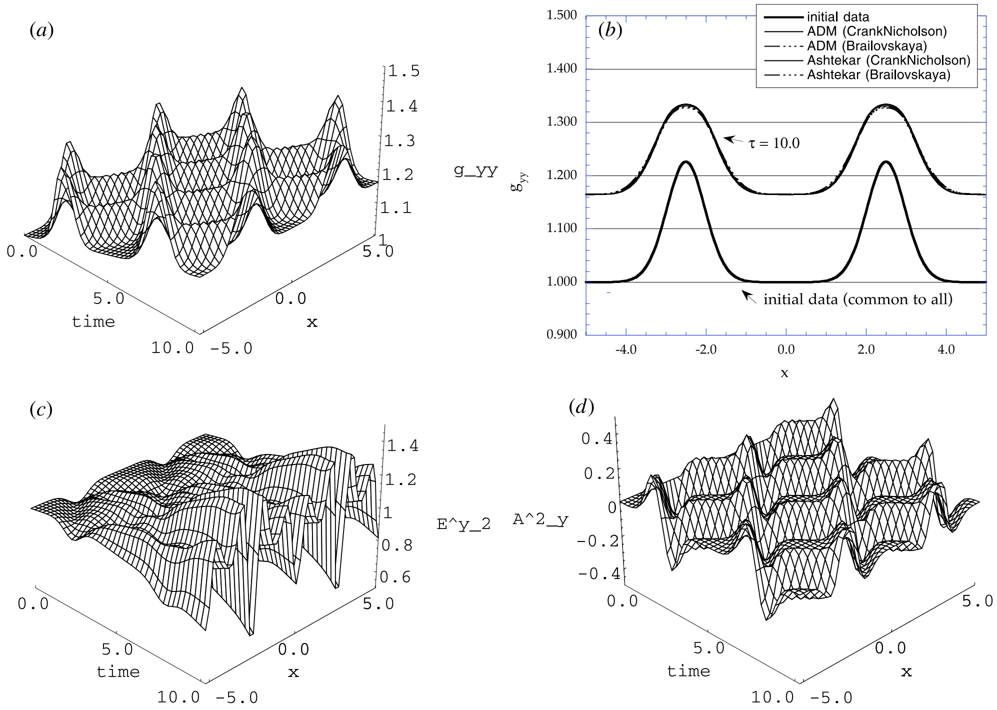

Figure 2. Images of gravitational wave propagation and comparisons of dynamical behaviour of Ashtekar's variables and ADM variables. We applied the same initial data of two +-mode pulse waves ($a = 0.2, b = 2.0, c = \pm 2.5$ in equation (21) and $K_0 = -0.025$), and the same slicing condition, the standard geodesic slicing condition ($N = 1$). (*a*) Image of the 3-metric component g_{yy} of a function of proper time τ and coordinate x. This behaviour can be seen identically both in ADM and Ashtekar evolutions, and both with the Brailovskaya and Crank–Nicholson time-integration scheme. Part (*b*) explains this fact by comparing the snapshot of g_{yy} at the same proper time slice ($\tau = 10$), where four lines at $\tau = 10$ are looked at identically. Parts (*c*) and (*d*) are of the real part of the densitized triad $\tilde{E}^y_2$, and the real part of the connection $\mathcal{A}^2_y$, respectively, obtained from the evolution of the Ashtekar variables.

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Hyperbolic formulations and numerical relativity: experiments using Ashtekar's connection variables

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$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c\, \tilde{E}^i_c + \kappa\, P^i{}_{ab}\, \mathcal{C}_G^{\mathrm{ASH}\,b},$$

$$\text{where} \qquad P^i{}_{ab} \equiv N^i \delta_{ab} + \mathrm{i}\underset{\sim}{N}\epsilon_{ab}{}^c \tilde{E}^i_c,$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa\, Q^a_i \mathcal{C}_H^{\mathrm{ASH}} + \kappa\, R_i{}^{ja}\, \mathcal{C}_{Mj}^{\mathrm{ASH}},$$

$$\text{where} \qquad Q^a_i \equiv e^{-2}\underset{\sim}{N}\tilde{E}^a_i, \qquad R_i{}^{ja} \equiv \mathrm{i}e^{-2}\underset{\sim}{N}\epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c.$$

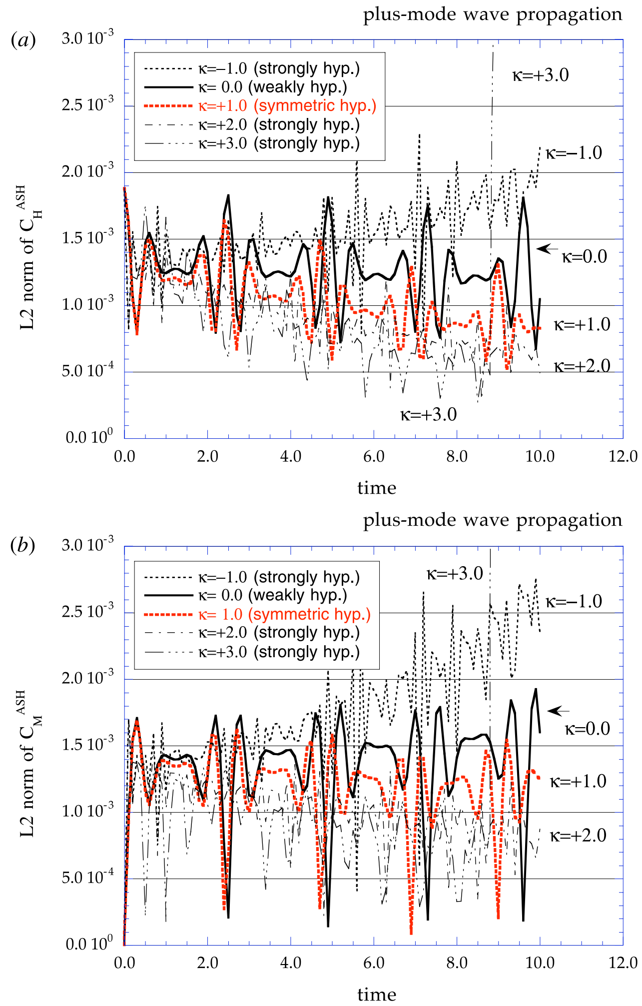


Figure 6. Comparisons of the 'adjusted' system with the different multiplier, κ, in equations (31) and (32). The model uses +-mode pulse waves ($a = 0.1$, $b = 2.0$, $c = \pm 2.5$) in equation (21) in a background $K_0 = -0.025$. Plots are of the L2 norm of the Hamiltonian and momentum constraint equations, $\mathcal{C}_H^{\mathrm{ASH}}$ and $\mathcal{C}_M^{\mathrm{ASH}}$ ((a) and (b), respectively). We see some κ produce a better performance than the symmetric hyperbolic system.

No drastic differences in stability between 3 levels of hyperbolicity.

