Introduction to Numerical Relativity

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Contents

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This file and viewgraphs of the lecture are available at `http://atlas.riken.go.jp/~shinkai/winterAPCTP/`

Notations:

- signature $(-+++)$.
- Covariant derivatives, Christoffel symbol

$$\nabla_\mu A^\alpha \equiv A^\alpha_{;\mu} \equiv A^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\nu} A^\nu \tag{0.1}$$
$$\nabla_\mu A_\alpha \equiv A_{\alpha;\mu} \equiv A_{\alpha,\mu} - \Gamma^\nu_{\alpha\mu} A_\nu \tag{0.2}$$
$$\Gamma^\alpha_{\mu\nu} = (1/2) g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{0.3}$$

- Riemann tensor, Ricci tensor, Weyl tensor

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{0.4}$$
$$R_{ab} \equiv R^\mu{}_{a\mu b} \equiv \Gamma^\mu_{ab,\mu} - \Gamma^\mu_{a\mu,b} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{ab} - \Gamma^\mu{}_{\nu b}\Gamma^\nu{}_{a\mu} \tag{0.5}$$
$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3} R g_{a[c}g_{d]b}, \tag{0.6}$$

- ADM decomposition, the extrinsic curvature (§2)

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu,\nu = 0,1,2,3) \\ \text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij} dx^i dx^j, \quad (i,j = 1,2,3) \end{aligned}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{0.7}$$

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = -\frac{1}{2} \pounds_n \gamma_{ij}. \tag{0.8}$$

1 Subjects of and for Numerical Relativity

1.1 Why Numerical Relativity?

The Einstein equation:

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.9}$$

What are the difficulties? (# 1)

- for 10-component metric, highly nonlinear partial differential equations.
- completely free to choose coordinates, gauge conditions, and even for decomposition of the space-time.
- mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- has singularity in its nature.

How to solve it?

- find exact solutions
  - assume symmetry in space-time, and decomposition of space-time
    spherically symmetric, cylindrical symmetric, ...
  - assume simple situation and matter
    time-dependency, homogeneity, algebraic speciality, ...

We know many exact solutions ($O(100)$) by this "Spherical Cow" approach.

- approximations
  - weak-field limit, linearization, perturbation, ...

We know correct prediction in the solar-system, binary neutron stars, ...
We know post-Newtonian behavior, first-order correction, BH stability, ...

Why don't we solve it using computers?

- dynamical behavior
- strong gravitational field
- no symmetry in space
- gravitational wave!
- higher-dimensional theories, and/or other gravitational theories, ...

The most robust way to study the strong gravitational field. Great.

Numerical Relativity **Box 1.1**
= Solve the Einstein equations numerically.
= Necessary for unveiling the nature of strong gravity. For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

What are the difficulties? (# 2)

- How to construct a realistic initial data?
- How to treat black-hole singularity?
- We cannot evolve the system stably in long-term evolution. Why?

More general and recent introductions to numerical relativity are available, e.g. by d'Inverno (1996) [3], Seidel (1996/98/99) [5], Brügmann (2000) [2], Lehner (2001) [4], van Putten (2001) [6], and Baumgarte-Shapiro (2002) [1].

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E. Seidel and W-M. Suen, gr-qc/9904014.

[6] M.H.P.M. van Putten, Proceedings of Asian Pacific CTP Winter School on black hole astrophysics, Pohang, Korea, available as gr-qc/0203076.

1.2 Overview of Numerical Relativity

Several milestones of NR
New proposals, developments, physical results.

1960s	Hahn-Lindquist	2 BH head-on collision	AnaPhys29(1964)304
	May-White	spherical grav. collapse	PR141(1966)1232
1970s	ÓMurchadha-York	conformal approach to initial data	PRD10(1974)428
	Smarr	3+1 formulation	PhD thesis (1975)
	Smarr-Cades-DeWitt-Eppley	2 BH head-on collision	PRD14(1976)2443
	Smarr-York	gauge conditions	PRD17(1978)2529
	ed. by L.Smarr	"Sources of Grav. Radiation"	Cambridge(1979)
1980s	Nakamura-Maeda-Miyama-Sasaki	axisym. grav. collapse	PTP63(1980)1229
	Miyama	axisym. GW collapse	PTP65(1981)894
	Bardeen-Piran	axisym. grav. collapse	PhysRep96(1983)205
	Stark-Piran	axisym. grav. collapse	unpublished
1990	Shapiro-Teukolsky	naked singularity formation	PRL66(1991)994
	Oohara-Nakamura	3D post-Newtonian NS coalesence	PTP88(1992)307
	Seidel-Suen	BH excision technique	PRL69(1992)1845
	Choptuik	critical behaviour	PRL70(1993)9
	NCSA group	axisym. 2 BH head-on collision	PRL71(1993)2851
	Cook et al	2 BH initial data	PRD47(1993)1471
	Shibata-Nakao-Nakamura	BransDicke GW collapse	PRD50(1994)7304
	Price-Pullin	close limit approach	PRL72(1994)3297
1995	NCSA group	event horizon finder	PRL74(1995)630
	NCSA group	hyperbolic formulation	PRL75(1995)600
	Anninos *et al*	close limit vs full numerical	PRD52(1995)4462
	Scheel-Shapiro-Teukolsky	BransDicke grav. collapse	PRD51(1995)4208
	Shibata-Nakamura	3D grav. wave collapse	PRD52(1995)5428
	Gunnersen-Shinkai-Maeda	ADM to NP	CQG12(1995)133
	Wilson-Mathews	NS binary inspiral, prior collapse?	PRL75(1995)4161
	Pittsburgh group	Cauchy-characteristic approach	PRD54(1996)6153
	Brandt-Brügmann	BH puncture data	PRL78(1997)3606
	Illinois group	synchronized NS binary initial data	PRL79(1997)1182
	Shibata-Baumgarte-Shapiro	2 NS inspiral, PN to GR	PRD58(1998)023002
	BH Grand Challenge Alliance	characteristic matching	PRL80(1998)3915
	Baumgarte-Shapiro	Shibata-Nakamura formulation	PRD59(1998)024007
	Brady-Creighton-Thorne	intermediate binary BH	PRD58(1998)061501
	Meudon group	irrotational NS binary initial data	PRL82(1999)892
	Shibata	2 NS inspiral coalesence	PRD60(1999)104052
	York	conformal thin-sandwich formulation	PRL82(1999)1350
	Brodbeck *et al*	λ-system	JMathPhys40(1999)909
2000	Kidder-Finn	BH, Spectral methods	PRD62(2000)084026
	Shinkai-Yoneda	planar GW, Ashtekar variables	CQG17(2000)4729
	AEI group	full numerical to close limit	CQG17(2000)L149
	AEI group	2 BH grazing collision	PRL87(2001)271103
	Shibata-Uryu	2 NS inspiral coalesence	PTP107(2002)265
	Shinkai-Yoneda	adjusted ADM systems	CQG19(2002)1027
	Meudon group	irrotational BH binary initial data	PRD65(2002)044020
	PennState group	isolated horizon	gr-qc/0206008

Issues to consider

Numerical Relativity – open issues **Box 1.2**

0. How to foliate space-time
   Cauchy $(3+1)$,
   Hyperboloidal $(3+1)$, ⇒ see e.g. [24]
   characteristic $(2+2)$, ⇒ see e.g. [6]
   or combined?

⇒ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data ⇒ see e.g. [3]

   Theoretical: Proper formulation for solving constraints?
   How to prepare realistic initial data?
   Effects of background gravitational waves?
   Connection to the post-Newtonian approximation?

   Numerical: Techniques for solving coupled elliptic equations?
   Appropriate boundary conditions?

2. How to evolve the data

   Theoretical: Free evolution or constrained evolution?
   Proper formulation for the evolution equations? ⇒ see e.g. [40]
   Suitable slicing conditions (gauge conditions)?

   Numerical: Techniques for solving the evolution equations?
   Appropriate boundary treatments?
   Singularity excision techniques?
   Matter and shock surface treatments?
   Parallelization of the code?

3. How to extract the physical information

   Theoretical: Gravitational wave extraction?
   Connection to other approximations?

   Numerical: Identification of black hole horizons?
   Visualization of simulations?

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[3] G. Cook, Livng Rev. Relativ. **2000-5** at `http://www.livingreviews.org/`

[4] H. Shinkai and G. Yoneda, in *Progress in Astronomy and Astrophysics* (Nova Science Publ) to be published, available as gr-qc/0209111.

1.3 Gravitational Wave Physics (Why Blackholes/Neutron Stars?)

1.3.1 General References

A resource guide by Centrella [1] might be quite useful. Essential references are by e.g. Thorne (1987/1997) [2], and Abramovici *et al* [3]. The latest review is by Cutler and Thorne [4]. Viewgraphs of the lecture (2002) by Dr. A. J. Weinstein (`http://www.ligo.caltech.edu/~ajw/`) may be also useful.

References

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[2] K. S. Thorne, in *300 Years of Gravitation*, ed. by S. Hawking and W. Israel (Cambridge, 1987). K. S. Thorne, gr-qc/9706079.

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[4] C. Cutler and K. S. Thorne, in Proceedings of GR16 (Durban, South Africa, 2001), available as gr-qc/0204090.