BSSN Pros:

- With Bona-Masso-type α (1+log), and frozon β ($\partial_t \Gamma^i \sim 0$), BSSN plus auxiliary variables form a 1st-order symmetric hyperbolic system,

  Heyer-Sarbach, [PRD 70 (2004) 104004]

- If we define 2nd order symmetric hyperbolic form, principal part of BSSN can be one of them,

  Gundlach-MartinGarcia, [PRD 70 (2004) 044031, PRD 74 (2006) 024016]

BSSN Cons:

- Existence of an ill-posed solution in BSSN (as well in ADM)

  Frittelli-Gomez [JMP 41 (2000) 5535]

- Gauge shocks in Bona-Masso slicing is inevitable. Current 3D BH simulation is lack of resolution.

  Garfinke-Gundlach-Hilditch [arXiv:0707.0726]

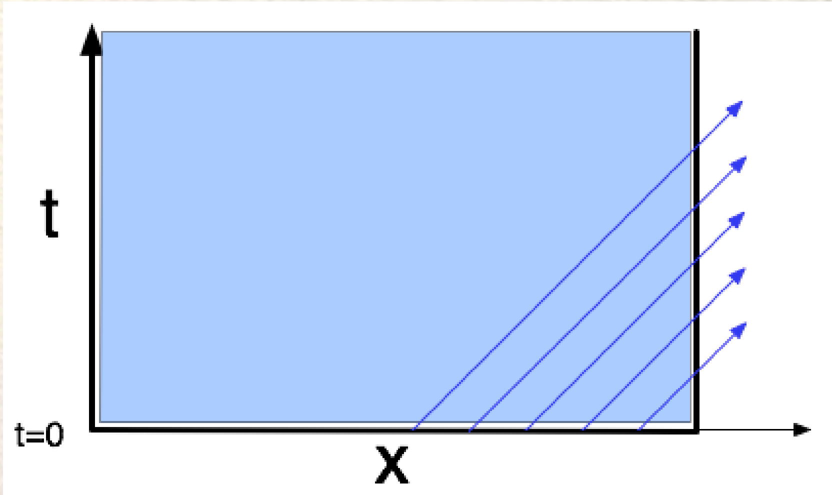
strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

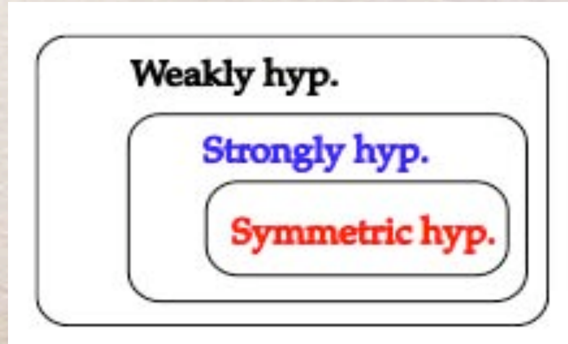
“YES” group
“Well-posed!”, $||u(t)|| \leq e^{\kappa t}||u(0)||$
Mathematically Rigorous Proofs
IBVP in future

Initial Boundary Value Problem

Consistent treatment is available only for symmetric hyperbolic systems.



- GR-IBVP
  - Stewart, CQG15 (98) 2865
- Tetrad formalism
  - Friedrich & Nagy, CMP201 (99) 619
- Linearized Bianchi eq.
  - Buchman & Sarbach, CQG 23 (06) 6709
- Constraint-preserving BC
  - Kreiss, Reula, Sarbach & Winicour, CQG 24 (07) 5973
- Higher-order absorbing BC
  - Ruiz, Rinne & Sarbach, CQG 24 (07) 6349



strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \leq e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \leq e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

Which level of hyperbolicity is necessary?

symmetric hyperbolic $\subset$ strongly hyperbolic $\subset$ weakly hyperbolic systems,

Advantages in Numerics (90s)	
Advantages in sym. hyp. – KST formulation by LSU	

strategy 2 Hyperbolic formulation (cont.)

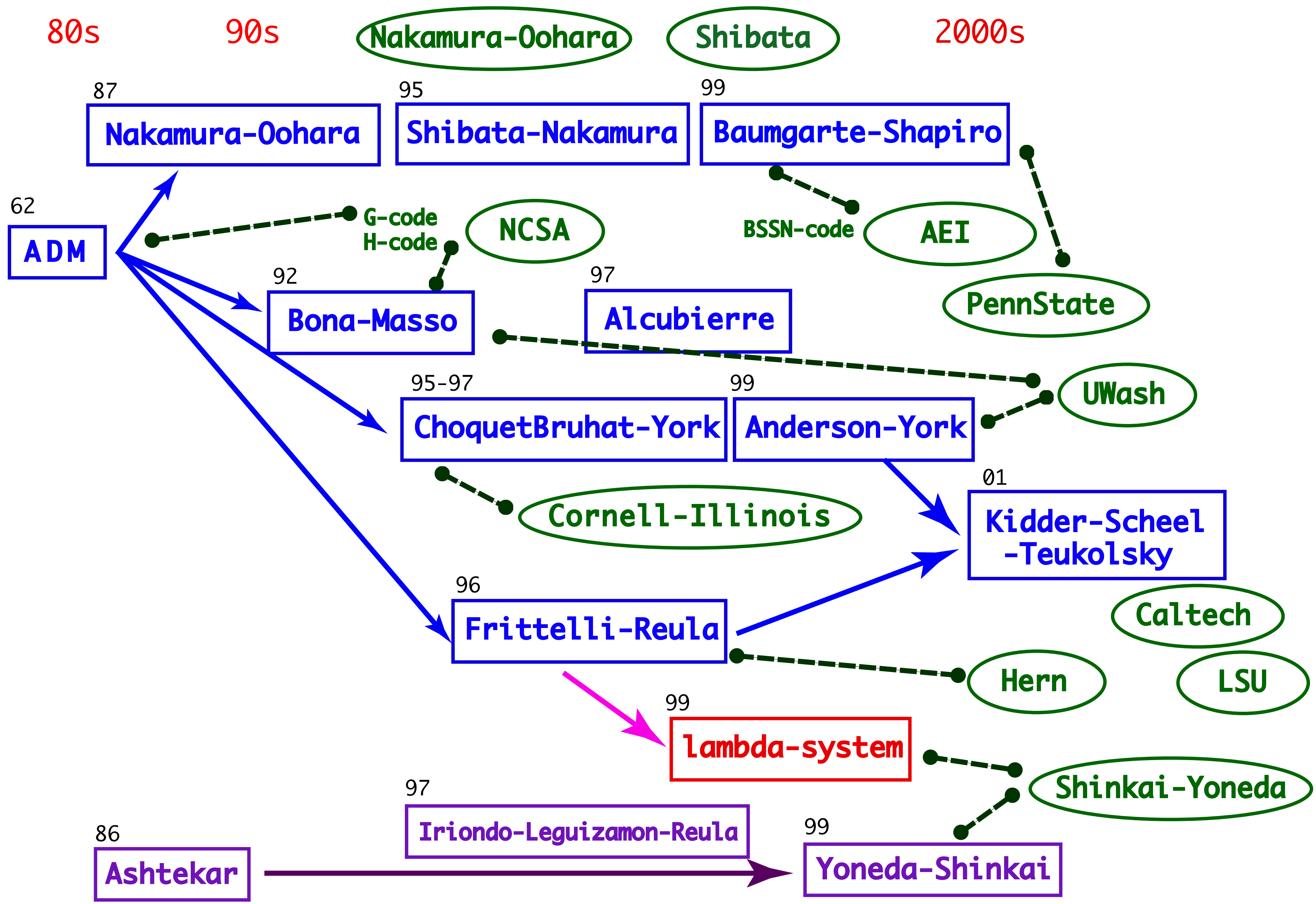
Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \le e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

Which level of hyperbolicity is necessary?

symmetric hyperbolic $\subset$ strongly hyperbolic $\subset$ weakly hyperbolic systems,

Advantages in Numerics (90s)	These were vs. ADM
Advantages in sym. hyp. – KST formulation by LSU	Not much differences in hyperbolic 3 levels – FR formulation, by Hern – Ashtekar formulation, by HS-Yoneda sym. hyp. is not always the best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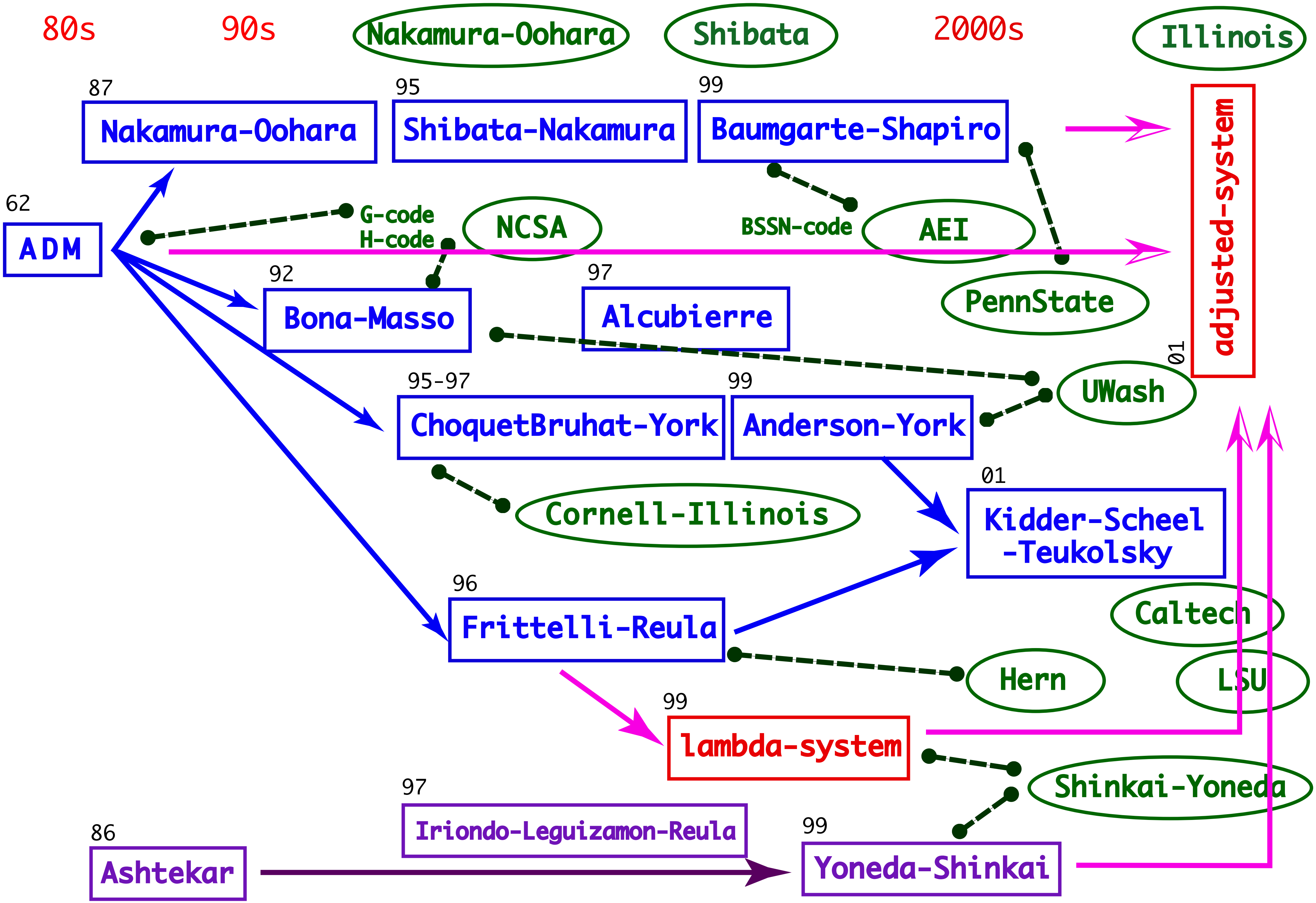
Yoneda-Shinkai



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Ashtekar
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Summary up to here (1st half)

[Keyword 1] Formulation Problem

Although mathematically equivalent, different set of equations shows different numerical stability.

[Keyword 2] ADM formulation

The starting formulation (Historically & Numerically).
Successes in 90s, but not for binary BH-BH/NS-NS problems.

[Keyword 3] BSSN formulation

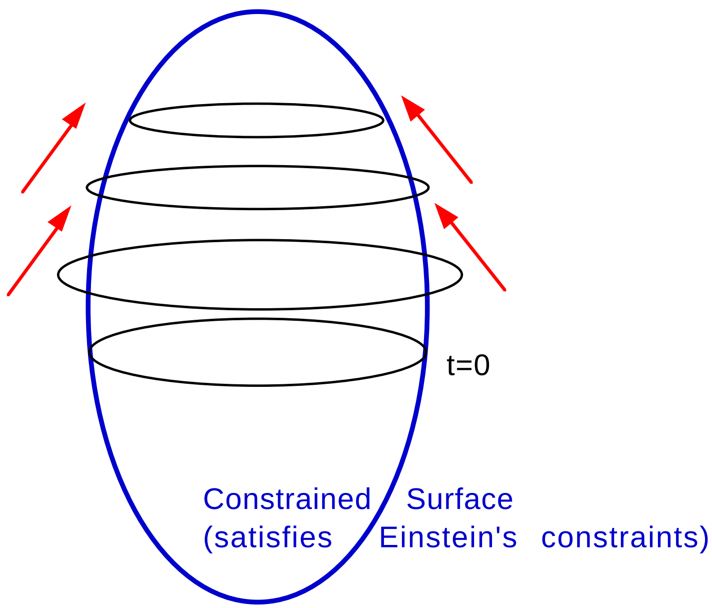
New variables and gauge fixing to ADM, shows better stability.
The reason why it is better was not known at first.
Many simulation groups uses BSSN. Technical tips are accumulated.