1.3.2 Laser Interferometers

Current Projects, under operations! (target frequency: $10 - 10^3$ Hz)

TAMA	Japan	300m @ Tokyo	1997-	http://tamago.mtk.nao.ac.jp/
LIGO	USA	4Km @ Hanford, WA	2001-	http://www.ligo-wa.caltech.edu/
LIGO	USA	4Km @ Livingston, LA	2001-	http://www.ligo-la.caltech.edu/
GEO	Germany/UK	600m @ Hannover	2001-	http://www.geo600.uni-hannover.de/
VIRGO	France/Italy	3Km @ Pisa	2002-	http://www.pi.infn.it/virgo/virgoHome.html

Future Planning

CLIO	JAPAN	100m @ Kamioka	2004?	
LCGT	JAPAN	3Km @ Kamioka	2006?	
LIGO 2	USA	4Km @ Hanford, WA	2006?	http://www.ligo.caltech.edu/
LIGO 2	USA	4Km @ Livingston, LA	2006?	
ACIGA	Australia	3Km @	2003?	http://www.anu.edu.au/Physics/ACIGA/

Future Planning as Space Satellites (target frequency: $10^{-4} - 10^{-1}$ Hz)

LISA	US/ESA	5 10^6 Km	2011?	http://lisa.jpl.nasa.gov/
DECIGO	JAPAN	5 10^5 Km	??	(not approved yet)

1.3.3 What Information can we extract from gravitational waveform of binary neutron star coalescence?

1. INSPIRAL phase [$\sim$ 3 mins.] (Post-Newtonian Approx.)

$$\begin{array}{lll} \text{'chirps'} \ \dfrac{df}{dt} & \Rightarrow & \text{'chirp mass'} \ M_c \equiv \dfrac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}} \\ \text{amplitude (incleasing)} & \Rightarrow & M_c, \text{distance} \\ \text{amplitude } (h_+/h_\times) & \Rightarrow & \text{inclination} \\ \text{waveform} & \Rightarrow & \text{eccentricity} \\ \text{moduration} & \Rightarrow & \text{spin}, \cdots \end{array}$$

2. ISCO phase (Post-Newtonian & Numerical Relativity)

$$\begin{array}{rcl} \text{frequency} & \Rightarrow & \text{Mass-Radius relation} \\ & \Rightarrow & \text{Equation of States} \end{array}$$

3. COALESCE phase [$\sim$ 3 millisecs.] (Numerical Relativity)

$$\begin{array}{lcl} \text{waveform} \cdots & \Rightarrow & ? \\ & \Rightarrow & \text{BH parameters } (m, a), \cdots \\ & \Rightarrow & \text{GR test} \\ \text{other elements?} & \Rightarrow & \gamma\text{-ray burst ?} \\ & \Rightarrow & \text{r-process elements?} \end{array}$$

4. BLACK HOLE formation phase [$\sim$ 10 msecs.] (Perturbation)

$$\text{Quasi-Normal Modes} \quad \Rightarrow \quad \text{Black Hole formation}$$

5. STATISTICS

$$\begin{array}{lcl} \text{with optical identification} & \Rightarrow & \text{Hubble parameter} \\ \text{statics} & \Rightarrow & \text{cosmological parameters} \end{array}$$

1.3.4 Requirements for Numerical Relativity

- Where to start the simulation? How to construct physically reasonable initial data?
- How can we evolve the system stably?
- How to treat black hole singularity if it appears?
- How to extract gravitational wave?
- How can we manage the large-scale simulations?

2 The Standard Cauchy Approach to Numerical Relativity

2.1 The ADM formulation

2.1.1 The 3+1 decomposition of space-time

The idea of space-time evolution was first formulated by Arnowitt, Deser, and Misner (ADM) [9]. The formulation was first motivated by a desire to construct a canonical framework in general relativity, but it also gave the community to the fundamental idea of time evolution of space and time: such as foliations of 3-dimensional hypersurface (Figure 1). This scheme is often called '3+1 formulation', 'the ADM formulation', or 'Cauchy approach'.

Let us denote the hypersurface $\Sigma(t)$ which is the three-dimensional spatial space with a parameter t. The evolution of spacetime is expressed as the dynamics of $\Sigma(t)$. The formulation begins by decomposing the metic as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu, \quad (\mu,\nu=0,1,2,3) \\ \text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij}dx^i dx^j, \quad (i,j=1,2,3) \end{aligned}$$

Let the unit normal vector of the slices be n^μ, where

$$n_\mu = (-\alpha, 0, 0, 0), \qquad n^\mu = g^{\mu\nu}n_\nu = (1/\alpha, -\beta^i/\alpha).$$

We then have a 3+1 decomposed metric as

$$\begin{aligned} ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (-\alpha^2 + \beta_l \beta^l)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j \end{aligned} \tag{2.1}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l\beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

where α and β_j are defined as

$$\alpha \equiv 1/\sqrt{-g^{00}}, \qquad \beta_j \equiv g_{0j}. \tag{2.2}$$

and called the lapse function and shift vector, respectively.

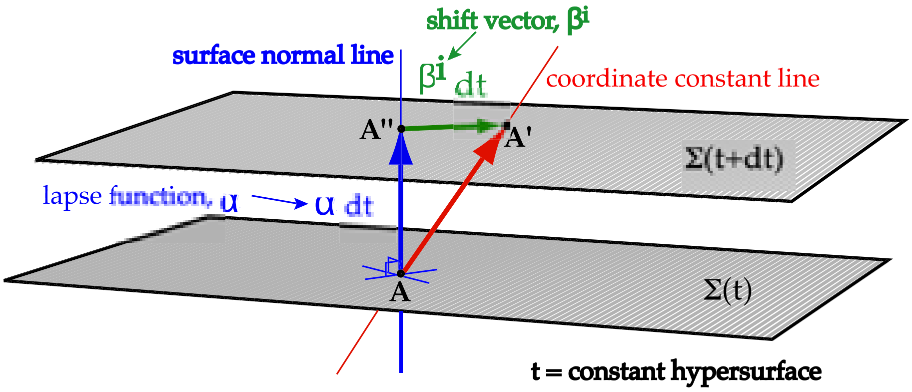

Figure 1: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted α or N, and β^i or N^i, respectively.

2.1.2 The Standard ADM formulation

In order to decompose the Einstein equation into 3+1, we introduce the projection operator $\perp^\mu_\nu$ normal to n^μ,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad \gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu. \tag{2.3}$$

We also call the spatial components of γ_{ij} the intrinsic 3-metric g_{ij}.[1]

The projections of the Einstein equation can be the following three:

$$\begin{aligned} G_{\mu\nu}\, n^\mu\, n^\nu &= 8\pi G\, T_{\mu\nu}\, n^\mu\, n^\nu \equiv 8\pi\rho_H & (2.4)\\ G_{\mu\nu}\, n^\mu \perp^\nu_i &= 8\pi G\, T_{\mu\nu}\, n^\mu \perp^\nu_i \equiv -8\pi J_i & (2.5)\\ G_{\mu\nu} \perp^\mu_i \perp^\nu_j &= 8\pi G\, T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv 8\pi S_{ij} & (2.6)\end{aligned}$$

To express these equation, we introduce the extrinsic curvature K_{ij} as

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = \cdots = \frac{1}{2\alpha}\left(-\partial_t\gamma_{ij} + \beta_{i|j} + \beta_{j|i}\right) = -\frac{1}{2}\pounds_n\gamma_{ij}. \tag{2.7}$$

Projection of the Einstein equation on to the 3-hypersurface Σ is given using the Gauss-Codacci relation: The Gauss equation,

$$^{(3)}R^\alpha{}_{\beta\gamma\delta} = {}^{(4)}R^\mu{}_{\nu\rho\sigma} \perp_\mu{}^\alpha \perp_\beta{}^\nu \perp_\gamma{}^\rho \perp_\delta{}^\sigma - K^\alpha{}_\gamma K_{\beta\delta} + K^\alpha{}_\delta K_{\beta\gamma}\,, \tag{2.8}$$

and the Codacci equation,

$$D_j K_i{}^j - D_i K = -{}^{(4)}R_{\rho\sigma} n^\sigma \perp_i{}^\rho\,, \tag{2.9}$$

where $K = K^i{}_i$, and D_μ is the covariant differentiation with respect to γ_{ij}.

The projections (2.4)-(2.6) can be derived as follows.

The Standard ADM formulation [42, 51]: **Box 2.1**

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned} \partial_t\gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, & (2.10)\\ \partial_t K_{ij} &= \alpha\, {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_i D_j\alpha \\ &\quad + (D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} & (2.11)\end{aligned}$$

  where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\begin{aligned} \mathcal{H}^{ADM} &:= {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, & (2.12)\\ \mathcal{M}^{ADM}_i &:= D_j K^j{}_i - D_i K \approx 0, & (2.13)\end{aligned}$$

  where ${}^{(3)}R = {}^{(3)}R^i{}_i$: these are called the Hamiltonian (or energy) and momentum constraint equations, respectively.

[1] If n_μ is space-like, then $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$

The formulation has 12 free first-order dynamical variables (γ_{ij}, K_{ij}), with 4 freedom of gauge choice (α, β_i) and with 4 constraint equations, (2.12) and (2.13). The rest freedom expresses 2 modes of gravitational waves.

We should remark here the 'original' ADM formulation. The evolution equations in Box 2.1 is the version by Smarr and York which is now the standard convention for numerical relativists. They adapted K_{ij} as a fundamental variable instead of the conjugate momentum π^{ij}, which was in the original Arnowitt-Deser-Misner's canonical formulation. Note that there is one replacement in (2.11) using (2.12) in the process of conversion from the original ADM to the standard ADM equations.

More detail description: The Hamiltonian density can be written as

$$\mathcal{H}_{GR} \;=\; \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \sqrt{-g}R = \alpha\sqrt{\gamma}[{}^{(3)}R - K^2 + K_{ij}K^{ij}],$$

where π^{ij} is the canonically conjugate momentum to γ_{ij},

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = -\sqrt{\gamma}(K^{ij} - K\gamma^{ij}),$$

omitting the boundary terms. The variation of $\mathcal{H}_{GR}$ with respect to α and β_i yields the constraints, and the dynamical equations are given by $\dot{\gamma}_{ij} = \frac{\delta \mathcal{H}_{GR}}{\delta \pi^{ij}}$ and $\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}_{GR}}{\delta h_{ij}}$.

$$\begin{aligned}
\partial_t \gamma_{ij} &= 2\frac{N}{\sqrt{\gamma}}(\pi_{ij} - (1/2)\gamma_{ij}\pi) + 2D_{(i}N_{j)}, \\
\partial_t \pi^{ij} &= -\sqrt{\gamma}N({}^{(3)}R^{ij} - (1/2){}^{(3)}R\gamma^{ij}) + (1/2)\frac{N}{\sqrt{\gamma}}h^{ij}(\pi_{mn}\pi^{mn} - (1/2)\pi^2) - 2\frac{N}{\sqrt{\gamma}}(\pi^{in}\pi_n{}^j - (1/2)\pi\pi^{ij}) \\
&\quad + \sqrt{\gamma}(D^iD^jN - \gamma^{ij}D^mD_mN) + \sqrt{\gamma}D_m(\gamma^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)}
\end{aligned}$$

The ADM formulation is a kind of constrained system, like Maxwell equations.