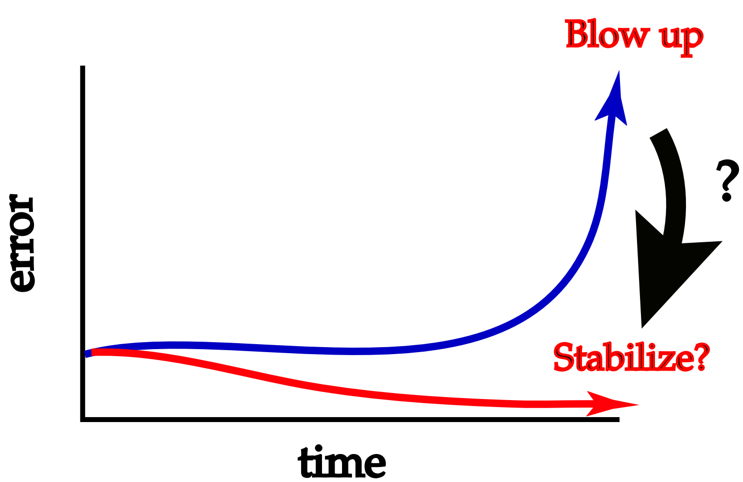
[Keyword 4] hyperbolic formulations

Mathematical classification of PDE shows "well-posedness", but its meaning is limited.
Many versions of hyperbolic Einstein equations are available.
Some group try to show the advantage of BSSN using "hyperbolicity".
But are they really helpful in numerics?

strategy 3 "Asymptotically Constrained" system / "Constraint Damping" system

Formulate a system which is "asymptotically constrained" against a violation of constraints
Constraint Surface as an Attractor





method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM/BSSN formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\partial_t u = J\partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\partial_t \lambda = \alpha C - \beta \lambda$ $(\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]
- The version for Z4 hyperbolic system by Gundlach-Calabrese-Hinder-MartinGarcia [CQG22(05)3767] $\Rightarrow$ Pretorius noticed the idea of "constraint damping" [PRL95(05)121101]

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Hyperbolic formulations and numerical relativity: II. asymptotically constrained systems of Einstein equations

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Maxwell-lambda system works as expected.

$$\partial_t \begin{pmatrix} E^i \\ B^i \\ \lambda_E \\ \lambda_B \end{pmatrix} = \begin{pmatrix} 0 & -c\epsilon^i{}_j{}^l & \alpha_1 \delta^{li} & 0 \\ c\epsilon^i{}_j{}^l & 0 & 0 & \alpha_2 \delta^{li} \\ \alpha_1 \delta^l_j & 0 & 0 & 0 \\ 0 & \alpha_2 \delta^l_j & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} E^j \\ B^j \\ \lambda_E \\ \lambda_B \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\beta_1 \lambda_E \\ -\beta_2 \lambda_B \end{pmatrix}.$$

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha_1 k^2 & 0 \\ 0 & 0 & 0 & -\alpha_2 k^2 \\ \alpha_1 & 0 & -\beta_1 & 0 \\ 0 & \alpha_2 & 0 & -\beta_2 \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix},$$

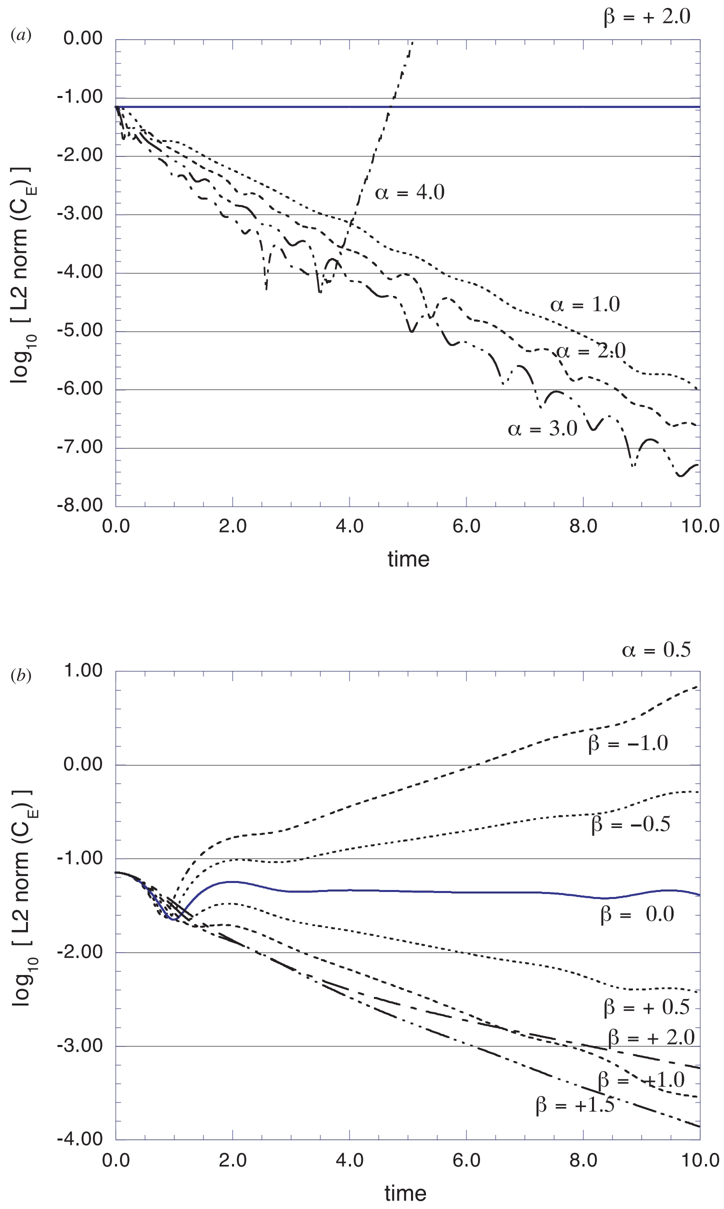

Figure 1. Demonstration of the λ system in the Maxwell equation. (*a*) Constraint violation (L2 norm of C_E) versus time with constant β (= 2.0) but changing α. Here $\alpha = 0$ means no λ system. (*b*) The same plot with constant α (= 0.5) but changing β. We see better performance for $\beta > 0$, which is the case of negative eigenvalues of the constraint propagation equation. The constants in (2.18) were chosen as $A = 200$ and $B = 1$.

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Ashtekar-lambda system works as expected, as well.