	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$	Hamiltonian constraint (2.12)
	div $\mathbf{B} = 0$	Momentum constraints (2.13)
evolution eqs.	$\partial_t \mathbf{E} = \text{rot } \mathbf{B} - 4\pi\mathbf{j}$	$\partial_t \gamma_{ij} = \cdots$ (2.10)
	$\partial_t \mathbf{B} = -\text{rot } \mathbf{E}$	$\partial_t K_{ij} = \cdots$ (2.11)

In order to see the constraints are conserved during the evolution or not, we have to check how the constraints evolve. The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (2.12) and the momentum constraints (2.13), can be written as [22, 39]

The Constraint Propagations of the Standard ADM: **Box 2.2**

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K\mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, &(2.14) \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad + \alpha K\mathcal{M}_i - \beta^k\gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. &(2.15)
\end{aligned}$$

From these equations, we know that **if the constraints are satisfied on the initial slice Σ, then the constraints are satisfied throughout evolution** (in principle).

2.1.3 Numerical Procedures

In numerical relativity, this free-evolution approach is also the standard. This is because solving the constraints (non-linear elliptic equations) is numerically expensive, and because free evolution allows us to monitor the accuracy of numerical evolution.

The normal numerical scheme (free evolution scheme):

1. preparation of the initial data
   solve the elliptic constraints for preparing the initial data (γ_{ij}, K_{ij}).
2. time evolution
   (a) specify the gauge conditions (slicing conditions) for the lapse α and shift β_i.
   (b) evolve (γ_{ij}, K_{ij}) by using the evolution equations.
   (c) monitor the accuracy of simulations by checking the constraints.
   (d) extract physical quantities.
3. step back to 2 and repeat.

References

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2.2 How to construct initial data 1: conformal approach

Initial Data Construction Problem **Box 2.3**

Prepare all metric and matter components by solving the two constraints:

- The Hamiltonian constraint equation

$$^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho + 2\Lambda \tag{2.16}$$

- The momentum constraint equations

$$D_j(K^{ij} - \gamma^{ij}\mathrm{tr}K) = \kappa J^i \tag{2.17}$$

Conformal Approach – York-ÓMurchadha (1974)

N.ÓMurchadha and J.W.York Jr., Phys. Rev. **D 10**, 428 (1974)

2.2.1 Formulation

The idea is

$$\boxed{\text{solution} \quad \gamma_{ij} = \psi^4 \hat{\gamma}_{ij} \quad \text{trial metric}} \tag{2.18}$$

We introduce the decomposition of K_{ij},

$$K_{ij} \Rightarrow \begin{cases} \mathrm{tr}K = \gamma^{ij} K_{ij} & \text{trace part} \\ A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}K & \text{trace-free part} \end{cases} \tag{2.19}$$

Then, other conformal transformations as consistent with (2.18) are:

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \qquad \gamma^{ij} = \psi^{-4}\hat{\gamma}^{ij}, \tag{2.20}$$

$$A^{ij} = \psi^{-10}\hat{A}^{ij}, \quad A_{ij} = \psi^{-2}\hat{A}_{ij}, \tag{2.21}$$

$$\rho = \psi^{-n}\hat{\rho}, \qquad J^i = \psi^{-10}\hat{J}^i, \tag{2.22}$$

and we suppose

$$\mathrm{tr}K = \hat{\mathrm{tr}}\hat{K}, \quad \mathrm{tr}A = \hat{\mathrm{tr}}\hat{A} = 0. \tag{2.23}$$

From (2.20), we get

$$\Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk} + 2\psi^{-1}(\delta^i{}_j \hat{D}_k\psi + \delta^i{}_k\hat{D}_j\psi - \hat{\gamma}_{jk}\hat{\gamma}^{im}\hat{D}_m\psi), \tag{2.24}$$

$$R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi. \tag{2.25}$$

where $\hat{\Delta} = \hat{\gamma}^{jk}\hat{D}_j\hat{D}_k$ and $\hat{R} = R(\hat{\gamma})$, and also $D_j A^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}$.

We further decompose $\hat{A}^{ij}$ to divergence-free (transverse-traceless, TT) part and longitudinal part:

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + (\hat{\mathrm{l}}W)^{ij}, \tag{2.26}$$

where we suppose

$$\hat{D}_j\hat{A}^{ij}_{TT} = 0 \qquad \text{and} \qquad \hat{\mathrm{tr}}\hat{A}_{TT} = 0. \tag{2.27}$$

and

$$(\hat{\mathrm{l}}W)^{ij} = \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \tag{2.28}$$

Using these terms, we can write

$$\begin{aligned} \hat{D}_j\hat{A}^{ij} &= \hat{D}_j(\hat{\mathrm{l}}W)^{ij} \equiv (\hat{\Delta}_{\mathrm{l}}W)^i, \\ &= (\hat{\Delta}W)^i + \frac{1}{3}\hat{D}^i(\hat{D}_j W^j) + \hat{R}^i{}_j W^j. \end{aligned} \tag{2.29}$$

With above transformation, the two constraints, (2.16) and (2.17), can be expressed as follows.

- The Hamiltonian constraint equation

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \tag{2.30}$$

- The momentum constraint equations

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6\hat{D}^i\mathrm{tr}K + 8\pi G\hat{J}^i \tag{2.31}$$

2.2.2 Summary

Conformal approach (York-ÓMurchadha, 1974) **Box 2.4**

One way to set up the metric and matter components $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints (2.16) and (2.17) is as follows.

1. Specify metric components $\hat{\gamma}_{ij}$, $\mathrm{tr}K$, $\hat{A}_{ij}^{TT}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.

2. Solve the next equations for $(\psi,\ W^i)$

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \qquad (2.30)$$

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6\hat{D}^i\mathrm{tr}K + 8\pi G\hat{J}^i \qquad (2.31)$$

   where

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \qquad (2.32)$$

3. Apply the inverse conformal transformation and get the metric and matter components γ_{ij}, K_{ij}, ρ, J^i in the physical frame:

$$\gamma_{ij} = \psi^4\hat{\gamma}_{ij}, \qquad (2.33)$$

$$K_{ij} = \psi^{-2}[\hat{A}_{ij}^{TT} + (\hat{\mathbb{l}}W)_{ij}] + \frac{1}{3}\psi^4\hat{\gamma}_{ij}\mathrm{tr}K, \qquad (2.34)$$

$$\rho = \psi^{-n}\hat{\rho}, \qquad (2.35)$$

$$J^i = \psi^{-10}\hat{J}^i \qquad (2.36)$$

Comments

- Using the idea of conformal rescaling, we have a way to fix 12 components of (γ_{ij}, K_{ij}) that satisfy 4 constraints.
- The Hamiltonian constraint, (2.30), is a non-linear elliptic equation for ψ, so that we have to solve it by an iterative method.
- The momentum constraints, (2.31), are PDEs for W^i and coupled with (2.30). If we assume $\mathrm{tr}K = 0$, then two constraints are decoupled. Normally people assume $\mathrm{tr}K = 0$ (maximal slicing condition) or $(\mathrm{tr}K)$ =const. (constant mean curvature slicing) for this purpose.
- For simplicity, people assume the background metric $\hat{\gamma}_{ij}$ is conformally flat $\hat{\gamma}_{ij} = \delta_{ij}$. The physical appropriateness of conformal flatness is often debatable.
- Two freedom of $\hat{A}_{ij}^{TT}$ corresponds to the one of gravitational wave. However, there have been no systematic discussion how to specify them, except applying tensor harmonics in a linearized situation.

2.2.3 Numerical procedures – Several tips

Solving the Hamiltonian constraint Two Methods:

1. Solve the non-linear equation (2.30) directly.
2. Solve the linearized equation $\psi = \psi_0 + \delta\psi$ iteratively.

$$\begin{aligned} 8\hat{\Delta}\psi &= E\,\psi + F\,\psi^{-7} + G\,\psi^5 + H\,\psi^{-3} + I\,\psi^{-1} \\ &= [E - 7F\psi_0^{-8} + 5G\psi_0^4 - 3H\psi_0^{-4} - 2I\psi_0^{-2}]\,\psi + [8F\psi_0^{-7} - 4G\psi_0^5 + 4H\psi_0^{-3} + 2I\psi_0^{-1}] \end{aligned}$$

Under an appropriate boundary condition, such as Robin BC $\psi = 1 + const./r$, or Dirichlet BC $\psi = 1 + M_{total}/2r$.

Solve the momentum constraints A couple of methods:

1. Solve the non-linear equations (2.31) directly.
2. Bowen's method for conformally flat case [GRG14(1982)1183]
   Under the ($\nabla^i K = 0$) condition, (2.31) becomes

$$\Delta W^i + \frac{1}{3}\nabla^i \nabla_j W^j = 8\pi S^i.$$

By introducing a decomposition of W^i into vector and gradient terms

$$W^i = V^i - \frac{1}{4}\nabla^i \theta,$$

the equations to solve are:

$$\begin{aligned} \Delta V^i &= 8\pi S^i, && (2.37) \\ \Delta\theta &= \nabla_i V^i, && (2.38) \end{aligned}$$

If the source is of finite extent, then the the asymptotic behavior of V^i and θ are given by

$$V^i = -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}, \qquad (2.39)$$

$$\begin{aligned} \theta =& -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} + \sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}} \\ &+ \sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}} \qquad (2.40) \end{aligned}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles Q and M are defined as

$$\begin{aligned} Q^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int S^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV, \\ M^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int r^2 S^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV, \end{aligned}$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)}$$

2.3 How to construct initial data 2: thin sandwich approach

J.W.York Jr., Phys. Rev. Lett. **82**, 1350 (1999)

The name "sandwich" comes from the proposal that this method prepares two spatial slices at $t = 0$ and $t = \Delta t$. There may be the following benefits:

- The input function is more friendly (3-metric and its time derivative) than the previous conformal approach.
- The input quantity also requires the lapse function, N. (Actually this is the inverse and densitized lapse function. See below.)
- The similar conformal transformation is applied. But the relation $\bar{A}^{ij} = \psi^{-10} A^{ij}$ is *derived* in this version.