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} \mathcal{M}^l{}_a{}^{bi}{}_j & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il} \delta_a{}^b \\ 0 & \mathcal{N}^{la}{}_{ib}{}^j & \mathrm{i}\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^j_i \tilde{E}^{la} - \gamma^{lj} \tilde{E}^a_i) & 0 \\ 0 & -\mathrm{i}\alpha_1 \epsilon_b{}^{cd} \tilde{E}^j_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^j_i \tilde{E}^l_b - \delta^l_i \tilde{E}^j_b) & 0 & 0 & 0 \\ \alpha_3 \delta^b_a \delta^l_j & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^j_b \\ \mathcal{A}^b_j \\ \lambda \\ \lambda_j \\ \lambda_b \end{pmatrix}.$$

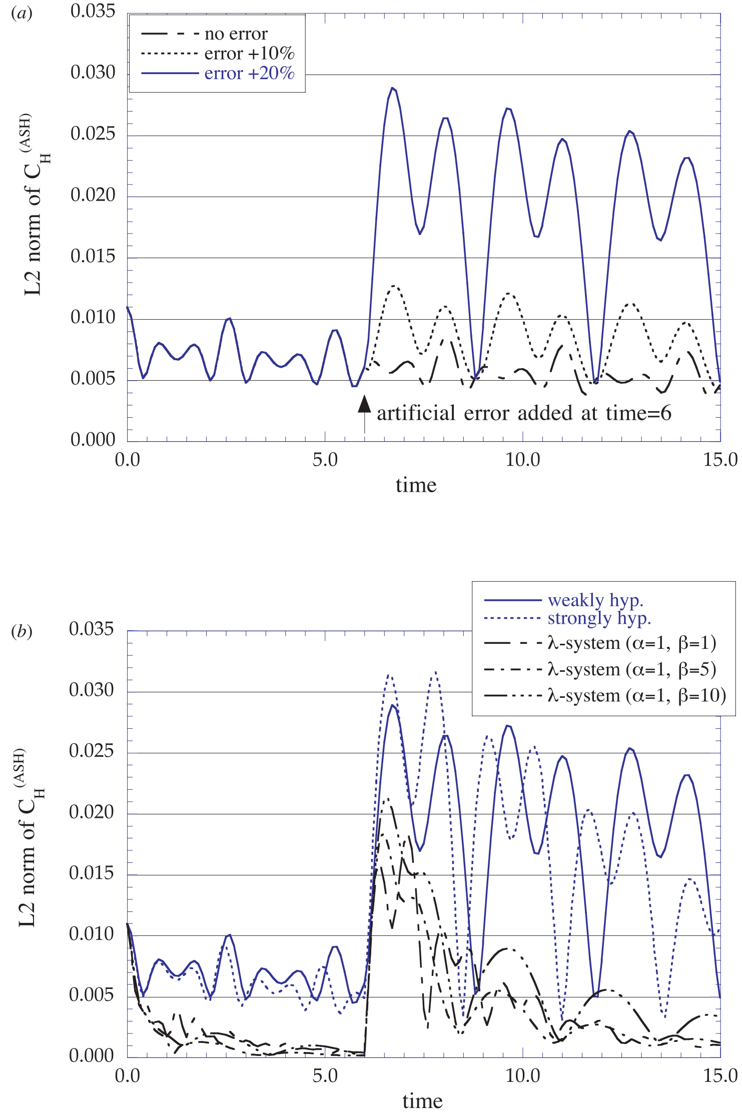

Figure 3. Demonstration of the λ system in the Ashtekar equation. We plot the violation of the constraint (the L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$) for the cases of plane-wave propagation under the periodic boundary. To see the effect more clearly, we added an artificial error at $t = 6$. Part (*a*) shows how the system goes bad depending on the amplitude of artificial error. The error was of the form $\mathcal{A}^2_y \to \mathcal{A}^2_y(1+\text{ error})$. All the curves are of the evolution of Ashtekar's original equation (no λ system). Part (*b*) shows the effect of the λ system. All the curves have 20% error amplitude, but show the difference of the evolution equations. The full curve is for Ashtekar's original equation (the same as in (*a*)), the dotted curve is for the strongly hyperbolic Ashtekar equation. Other curves are of λ systems, which produce a better performance than that of the strongly hyperbolic system.

Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)} \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$\begin{aligned} \partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + Q_i C_B, \\ \partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + S_i C_B, \\ & C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0, \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

Constraint propagation equations

$$\begin{aligned} \partial_t C_E &= (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B), \\ \partial_t C_B &= (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B), \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

CAFs?

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}$$

$$\Rightarrow \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2$$

Therefore CAFs become negative-real when

$$P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0$$

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Adjusted-Maxwell system works as well.

3.2.1. Adjusted system. Here we again consider the Maxwell equations (2.9)–(2.11). We start from the adjusted dynamical equations

$$\partial_t E_i = c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + p^j{}_i(\partial_j C_E) + Q_i C_B + q^j{}_i(\partial_j C_B), \qquad (3.7)$$

$$\partial_t B_i = -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + r^j{}_i(\partial_j C_E) + S_i C_B + s^j{}_i(\partial_j C_B), \qquad (3.8)$$

where P, Q, R, S, p, q, r and s are multipliers. These dynamical equations adjust the constraint propagation equations as

$$\partial_t C_E = (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B)$$
$$+(\partial_i p^{ji})(\partial_j C_E) + p^{ji}(\partial_i\partial_j C_E) + (\partial_i q^{ji})(\partial_j C_B) + q^{ji}(\partial_i\partial_j C_B), \qquad (3.9)$$

$$\partial_t C_B = (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B)$$
$$+(\partial_i r^{ji})(\partial_j C_E) + r^{ji}(\partial_i\partial_j C_E) + (\partial_i s^{ji})(\partial_j C_B) + s^{ji}(\partial_i\partial_j C_B). \qquad (3.10)$$

This will be expressed using Fourier components by

$$\partial_t \begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} = \begin{pmatrix}\partial_i P^i + \mathrm{i}P^i k_i + \mathrm{i}k_j(\partial_i p^{ji}) - k_i k_j p^{ji} & \partial_i Q^i + \mathrm{i}Q^i k_i + \mathrm{i}k_j(\partial_i q^{ji}) - k_i k_j q^{ji}\\ \partial_i R^i + \mathrm{i}R^i k_i + \mathrm{i}k_j(\partial_i r^{ji}) - k_i k_j r^{ji} & \partial_i S^i + \mathrm{i}S^i k_i + \mathrm{i}k_j(\partial_i s^{ji}) - k_i k_j s^{ji}\end{pmatrix}$$
$$\times \begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} =: T\begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix}. \qquad (3.11)$$

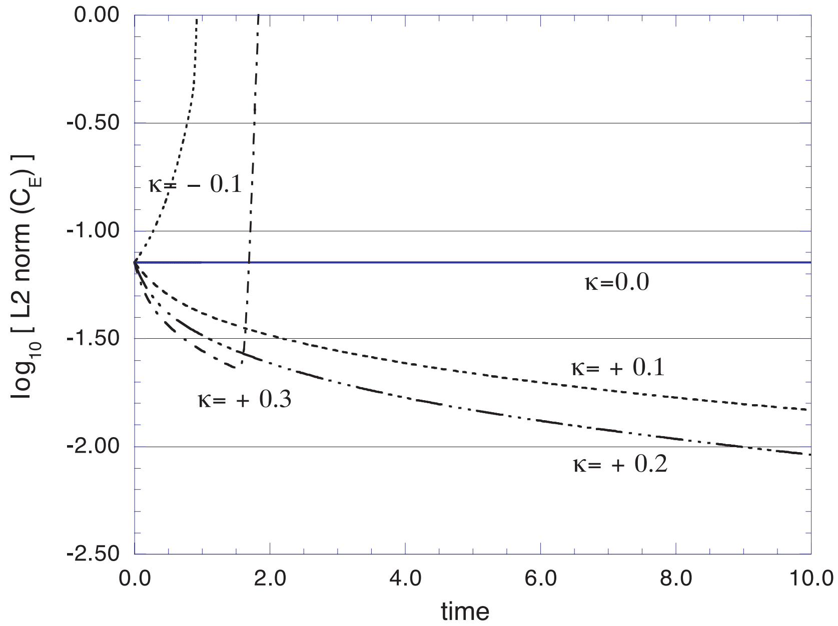

Figure 4. Demonstrations of the adjusted system in the Maxwell equation. We perform the same experiments with section 2.2.3 (figure 1). Constraint violation (L2 norm of C_E) versus time are plotted for various κ $(= p^j{}_i = s^j{}_i)$. We see that $\kappa > 0$ gives a better performance (i.e. negative real part eigenvalues for the constraint propagation equation), while excessively large positive κ makes the system divergent again.

Example: the Ashtekar equations

HS Yoneda, CQG 17 (2000) 4799

Adjusted dynamical equations:

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \underbrace{+X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}}_{adjust}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \underbrace{+Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}}_{adjust}$$

Adjusted and linearized:

$$X = Y = Z = 0,\ P^{ia}_b = \kappa_1(iN^i \delta^a_b),\ Q^a_i = \kappa_2(e^{-2}\underset{\sim}{N}\tilde{E}^a_i),\ R^{aj}{}_i = \kappa_3(-ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c)$$

Fourier transform and extract 0th order of the characteristic matrix:

$$\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \end{pmatrix} = \begin{pmatrix} 0 & i(1+2\kappa_3)k_j & 0 \\ i(1-2\kappa_2)k_i & \kappa_3 \epsilon^{kj}{}_i k_k & 0 \\ 0 & 2\kappa_3 \delta^j_a & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \end{pmatrix}$$

Eigenvalues:

$$\left(0, 0, 0, \pm\kappa_3\sqrt{-kx^2 - ky^2 - kz^2}, \pm\sqrt{(-1+2\kappa_2)(1+2\kappa_3)(kx^2+ky^2+kz^2)}\right)$$

In order to obtain non-positive real eigenvalues:

$$(-1+2\kappa_2)(1+2\kappa_3) < 0$$

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Adjusted-Ashtekar system works as well.

3.3.1. Adjusted system for controlling constraint violations. Here we only consider the adjusted system which controls the departures from the constraint surface. In the appendix, we present an advanced system which controls the violation of the reality condition together with a numerical demonstration.

Even if we restrict ourselves to adjusted equations of motion for $(\tilde{E}^i_a, \mathcal{A}^a_i)$ with constraint terms (no adjustment with derivatives of constraints), generally, we could adjust them as

$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}, \tag{3.14}$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i + Q^a_i \mathcal{C}_H + R_i{}^{ja} \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}, \tag{3.15}$$

where $X^i_a, Y^{ij}_a, Z^{ab}_i, P^{ib}_a, Q^a_i$ and R^{aj}_i are multipliers. However, in order to simplify the discussion, we restrict multipliers so as to reproduce the symmetric hyperbolic equations of motion [10, 11], i.e.

$$\begin{aligned} X &= Y = Z = 0, \\ P^{ib}_a &= \kappa_1 (N^i \delta^b_a + \mathrm{i}\underset{\sim}{N} \epsilon_a{}^{bc} \tilde{E}^i_c), \\ Q^a_i &= \kappa_2 (e^{-2} \underset{\sim}{N} \tilde{E}^a_i), \\ R_i{}^{ja} &= \kappa_3 (\mathrm{i} e^{-2} \underset{\sim}{N} \epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c). \end{aligned} \tag{3.16}$$

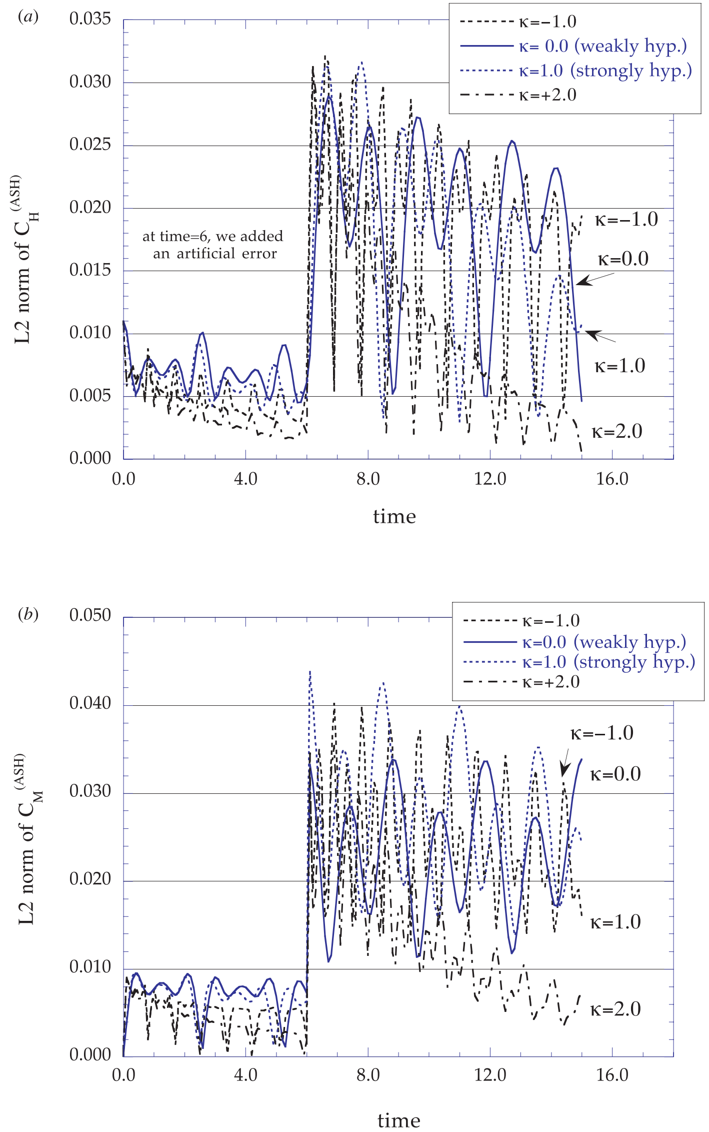


Figure 5. Demonstration of the adjusted system in the Ashtekar equation. We plot the violation of the constraint for the same model as figure 3(*b*). An artificial error term was added at $t = 6$, in the form of $\mathcal{A}^2_y \to \mathcal{A}^2_y(1+\text{ error})$, where error is + 20% as before. (*a*), (*b*) L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$, and momentum constraint equation, $\mathcal{C}_{Mx}$, respectively. The full curve is the case of $\kappa = 0$, that is the case of 'no adjusted' original Ashtekar equation (weakly hyperbolic system). The dotted curve is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see that the other curve ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case.