However, the numerical solvability is still debatable. Partial applications are seen in constructing quasi-equilibrium binary neutron stars/black-holes. Matter terms are inserted by H. Shinkai.

2.3.1 metric, conformal metric, weighted conformal metric

First, I list three types of 3-metric along to the conformal transformation.

(a) The metric, $\overline{g}_{ij}$ which satisfies the constraints. (That is, the solution to seek.)

(b) The conformal metric g_{ij}, where

$$\boxed{\text{“this world”} \qquad \overline{g}_{ij} = \psi^4 g_{ij} \qquad \text{“that world”}}$$

(c) The "weighted" $(-2/3)$ conformal metric

$$\hat{g}_{ij} = \overline{g}^{-1/3}\overline{g}_{ij} = g^{-1/3} g_{ij} \tag{2.41}$$

where $\overline{g} = \det(\overline{g}_{ij})$ and $g = (\det g_{ij})$. For small variation, $\overline{g}^{ij}\delta\hat{g}_{ij} = 0$ is always hold, and we get

$$\overline{g}^{ij}\partial_t \hat{g}_{ij} = g^{ij}\partial_t \hat{g}_{ij} = \hat{g}^{ij}\partial_t \hat{g}_{ij} = 0\ . \tag{2.42}$$

In York's paper, he does not use the weighted conformal metric, but he imposes that the conformal metric does have the property similar to (2.42).

2.3.2 introduction of velocity tensor

On the second slice $t = \delta t$, we write the conformal metric

$$g'_{ij} = g_{ij} + u_{ij}\delta t, \tag{2.43}$$

where we introduced the velocity tensor (suppose to be a given quantity)

$$\boxed{u_{ij} = \dot{g}_{ij}, \qquad (2.44)}$$

together with the "weighted" condition,

$$g^{ij}u_{ij} = 0, \qquad \text{and} \qquad g^{ij}\dot{g}_{ij} = 0. \tag{2.45}$$

By taking the traceless part of the evolution equation,

$$\partial_t \overline{g}_{ij} \equiv \dot{\overline{g}}_{ij} = -2\overline{NK}_{ij} + (\overline{D}_i\overline{\beta}_j + \overline{D}_j\overline{\beta}_i) \tag{2.46}$$

together with (2.24) and (2.25), we get

$$\dot{\overline{g}}_{ij} - \frac{1}{3}\overline{g}_{ij}\overline{g}^{kl}\dot{\overline{g}}_{kl} \equiv \overline{u}_{ij} = -2\overline{NA}_{ij} + (\overline{L\beta})_{ij} \tag{2.47}$$

$$\text{where } \overline{A}_{ij} \equiv \overline{K}_{ij} - (1/3)\overline{K}\overline{g}_{ij}, \quad \text{and} \quad (\overline{L\beta})_{ij} \equiv \overline{D}_i\overline{\beta}_j + \overline{D}_j\overline{\beta}_i - (2/3)\overline{g}_{ij}\overline{D}^k\overline{\beta}_k. \tag{2.48}$$

From (2.47), we obtain

$$\boxed{\overline{u}_{ij} = \psi^4 u_{ij}} \tag{2.49}$$

Similarly, we obtain

$$\overline{\beta}^i = \beta^i, \qquad \overline{\beta}_i = \psi^4\beta_i, \tag{2.50}$$

$$(\overline{L\beta})_{ij} = \psi^4(L\beta)_{ij}, \qquad (\overline{L}\beta)^{ij} = \psi^{-4}(L\beta)^{ij}\ . \tag{2.51}$$

2.3.3 Redefinition of the lapse and its conformal transformation

We call the standard $\alpha(t,x) > 0$ slicing function, and define the lapse function $\overline{N}$ as

$$\boxed{\overline{N} = \overline{g}^{1/2}\alpha.} \tag{2.52}$$

The slicing function is now $\alpha = \overline{g}^{-1/2}\overline{N} = \overline{\underset{\sim}{N}}$, which may be called the inverse densitized lapse. Note that the lapse here, $\overline{N}$, depends g, so that $\overline{N}$ is not a pure gauge quantity. Therefore we treat that α is the freely specified function and let $\boxed{\overline{\alpha} = \alpha}$. In result, we obtain a new relation $\boxed{\overline{N} = \psi^6 N}$ from (2.52).

We also impose $\boxed{\overline{K} = K}$ as before. The relation (2.47) then *derives*

$$\begin{aligned} \overline{A}^{ij} &= \psi^{-6}(2N)^{-1}\left[\psi^{-4}(L\beta)^{ij} - \psi^{-4}u^{ij}\right] \\ &= \psi^{-10}\left\{(2N)^{-1}\left[(L\beta)^{ij} - u^{ij}\right]\right\} = \psi^{-10}A^{ij} \qquad \text{that is} \qquad \boxed{\overline{A}^{ij} = \psi^{-10}A^{ij}} \end{aligned}$$

2.3.4 Constraints to solve

By using above boxed conformal transformations, two constraints can be transformed as [2]

- The Hamiltonian constraint equation (the same with before)

$$8\Delta_g\psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - [\frac{2}{3}K - 2\Lambda]\psi^5 - 16\pi G\rho\psi^{5-n} = 0\ , \tag{2.53}$$

- The momentum constraint equations

$$D_j\left[(2N)^{-1}(L\beta)^{ij}\right] = D_j\left[(2N)^{-1}u^{ij}\right] + \frac{2}{3}\psi^6 D^i K + 8\pi G J^i \tag{2.54}$$

[2]LHS of (2.54) is

$$D_a\left[(2N)^{-1}\right](D^i\beta^a + D^a\beta^i - \frac{1}{3}g^{ia}D^k\beta_k) + (2N)^{-1}\left[\Delta\beta^i + \frac{1}{3}D^iD_k\beta^k + R^i{}_k\beta^k\right]$$

2.3.5 Summary

Thin-Sandwich approach (York, 1999) **Box 2.5**

One way to set up the metric, gauge values and matter components $(\overline{g}_{ij}, \overline{K}_{ij}, \overline{N}, \overline{\beta}^i, \overline{\rho}, \overline{J}^i)$ so as to satisfy the constraints (2.16) and (2.17) is as follows.

1. Specify metric components $g_{ij}, u_{ij}(= \dot{g}_{ij}), K$, the lapse function N, and matter distribution $\rho,\ J^i$ in the conformal frame.

2. Solve the next equations for (ψ, β^i)

$$8\Delta_g\psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - [\frac{2}{3}K - 2\Lambda]\psi^5 - 16\pi G\rho\psi^{5-n} = 0\ , \tag{2.53}$$

$$D_j\left[(2N)^{-1}(L\beta)^{ij}\right] = D_j\left[(2N)^{-1}u^{ij}\right] + \frac{2}{3}\psi^6 D^i K + 8\pi G J^i, \tag{2.54}$$

where

$$A^{ij} = (2N)^{-1}\left[(L\beta)^{ij} - u^{ij}\right]. \tag{2.55}$$

3. Apply the inverse conformal transformation and get the metric and matter components $(\overline{\gamma}_{ij}, \overline{K}_{ij}, \overline{N}, \overline{\beta}^i, \overline{\rho}, \overline{J}^i)$ in the physical frame:

$$\overline{N} = \psi^6 N, \tag{2.56}$$

$$\overline{\beta}^i = \beta^i, \tag{2.57}$$

$$\overline{g}_{ij} = \psi^4 g_{ij}, \tag{2.58}$$

$$\overline{K}_{ij} = \psi^{-2}A_{ij} + \frac{1}{3}\psi^4 g_{ij}K, \tag{2.59}$$

$$\overline{\rho} = \psi^{-8}\rho, \tag{2.60}$$

$$\overline{J}^i = \psi^{-10}J^i. \tag{2.61}$$

Comments

- The two equations, (2.53) and (2.54), are coupled, but they will be decoupled if we assume the constant mean curvature condition, (tr$K$) =const. (This is the same as the conformal approach, but we have to solve the momentum constraints first here.)
- The (general) solvability of (2.54) is still debatable.

Comparison between two approaches

	conformal approach	thin-sandwich approach
input functions	g_{ij}, K, A_{ij}^{TT} (components: 6, 1, 2) GW components are separated out	g_{ij}, K, u_{ij}, N (comp.: 6, 1, 5, 1) can specify time-derivatives
treatment of gauge functions	lapse and shift are not appearing in the formulation.	lapse is given by the conformal transformation. shift is given by solving the constraints.
usage of the constraints	Hamiltonian constraint is for the conformal factor ψ momentum constraints are for the longitudinal part of A_{ij}.	Hamiltonian constraint is for the conformal factor ψ momentum constraints are for shift function β^i.
counting the freedom	(input 9 functions) plus (3 functions by solving momentum constraints) = 12 = (3-metric) plus (extrinsic curvature).	(input 13 functions) plus (3 functions by solving momentum constraints) = 16 = (3-metric) plus (extrinsic curvature) plus (gauge functions).

Interpretation of quasi-equilibrium data construction

cf) G. Cook, Living Reviews in Relativity, 2000-5

The thin-sandwich formulation offers the easiest way to interpret a method constructing the initial data for binary NSs or BHs. For example, if we assume

- the binary has a constant orbital angular velocity of Ω.
- the existence of the helical Killing vector, $\xi^\mu = t^\mu + \Omega\phi^\mu$. ($\xi^\mu$ is also assumed to the time vector in the rotating frame that we are on, while t^μ is in the rest frame).
- align shift vector with Killing vector

$$\Rightarrow \qquad \psi^4(L\beta)_{ij} = (\overline{L\beta})_{ij} \equiv \overline{D}_i\overline{\beta}_j + \overline{D}_j\overline{\beta}_i - (2/3)\overline{g}_{ij}\overline{D}^k\overline{\beta}_k = 0$$

- gravitational radiation is negligible, i.e. conformally flat background, $R = 0$.
- maximally slicing condition, $K = \partial_t K = 0$.