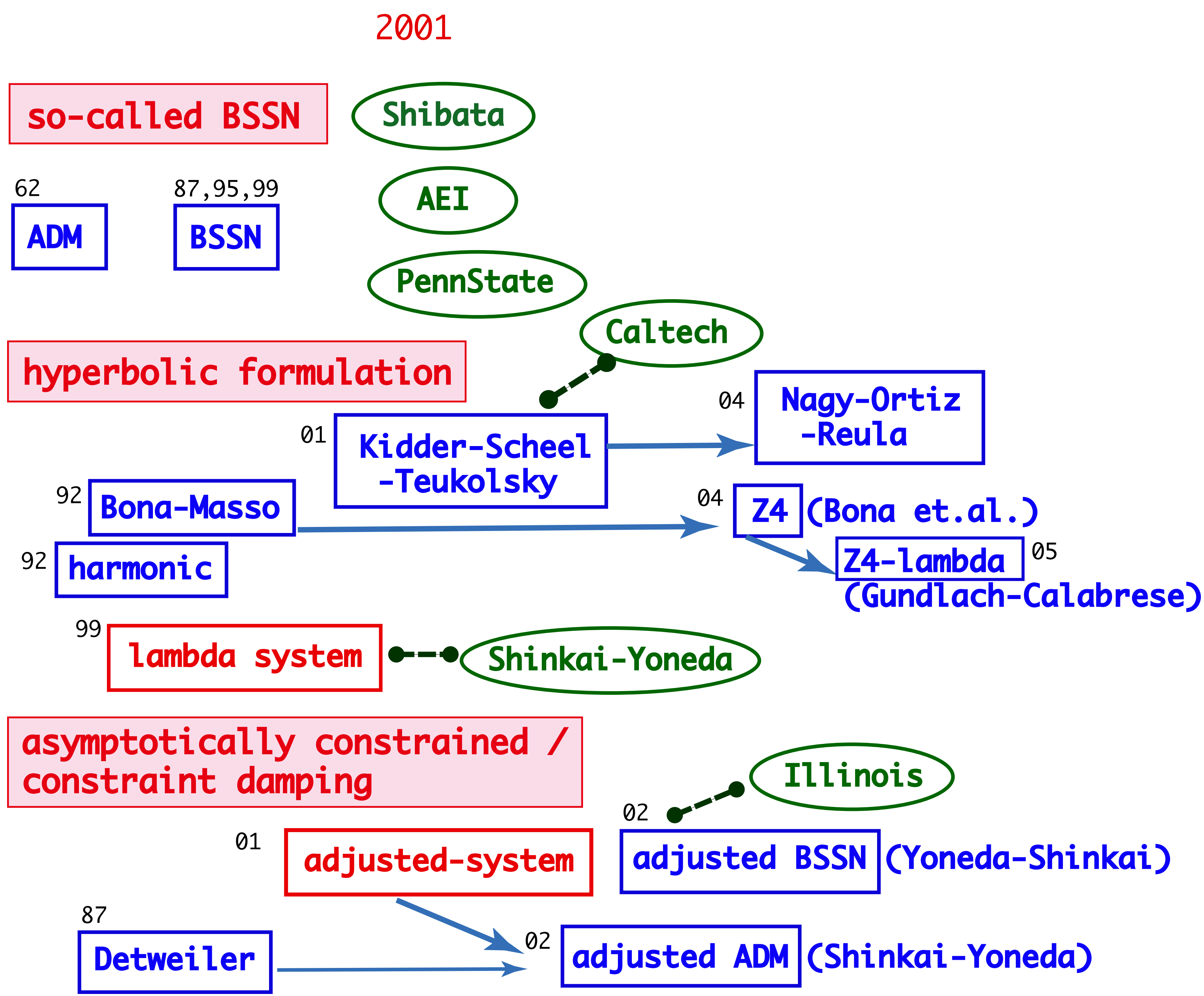
The Adjusted system (essentials):

Purpose:	Control the violation of constraints by reformulating the system so as to have a constrained surface an attractor.
Procedure:	Add a particular combination of constraints to the evolution equations, and adjust its multipliers.
Theoretical support:	Eigenvalue analysis of the constraint propagation equations.
Advantages:	Available even if the base system is not a symmetric hyperbolic.
Advantages:	Keep the number of the variable same with the original system.

Conjecture on Constraint Amplification Factors (CAFs):

(A) If CAF has a negative real-part (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a non-zero imaginary-part (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.