Thin-sandwich formulation gives us following explicit relations:

1. The momentum constraints, $D_j\left[(2N)^{-1}(L\beta)^{ij}\right] = D_j\left[(2N)^{-1}u^{ij}\right] + \frac{2}{3}\psi^6 D^i K + 8\pi G J^i$,
   $$\Rightarrow \qquad \boxed{u_{ij} = 0}$$
2. The Hamiltonian constraint, $8\Delta_g\psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - [\frac{2}{3}K - 2\Lambda]\psi^5 - 16\pi G\rho\psi^{5-n} = 0$,
   $$\Rightarrow \qquad \boxed{\Delta(\alpha\psi^7) = (\alpha\psi^7)\left[\frac{7}{8}A_{ij}A^{ij}\psi^{-8} + 2\pi G\psi^4(\rho + 2S)\right]}$$

2.4 How to choose gauge conditions

The standard 3+1 formulation allows us to choose gauge conditions (slicing conditions) for every time step. The fundamental guidelines for fixing the lapse function α and the shift vector β_i:

- to avoid the foliation hitting the physical and coordinate singularity in its evolution.
- to make system suitable for physical situation.
- to make the evolution system as simple as possible.
- to enable the gravitational wave extraction easy.

I list several essential slicing conditions below. The notations hereafter follows those of §2.1 (ADM formulation).

2.4.1 Lapse conditions

geodesic slice	$\alpha = 1$	`GOOD` `BAD`	simple, easy to understand no singularity avoidance	

harmonic slice	$\nabla_a \nabla^a x^b = 0$	`GOOD`
		`GOOD`
		`BAD`
			simplify eqs.,
			easy to compare analytical investigations
			no singularity avoidance or coordinate pathologies
				[2]-[7]

maximal slice	$K = 0$	`GOOD` `BAD`	singularity avoidance have to solve an elliptic eq.	[1],[8]-[15]
maximal slice (K-driver)	$\partial_t K = -c^2 K$	`G&B` `GOOD`	same with maximal slice, easy to maintain $K = 0$	[12]
constant mean curvature	$K = const.$	`G&B` `GOOD`	same with maximal slice, suitable for cosmological situation	[16]-[18]
polar slicing	$K_\theta^\theta + K_\varphi^\varphi = 0$, or $K = K_r^r$	`GOOD` `BAD`	singularity avoidance in isotropic coord. trouble in Schwarzschild coord.	[19]-[21]
algebraic	$\alpha \sim \sqrt{\gamma}$, $\alpha \sim 1 + \log \gamma$	`GOOD` `BAD`	easy to implement not avoiding singularity	

Maximal slicing This is always the first one to be mentioned as a singularity avoiding gauge condition. The name of 'maximal' comes from the fact that the deviation of the 3-volume $V = \int \sqrt{\gamma} d^3x$ along to the normal line becomes maximal when we set $K = 0$. This is simply written as

$$K = 0 \qquad \text{on} \quad \Sigma(t). \tag{2.62}$$

Pioneering idea can be seen in Lichnerowicz [8], and it was extended by York [1]. This condition is supposed to be applied in simulations that a singularity will appear during evolutions such as gravitational collapses. The actual equation for determining the lapse function α can be obtained from $\partial_t K = \partial_t(K_{ij}\gamma^{ij}) = 0$. By substituting the evolution equations, we get

$$D^i D_i \alpha = \{ {}^{(3)}R + K^2 + 4\pi G(S - 3\rho_H) - 3\Lambda\}\alpha, \tag{2.63}$$

or by using the Hamiltonian constraint further,

$$D^i D_i \alpha = \{K_{ij}K^{ij} + 4\pi G(S + \rho_H) - \Lambda\}\alpha. \tag{2.64}$$

This is an elliptic equation. When the curvature is strong (i.e. close to the appearance of a singularity), the RHS of equation become larger, hence the lapse becomes smaller. Therefore the foliation near the singularity evolves slowly.

For Schwarzschild black-hole space-time, Estabrook *et al.* [10] showed that the maximal slicing condition allows the 3-surface to reach into $r = 1.5M$ in the limit $t \to \infty$, that is inside of the event horizon, $r = 2M$. However, it is also reported that the difference of α-evolution causes the grid-stretching problem.

2.4.2 Shift conditions

geodesic slice	$\beta^i = 0$	`GOOD`	simple, easy to understand	
		`BAD`	too simple	
minimal distortion	$\min \Sigma^{ij}\Sigma_{ij}$	`GOOD`	geometrical meaning	[1]
		`BAD`	elliptic eqs., hard to solve	
minimal strain	$\min \Theta^{ij}\Theta_{ij}$	`G&B`	same with minimal distortion	[1]

Minimal distortion condition, minimal strain condition Any singularity avoiding slice conditions causes the grid stretching problem. Smarr and York [1] proposed the condition which minimize the distortion in a global sense.

Let us define the expansion tensor $\Theta_{\mu\nu}$ and the distortion tensor Σ_{ij}. Let the normal direction to the surface n^μ, and the coordinate-constant congruence $t_\mu = \alpha n_\mu + \beta_\mu$. By projecting t^μ onto the hypersurface using the projection operator $\perp^a_b = \delta^a_b + n^a n_b$,

$$\Theta_{\mu\nu} = \perp\nabla_{(\nu} t_{\mu)} = -\alpha K_{\mu\nu} + \frac{1}{2} D_{(\mu}\beta_{\nu)} \tag{2.65}$$

We then extract this traceless part and define,

$$\Sigma_{ij} \;=\; \Theta_{ij} - \frac{1}{3}\Theta\gamma_{ij} = -2\alpha\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) + \frac{1}{2}\left(D_{(i}\beta_{j)} - \frac{1}{3}D^k\beta_k\right). \tag{2.66}$$

The minimal distortion condition is to choose β^i which minimize the action

$$\delta S[\beta] = \delta\{\frac{1}{2}\int \Sigma_{ij}\Sigma^{ij} d^3x\} = 0. \tag{2.67}$$

This condition can be written as $D^j\Sigma_{ij} = 0$, or

$$D^j D_j \beta_i + D^j D_i \beta_j - \frac{2}{3} D_i D_j \beta^j = D^j\left[2\alpha\left(K_{ij} - \frac{1}{3} trK\gamma_{ij}\right)\right], \tag{2.68}$$

or

$$\Delta\beta_i + \frac{1}{3}D_i(D^j\beta_j) + R^j_i\beta_j = D^j\left[2\alpha\left(K_{ij} - \frac{1}{3} trK\gamma_{ij}\right)\right], \tag{2.69}$$

where $\Delta = D^i D_i$.

Similarly, we can define the minimal strain condition by minimizing $\Theta^{ij}\Theta_{ij}$.

The both requires non-linear elliptic equations and hard to solve. Several group solves “pseudo”-minimal distortion condition by replacing the covariant derivatives to the partial derivatives [22]. This simplification also works for inspiral binary neutron star evolution.

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2.5 How to evolve the system

Many trials for longterm stable and accurate simulations of binary compact objects have revealed that mathematically equivalent sets of evolution equations show different numerical stability in free evolution schemes. Thus, the stability problem or the formulation problem is now shedding light on the mathematical structure of the Einstein equations. More detail review is available as [40].

2.5.1 Overview

Up to a couple of years ago, the "standard ADM" decomposition (§2.1) of the Einstein equation was taken as the standard formulation for numerical relativists. However, numerical simulations were often interrupted by unexplained blow-ups. This was thought due to the lack of resolution, or inappropriate gauge choice, or the particular numerical scheme which was applied. However, after the accumulation of much experience, people have noticed the importance of the formulation of the evolution equations, since there are apparent differences in numerical stability although the equations are mathematically equivalent Figure 2 is a chronological map of the researches. See Column 1 for the meaning of "stability".

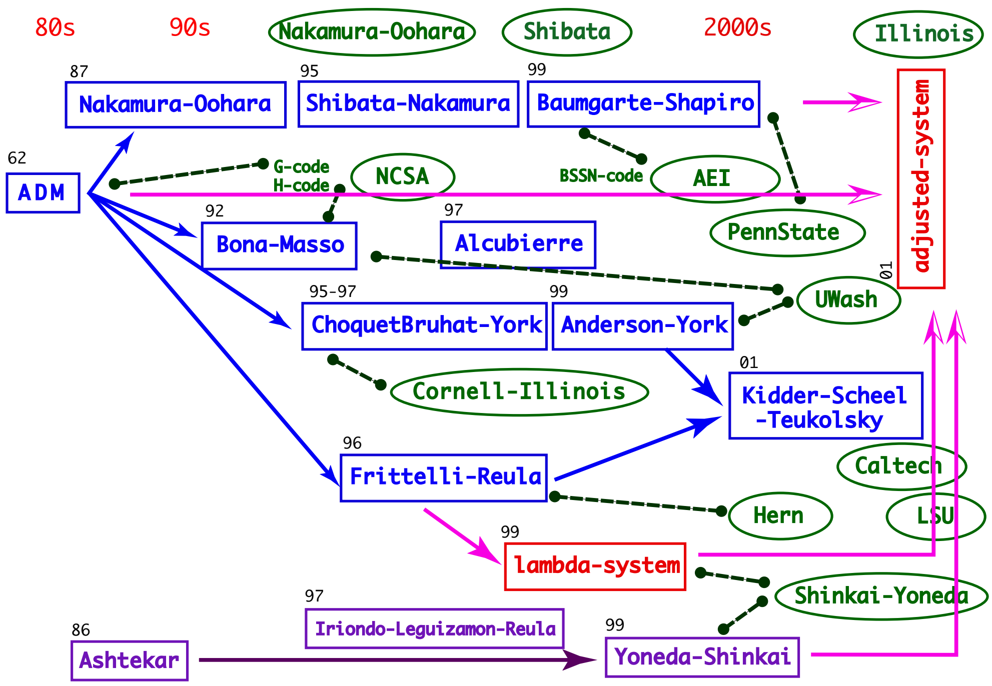

Figure 2: Chronological table of formulations and their numerical tests. Boxed ones are of proposals of formulation, circled ones are related numerical experiments. Please refer Table 1 in [40] for references.

At this moment, there are three major ways to obtain longer time evolutions: (1) modifications of the standard Arnowitt-Deser-Misner equations initiated by the Kyoto group, (2) rewriting of the evolution equations in hyperbolic form, and (3) construction of an "asymptotically constrained" system. Of course, the ideas, procedures, and problems are mingled with each other. The purpose of this section is to review all three approaches and to introduce our idea to view them in a unified way. The third idea has been generalized by us as an asymptotically constrained system. The main procedure is to adjust the evolution equations using the constraint equations [47, 48, 39]. The method is also applied to explain why the above approach (1) works, and also to propose alternative systems based on the ADM [48, 39] and BSSN [49] equations.

Column 1

The word **stability** is used quite different ways in the community.

- We mean by *numerical stability* a numerical simulation which continues without any blow-ups and in which data remains on the constrained surface.
- *Mathematical stability* is defined in terms of the well-posedness in the theory of partial differential equations, such that the norm of the variables is bounded by the initial data. See eq. (2.83) and around.
- For numerical treatments, there is also another notion of *stability*, the stability of finite differencing schemes. This means that numerical errors (truncation, round-off, etc) are not growing by evolution, and the evaluation is obtained by von Neumann's analysis. Lax's equivalence theorem says that if a numerical scheme is consistent (converging to the original equations in its continuum limit) and stable (no error growing), then the simulation represents the right (converging) solution. See [18] for the Einstein equations.

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2.5.2 Strategy 0: The ADM formulation

As we see in §2.1, we know that *if the constraints are satisfied on the initial slice* Σ*, then the constraints are satisfied throughout evolution.* The normal numerical scheme is to solve the elliptic constraints for preparing the initial data, and to apply the free evolution (solving only the evolution equations). The constraints are used to monitor the accuracy of simulations.

The origin of the problem was that the above statement in *Italics* is true in principle, but is not always true in numerical applications. A long history of trial and error began in the early 90s. Shinkai and Yoneda showed that the standard ADM equations *has* a constraint violating mode in its constraint propagation equations even for a single black-hole (Schwarzschild) spacetime [39].

2.5.3 Strategy 1: Modified ADM formulation by Nakamura et al

Up to now, the most widely used formulation for large scale numerical simulations is a modified ADM system, which is now often cited as the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation. This reformulation was first introduced by Nakamura *et al.* [29, 30, 35]. The usefulness of this reformulation was re-introduced by Baumgarte and Shapiro [11], then was confirmed by other groups to show a long-term stable numerical evolution [4, 6].

2.5.4 Basic variables and equations

The widely used notation[11] introduces the variables $(\varphi, \tilde\gamma_{ij}, K, \tilde A_{ij}, \tilde\Gamma^i)$ instead of (γ_{ij}, K_{ij}), where

$$\varphi = (1/12)\log(\det\gamma_{ij}), \qquad \tilde\gamma_{ij} = e^{-4\varphi}\gamma_{ij}, \qquad K = \gamma^{ij}K_{ij}, \tag{2.70}$$

$$\tilde A_{ij} = e^{-4\varphi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde\Gamma^i = \tilde\Gamma^i_{jk}\tilde\gamma^{jk}. \tag{2.71}$$

The new variable $\tilde\Gamma^i$ was introduced in order to calculate Ricci curvature more accurately. In BSSN formulation, Ricci curvature is not calculated as $R^{ADM}_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li}$, but as $R^{BSSN}_{ij} = R^\varphi_{ij} + \tilde R_{ij}$, where the first term includes the conformal factor φ while the second term does not. These are approximately equivalent, but R_{ij}^{BSSN} does have wave operator apparently in the flat background limit, so that we can expect more natural wave propagation behavior.

Additionally, the BSSN requires us to impose the conformal factor as $\tilde{\gamma}(:= \det\tilde{\gamma}_{ij}) = 1$, during evolution. This is a kind of definition, but can also be treated as a constraint. [3]

The BSSN formulation [29, 30, 35, 11]: **Box 2.3**

The fundamental dynamical variables are $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$.

The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i), && (2.72)\\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}), && (2.73)\\
\partial_t^B K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), && (2.74)\\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R_{ij}^{BSSN})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j \\
&\quad +(\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} - (2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}), && (2.75)\\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi)) \\
&\quad -\partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)). && (2.76)
\end{aligned}$$

- Constraint equations:

$$\begin{aligned}
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, && (2.77)\\
\mathcal{M}_i^{BSSN} &= \mathcal{M}_i^{ADM}, && (2.78)\\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, && (2.79)\\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij}, && (2.80)\\
\mathcal{S} &= \tilde{\gamma} - 1. && (2.81)
\end{aligned}$$

(2.77) and (2.78) are the Hamiltonian and momentum constraints (the “kinematic” constraints), while the latter three are “algebraic” constraints due to the requirements of BSSN formulation.

Remarks Why BSSN is better than the standard ADM? Together with numerical comparisons with the standard ADM case[6], this question has been studied by many groups using different approaches. Using numerical test evolution, Alcubierre et al [4] found that the essential improvement is in the process of replacing terms by the momentum constraints. They also pointed out that the eigenvalues of BSSN *evolution equations* have fewer “zero eigenvalues” than those of ADM, and they conjectured that the instability might be caused by these “zero eigenvalues”. An effort was made to understand the advantage of BSSN from the point of hyperbolization of the equations in its linearized limit [4, 32]. These studies provide some support regarding the advantage of BSSN, while it is also shown an example of an ill-posed solution in BSSN (as well in ADM) by Frittelli and Gomez [23].

As we discussed in [49], the stability of the BSSN formulation is due not only to the introductions of new variables, but also to the replacement of terms in the evolution equations using the constraints. Further, we will show several additional adjustments to the BSSN equations which are expected to give us more stable numerical simulations.

[3] The box/column numbers in this subsection are numerated so as to fit with [40].

2.5.5 Strategy 2: Hyperbolic reformulations

Definitions, properties, mathematical backgrounds The second effort to re-formulate the Einstein equations is to make the evolution equations reveal a first-order hyperbolic form explicitly. This is motivated by the expectation that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and also that the boundary treatment can be improved if we know the characteristic speed of the system.

Hyperbolic formulations **Box 2.5**

We say that the system is *a first-order (quasi-linear) partial differential equation system*, if a certain set of (complex-valued) variables u_α ($\alpha = 1, \cdots, n$) forms

$$\partial_t u_\alpha = \mathcal{M}^{l\beta}{}_\alpha(u)\,\partial_l u_\beta + \mathcal{N}_\alpha(u), \tag{2.82}$$

where $\mathcal{M}$ (the characteristic matrix) and $\mathcal{N}$ are functions of u but do not include any derivatives of u. Further we say the system is

- *a weakly hyperbolic system*, if all the eigenvalues of the characteristic matrix are real.
- *a strongly hyperbolic system* (or a diagonalizable / symmetrizable hyperbolic system), if the characteristic matrix is diagonalizable (has a complete set of eigenvectors) and has all real eigenvalues.
- *a symmetric hyperbolic system*, if the characteristic matrix is a Hermitian matrix.

Writing the system in a hyperbolic form is a quite useful step in proving that the system is *well-posed*. The mathematical well-posedness of the system means (1°) local existence (of at least one solution u), (2°) uniqueness (i.e., at most solutions), and (3°) stability (or continuous dependence of solutions $\{u\}$ on the Cauchy data) of the solutions. The resultant statement expresses the existence of the energy inequality on its norm,

$$||u(t)|| \le e^{\alpha\tau}||u(t=0)||, \qquad \text{where } 0 < \tau < t, \quad \alpha = const. \tag{2.83}$$

This indicates that the norm of $u(t)$ is bounded by a certain function and the initial norm. Remark that this mathematical boundness does not mean that the norm $u(t)$ decreases along the time evolution.

The inclusion relation of the hyperbolicities is,

$$\text{symmetric hyperbolic} \subset \text{strongly hyperbolic} \subset \text{weakly hyperbolic}. \tag{2.84}$$

The Cauchy problem under weak hyperbolicity is not, in general, C^∞ well-posed. At the strongly hyperbolic level, we can prove the finiteness of the energy norm if the characteristic matrix is independent of u (cf [43]), that is one step definitely advanced over a weakly hyperbolic form. Similarly, the well-posedness of the symmetric hyperbolic is guaranteed if the characteristic matrix is independent of u, while if it depends on u we have only limited proofs for the well-posedness.

From the point of numerical applications, to hyperbolize the evolution equations is quite attractive, not only for its mathematically well-posed features. The expected additional advantages are the following.

(a) It is well known that a certain flux conservative hyperbolic system is taken as an essential formulation in the computational Newtonian hydrodynamics when we control shock wave formations due to matter.

(b) The characteristic speed (eigenvalues of the principal matrix) is supposed to be the propagation speed of the information in that system. Therefore it is naturally imagined that these magnitudes are equivalent to the physical information speed of the model to be simulated.

(c) The existence of the characteristic speed of the system is expected to give us an improved treatment of the numerical boundary, and/or to give us a new well-defined Cauchy problem within a finite region (the so-called initial boundary value problem, IBVP).

These statements sound reasonable, but have not yet been generally confirmed in actual numerical simulations. But we are safe in saying that the formulations are not yet well developed to test these issues.

Hyperbolic formulations of the Einstein equations Most physical systems can be expressed as symmetric hyperbolic systems. In order to prove that the Einstein's theory is a well-posed system, to hyperbolize the Einstein equations is a long-standing research area in mathematical relativity.

The standard ADM system does not form a first order hyperbolic system. This can be seen immediately from the fact that the ADM evolution equation (2.11) has Ricci curvature in RHS. So far, several first order hyperbolic systems of the Einstein equation have been proposed. In constructing hyperbolic systems, the essential procedures are (1°) to introduce new variables, normally the spatially derivatived metric, (2°) to adjust equations using constraints. Occasionally, (3°) to restrict the gauge conditions, and/or (4°) to rescale some variables. Due to process (1°), the number of fundamental dynamical variables is always larger than that of ADM.

Due to the limitation of space, we can only list several hyperbolic systems of the Einstein equations.

- The Bona-Massó formulation [13, 14]
- The Einstein-Ricci system [19, 1] / Einstein-Bianchi system [7]
- The Einstein-Christoffel system [8]
- The Ashtekar formulation [10]
- The Frittelli-Reula formulation [24, 43]
- The Conformal Field equations [21]
- The Kidder-Scheel-Teukolsky (KST) formulation [26]

Please refer [40] for each brief introductions.