2001
so-called BSSN
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ADM
87,95,99
BSSN
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PennState
Caltech
hyperbolic formulation
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Kidder-Scheel -Teukolsky
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Nagy-Ortiz -Reula
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Bona-Masso
04
Z4 (Bona et.al.)
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harmonic
Z4-lambda
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lambda system
Shinkai-Yoneda
asymptotically constrained / constraint damping
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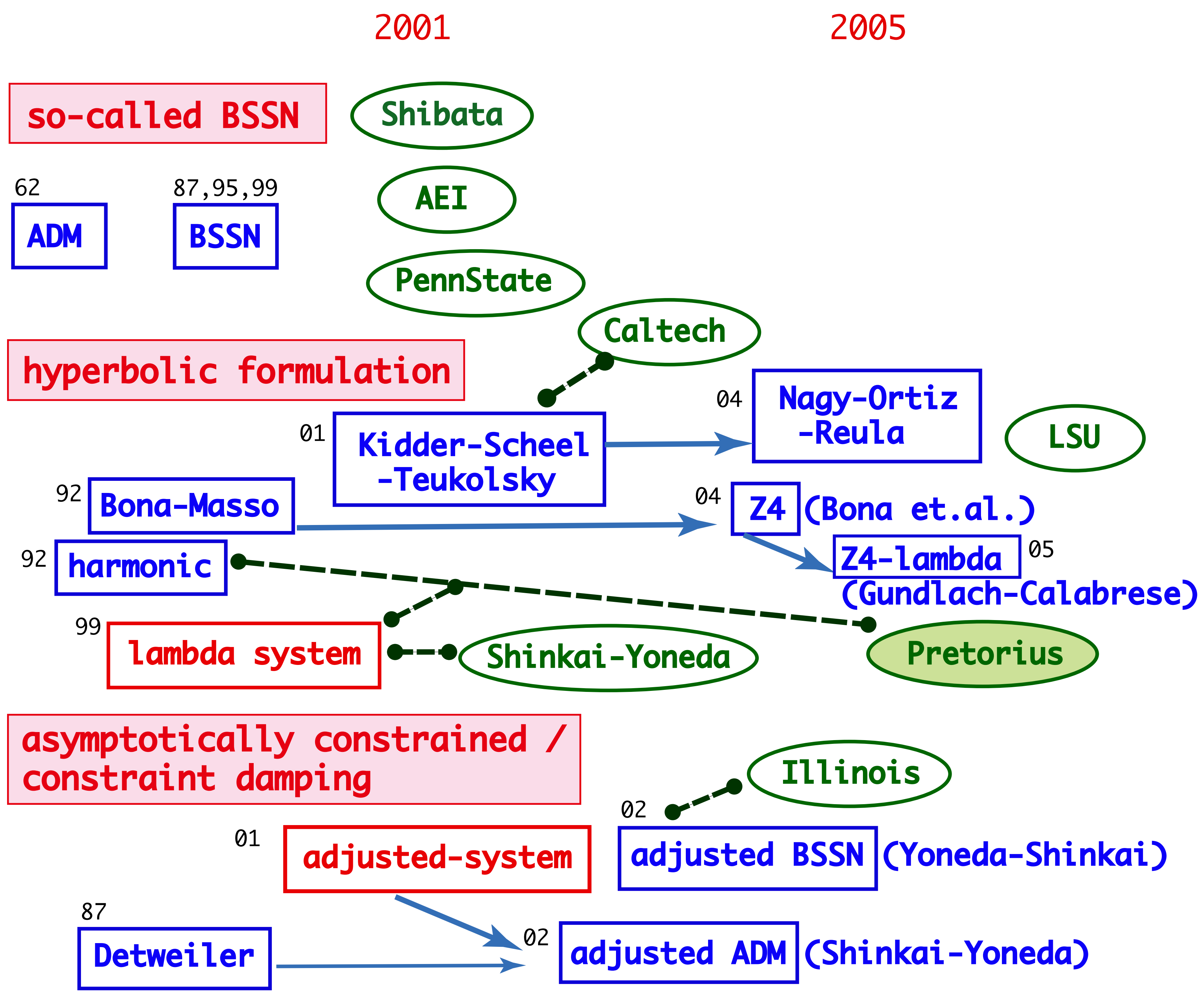
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so-called BSSN
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Caltech
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04
Nagy-Ortiz -Reula
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Bona-Masso
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(Bona et.al.)
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Z4-lambda
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lambda system
Shinkai-Yoneda
Pretorius
asymptotically constrained / constraint damping
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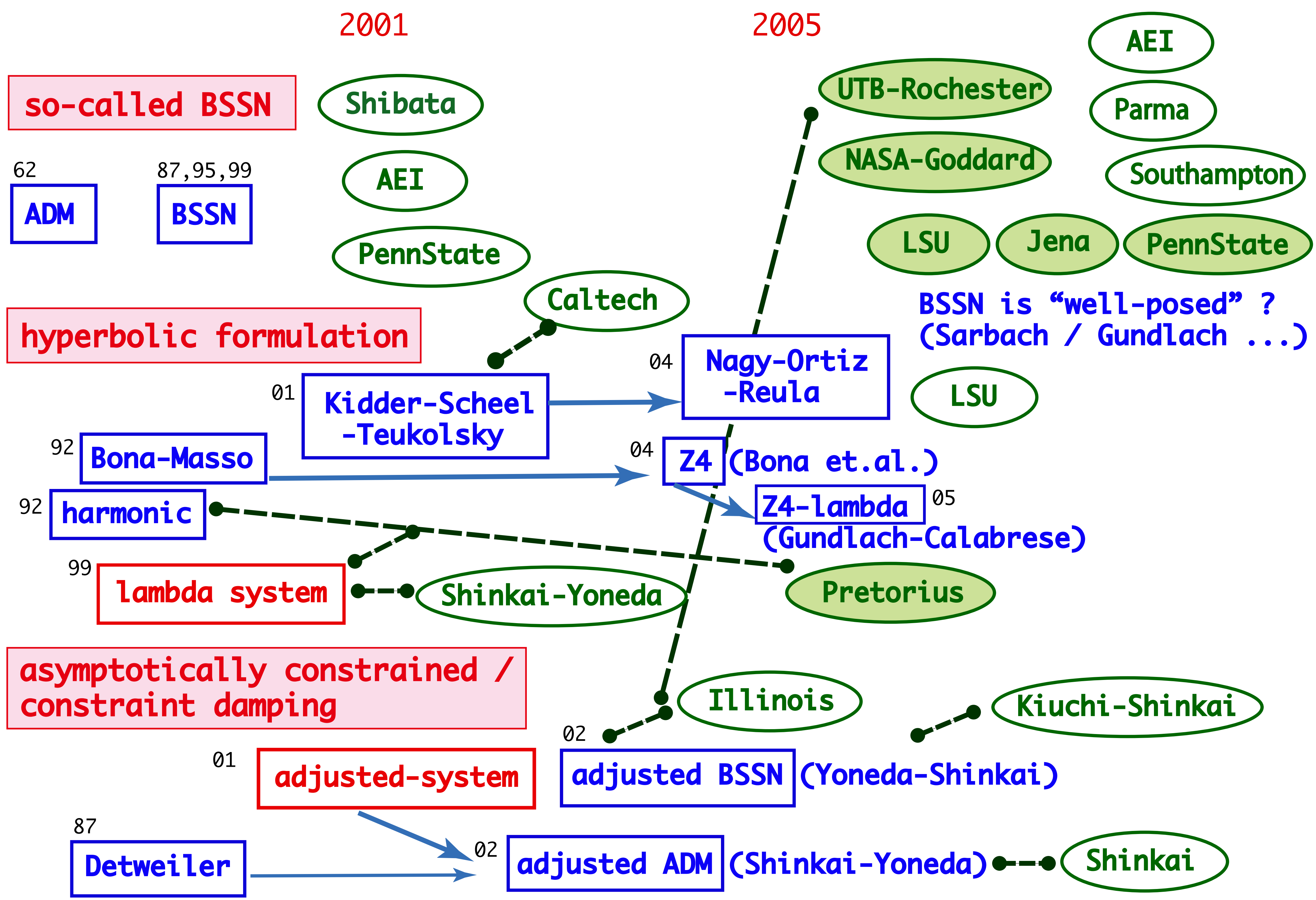
Detweiler
02
adjusted ADM
(Shinkai-Yoneda)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AEI
PennState
UTB-Rochester
NASA-Goddard
AEI
Parma
Southampton
LSU
Jena
PennState
BSSN is "well-posed" ?
(Sarbach / Gundlach ...)
hyperbolic formulation
Caltech
01
Kidder-Scheel -Teukolsky
04
Nagy-Ortiz -Reula
LSU
92
Bona-Masso
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Z4
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Shinkai