Remarks When we discuss hyperbolic systems in the context of numerical stability, the following questions should be considered:

Q From the point of the set of evolution equations, does hyperbolization actually contribute to numerical accuracy and stability? Under what conditions/situations will the advantages of hyperbolic formulation be observed?

Unfortunately, we do not have conclusive answers to these questions, but many experiences are being accumulated. Several earlier numerical comparisons reported the stability of hyperbolic formulations [14, 15, 33, 34]. But we have to remember that this statement went against the standard ADM formulation, which has a constraint-violating mode for Schwarzschild spacetime as has been shown recently[39].

These partial numerical successes encouraged the community to formulate various hyperbolic systems. Recently, Calabrese et al [17] reported there is a certain differences in the long-term convergence features between weakly and strongly hyperbolic systems on the Minkowskii background space-time. However, several numerical experiments also indicate that this direction is not a complete success.

<u>Objections from numerical experiments</u>

- Above earlier numerical successes were also terminated with blow-ups.

- If the gauge functions are evolved according to the hyperbolic equations, then their finite propagation speeds may cause pathological shock formations in simulations [2, 3].
- There are no drastic differences in the evolution properties *between* hyperbolic systems (weakly, strongly and symmetric hyperbolicity) by systematic numerical studies by Hern [25] based on Frittelli-Reula formulation [24], and by the authors [38] based on Ashtekar's formulation [10, 46].
- Proposed symmetric hyperbolic systems were not always the best ones for numerical evolution. People are normally still required to reformulate them for suitable evolution. Such efforts are seen in the applications of the Einstein-Ricci system [34], the Einstein-Christoffel system [12], and so on.

Of course, these statements only casted on a particular formulation, and therefore we have to be careful not to over-emphasize the results. In order to figure out the reasons for the above objections, it is worth stating the following cautions:

Remarks on hyperbolic formulations

(a) Rigorous mathematical proofs of well-posedness of PDE are mostly for simple symmetric or strongly hyperbolic systems. If the matrix components or coefficients depend on dynamical variables (as in all any versions of hyperbolized Einstein equations), almost nothing was proved in more general situations.

(b) The statement of "stability" in the discussion of well-posedness refers to the bounded growth of the norm, and does not indicate a decay of the norm in time evolution.

(c) The discussion of hyperbolicity only uses the characteristic part of the evolution equations, and ignores the rest.

We think the origin of confusion in the community results from over-expectation on the above issues. Mostly, point (c) is the biggest problem. The above numerical claims from Ashtekar and Frittelli-Reula formulations were mostly due to the contribution (or interposition) of non-principal parts in evolution. Regarding this issue, the recent KST formulation finally opens the door. KST's "kinematic" parameters enable us to reduce the non-principal part, so that numerical experiments are hopefully expected to represent predicted evolution features from PDE theories. At this moment, the agreement between numerical behavior and theoretical prediction is not yet perfect but close [27].

If further studies reveal the direct correspondences between theories and numerical results, then the direction of hyperbolization will remain as the essential approach in numerical relativity, and the related IBVP researches will become a main research subject in the future. Meanwhile, it will be useful if we have an alternative procedure to predict stability including the effects of the non-principal parts of the equations. Our proposal of adjusted system in the next subsection may be one of them.

2.5.6 Strategy 3: Asymptotically constrained systems

The third strategy is to construct a robust system against the violation of constraints, such that the constraint surface is an attractor. The idea was first proposed as "λ-system" by Brodbeck et al [16], and then developed in more general situations as "adjusted system" by the authors [47].

The "λ-system" Brodbeck et al [16] proposed a system which has additional variables λ that obey artificial dissipative equations. The variable λs are supposed to indicate the violation of constraints and the target of the system is to get $\lambda = 0$ as its attractor.

The "λ-system" (Brodbeck-Frittelli-Hübner-Reula) [16]: **Box 2.7**

For a symmetric hyperbolic system, add additional variables λ and artificial force to reduce the violation of constraints.

The procedure:

1.	Prepare a symmetric hyperbolic evolution system	$\partial_t u = M\partial_i u + N$
2.	Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion	$\partial_t \lambda = \alpha C - \beta\lambda$ $(\alpha \neq 0, \beta > 0)$
3.	Take a set of (u, λ) as dynamical variables	$\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4.	Modify evolution eqs so as to form a symmetric hyperbolic system	$\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Since the total system is designed to have symmetric hyperbolicity, the evolution is supposed to be unique. Brodbeck et al showed analytically that such a decay of λs can be seen for sufficiently small $\lambda(> 0)$ with a choice of appropriate combinations of αs and βs.

Brodbeck et al presented a set of equations based on Frittelli-Reula's symmetric hyperbolic formulation [24]. The version of Ashtekar's variables was presented by the authors [37] for controlling the constraints or reality conditions or both. The numerical tests of both the Maxwell-λ-system and the Ashtekar-λ-system were performed [47], and confirmed to work as expected. Although it is questionable whether the recovered solution is true evolution or not [41], we think the idea is quite attractive. To enforce the decay of errors in its initial perturbative stage seems the key to the next improvements, which are also developed in the next section on "adjusted systems".

However, there is a high price to pay for constructing a λ-system. The λ-system can not be introduced generally, because (i) the construction of λ-system requires the original evolution equations to have a symmetric hyperbolic form, which is quite restrictive for the Einstein equations, (ii) the final system requires many additional variables and we also need to evaluate all the constraint equations at every time step, which is a hard task in computation. Moreover, (iii) it is not clear that the λ-system is robust enough for non-linear violation of constraints, or that λ-system can control constraints which do not have any spatial differential terms.

The "adjusted system" Next, we propose an alternative system which also tries to control the violation of constraint equations actively, which we named "adjusted system". We think that this system is more practical and robust than the previous λ-system.

The Adjusted system (procedures): **Box 2.9**

1.	Prepare a set of evolution eqs.	$\partial_t u = J\partial_i u + K$
2.	Add constraints in RHS	$\partial_t u = J\partial_i u + K \underbrace{+\kappa C}$
3.	Choose the coeff. κ so as to make the eigenvalues of the homogenized adjusted $\partial_t C$ eqs negative reals or pure imaginary.	$\partial_t C = D\partial_i C + EC$ $\partial_t C = D\partial_i C + EC \underbrace{+F\partial_i C + GC}$

The process of adjusting equations is a common technique in other re-formulating efforts as we reviewed. However, we try to employ the evaluation process of constraint amplification factors as an alternative guideline to hyperbolization of the system. We will explain these issues in the next section.

2.5.7 A unified treatment: Adjusted System

This section is devoted to present our idea of "asymptotically constrained system". Original references can be found in [47, 48, 39, 49].

Procedures : Constraint propagation equations and Proposals Suppose we have a set of dynamical variables $u^a(x^i, t)$, and their evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \tag{2.85}$$

and the (first class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \tag{2.86}$$

Note that we do not require (2.85) forms a first order hyperbolic form. We propose to investigate the evolution equation of C^α (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots), \tag{2.87}$$

for predicting the violation behavior of constraints in time evolution. We do not mean to integrate (2.87) numerically together with the original evolution equations (2.85), but mean to evaluate them analytically in advance in order to reformulate the equations (2.85).

There may be two major analyses of (2.87); (a) the hyperbolicity of (2.87) when (2.87) is a first order system, and (b) the eigenvalue analysis of the whole RHS in (2.87) after a suitable homogenization. However, as we critically viewed the hyperbolization road in the previous section, we prefer to proceed the road (b).

Amplification Factors of Constraint Propagation equations: **Box 3.1**
We propose to homogenize (2.87) by a Fourier transformation, e.g.

$$\begin{aligned} \partial_t \hat{C}^\alpha &= \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \\ &\text{where } C(x,t)^\rho = \int \hat{C}(k,t)^\rho \exp(ik\cdot x) d^3k, \end{aligned} \tag{2.88}$$

then to analyze the set of eigenvalues, say Λs, of the coefficient matrix, $M^\alpha{}_\beta$, in (2.88). We call Λs the constraint amplification factors (CAFs) of (2.87).

The CAFs predict the evolution of constraint violations. We therefore can discuss the "distance" to the constraint surface using the "norm" or "compactness" of the constraint violations (although we do not have exact definitions of these "$\cdots$" words).

The next conjecture seems to be quite useful to predict the evolution feature of constraints:

Conjecture on Constraint Amplification Factors (CAFs): **Box 3.2**

(A) If CAF has a *negative real-part* (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a *non-zero imaginary-part* (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.

We found that the system becomes more stable when more Λs satisfy the above criteria. A general feature of the constraint propagation is reported in [50].

The above features of the constraint propagation, (2.87), will differ when we modify the original evolution equations. Suppose we add (adjust) the evolution equations using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots), \tag{2.89}$$

then (2.87) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots). \tag{2.90}$$

Therefore, the problem is how to adjust the evolution equations so that their constraint propagations satisfy the above criteria as much as possible.

Applications For the Maxwell equation and the Ashtekar version of the Einstein equations, we numerically found that this idea works to reduce the violation of constraints, and that the effects are much better than by constructing its symmetric hyperbolic versions [38, 47].

Applications to ADM The idea was applied to the standard ADM formulation which is not hyperbolic and several attractive adjustments were proposed [48, 39]. We made various predictions how additional adjusted terms will change the constraint propagation. Systematic numerical comparisons are also progressing, and we show two sample plots here.

Figure 4 (a) is a test numerical evolution of Detweiler-type adjustment [20] on the Minkowskii background. We see the adjusted version gives convergence on to the constraint surface by arranging the magnitude of the adjusting parameter, κ. Figure 4 (b) is obtained by a 3-dimensional numerical evolution of weak gravitational wave, the so-called Teukolsky wave [44]. The lines are of the original/standard ADM evolution equations, Detweiler-type adjustment, and a part of Detweiler-type adjustment. For a particular choice of κ, we observe again the L2 norm of constraint (violation of constraints) is reduced than the standard ADM case, and can evolve longer than that.

Notion of Time Reversal Symmetry During the comparisons of adjustments, we found that it is necessary to create time asymmetric structure of evolution equations in order to force the evolution on to the constraint surface. There are infinite ways of adjusting equations, but we found that if we follow the guideline Box 3.5, then such an adjustment will give us time asymmetric evolution.

Trick to obtain asymptotically constrained system: **Box 3.5**
= Break the time reversal symmetry (TRS) of the evolution equation.

1. Evaluate the parity of the evolution equation.
   By reversing the time ($\partial_t \to -\partial_t$), there are variables which change their signatures (parity $(-)$) [e.g. $K_{ij}, \partial_t \gamma_{ij}, \mathcal{M}_i, \cdots$], while not (parity $(+)$) [e.g. $g_{ij}, \partial_t K_{ij}, \mathcal{H}, \cdots$].
2. Add adjustments which have different parity of that equation.
   For example, for the parity $(-)$ equation $\partial_t \gamma_{ij}$, add a parity $(+)$ adjustment $\kappa\mathcal{H}$.

One of our criteria, the negative real CAFs, requires breaking the time-symmetric features of the original evolution equations. Such CAFs are obtained by adjusting the terms which break the TRS of the evolution equations, and this is available even at the standard ADM system.

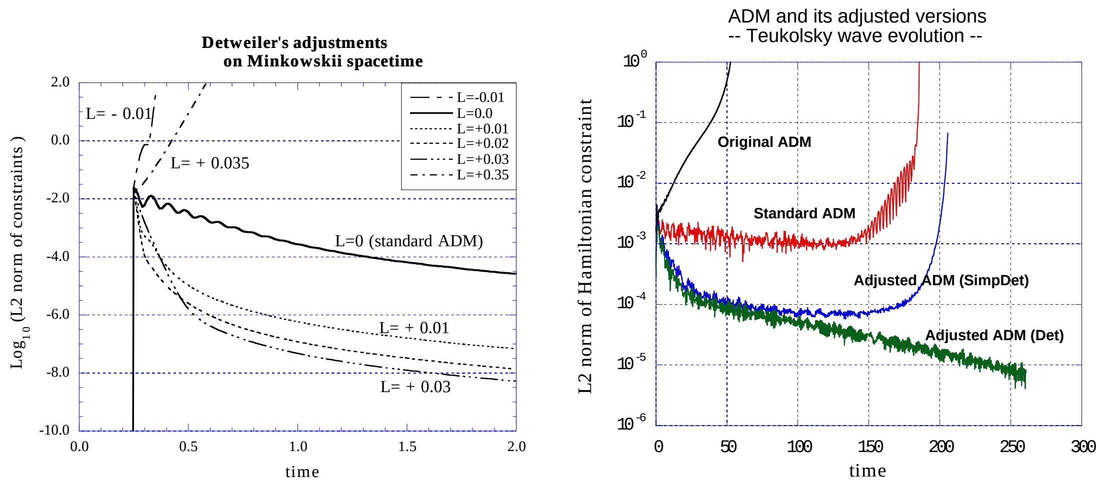

Figure 3: Comparisons of numerical evolution between adjusted ADM systems. (a) Demonstration of the Detweiler's modified ADM system on Minkowskii background spacetime, 1-dimensional simulation. The L2 norm of the constraints $\mathcal{H}^{ADM}$ and $\mathcal{M}^{ADM}$ is plotted in the function of time. Artificial error was added at $t = 0.25$. L is the parameter in Detweiler's adjustment. We see the evolution is asymptotically constrained for small $\kappa > 0$. (b) L2 norm of the Hamiltonian constraint $\mathcal{H}^{ADM}$ of evolution using ADM/adjusted ADM formulations for the case of Teukolsky wave, 3-dimensional simulation.

Applications to BSSN This analysis was also applied to explain the advantages of the BSSN formulation, and again several alternative adjustments to BSSN equations were proposed [49]. Recently Yo *et al*[45] reported their simulations of stationary rotating black hole, and mentioned that one of our proposal was contributed to maintain their evolution of Kerr black hole (J/M up to $0.9M$) for long time ($t \sim 6000M$). Their results also indicates that the evolved solution is closed to the exact one, that is, the constrained surface.

2.5.8 Outlook

What we have achieved

- The constraint propagation features become different by simply adding constraint terms to the original evolution equations (we call this the *adjustment* of the evolution equations).
- There *is* a constraint-violating mode in the standard ADM evolution system when we apply it to a single non-rotating black hole space-time, and its growth rate is larger near the black-hole horizon.
- Such a constraint-violating mode can be killed if we adjust the evolution equations with a particular modification using constraint terms (Box 2.7). An effective guideline is to adjust terms as they break the time-reversal symmetry of the equations (Box 3.5).
- Our expectations are borne out in simple numerical experiments using the Maxwell, Ashtekar, and ADM systems. However the modifications are not yet perfect to prevent non-linear growth of the constraint violation.
- We understand why the BSSN formulation works better than the ADM one in the limited case (perturbative analysis in the flat background), and further we proposed modified evolution equations along the lines of our previous procedure.

The common key to the problem is how to adjust the evolution equations with constraints. Any adjusted systems are mathematically equivalent if the constraints are completely satisfied, but this is not the case for numerical simulations. Replacing terms with constraints is one of the normal steps when people hyperbolize equations. Our approach is to employ the evaluation process of constraint amplification factors for an alternative guideline to hyperbolization of the system.

Final remarks If we say the final goal of this project is to find a robust algorithm to obtain long-term accurate and stable time-evolution method, then the recipe should be a combination of (a) formulations of the evolution equations, (b) choice of gauge conditions, (c) treatment of boundary conditions, and (d) numerical integration methods. We are in the stages of solving this mixed puzzle. The ideal almighty algorithm may not exit, but we believe our accumulating experience will make the ones we do have more robust and automatic.

I have written this review from the viewpoint that the general relativity is a constrained dynamical system. This is not only a proper problem in general relativity, but also in many physical systems such as electrodynamics, magnetohydrodynamics, molecular dynamics, mechanical dynamics, and so on. Therefore sharing the thoughts between different field will definitely accelerate the progress.

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3 Alternative Approaches to Numerical Relativity

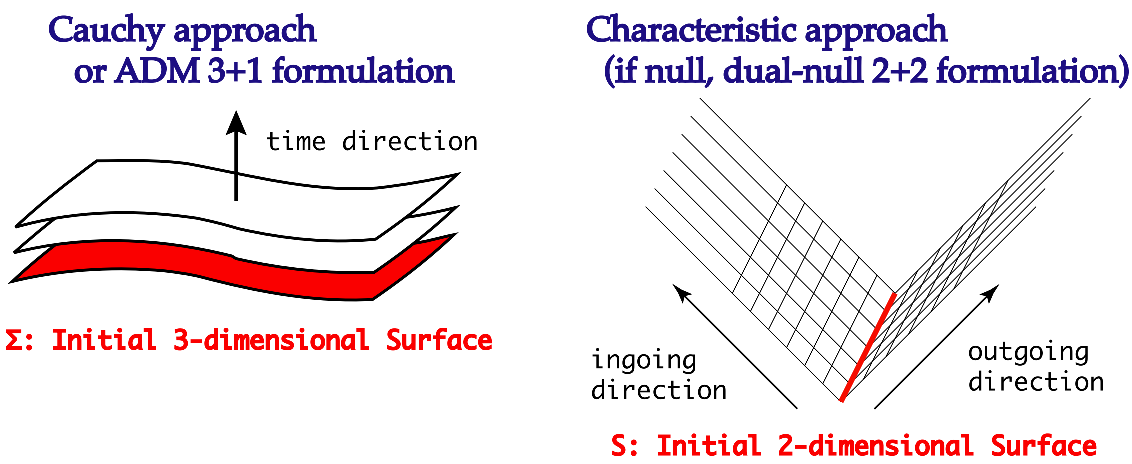
In the pioneer days of numerical relativity (70s - 80s), people set the destination of a code so as it simultaneously (1) avoids singularities, (2) handles black-holes, (3) maintains high accuracy, and (4) runs forever. This goal was also called "Holy Grail" of numerical relativity [1]. Various approaches have been proposed and tested for these purposes. In this chapter, I introduce several alternative or complemental approaches to the standard 3+1 (ADM, or Cauchy) approach.

3.1 Full numerical, but different foliations

3.1.1 Characteristic foliation

Cauchy versus Characteristic

Figure 4: Two major foliations for seeking evolution in general relativity.



	Cauchy (3+1) evolution	Characteristic (2+2) evolution
pioneers	ADM, York-Smarr	Bondi *et al* [2], Sachs [3], Penrose[4] Numerical works by Stewart et al [5]
variables	easy to understand the concept of time evolution	has geometrical meanings 1 complex function related to 2 GW polarization modes
foliation	has Hamilton structure	allows implementation of Penrose's space-time compactification
initial data	need to solve constraints	no constraints
evolution	PDEs need to avoid constraint violation	ODEs with consistent conditions propagation eqs along the light rays
singularity	need to avoid by some method	can truncate the grid
disadvantages	can not cover space-time globally	difficulty in treating caustics hard to treat matter

Table 1: Comparison of Cauchy and characteristic approaches. See reviews by Winicour [6].

A connection formula from ADM to Newman-Penrose The Newman-Penrose formulation [7, 8] has many advantages, especially for treating gravitational wave dynamics.

- Natural framework for calculations in radiative space-time.
- Variables have geometrical meanings.
- Practical advantages in treating Petrov type-D space-time.
- Closely related with spinor formalism.

Newman-Penrose's variables are based on real-valued null vectors $\mathbf{l}, \mathbf{n}$ and complex conjugate null vectors $\mathbf{m}, \overline{\mathbf{m}}$, which satisfy

$$\mathbf{l}\cdot\mathbf{n} = l^a n_a = l_a n^a = 1, \qquad \mathbf{m}\cdot\overline{\mathbf{m}} = m^a \overline{m}_a = m_a \overline{m}^a = -1, \qquad \text{and else } 0. \tag{3.1}$$

This set of null basis $(l^a, n^a, m^a, \overline{m}^a)$ have relations with orthogonal tetrad basis $(t^{\hat{a}}, x^{\hat{a}}, y^{\hat{a}}, z^{\hat{a}})$

$$l^a = o^A o^{A'} = \frac{1}{\sqrt{2}}(t^{\hat{a}} + z^{\hat{a}}), \qquad m^a = o^A \iota^{A'} = \frac{1}{\sqrt{2}}(x^{\hat{a}} - i y^{\hat{a}}) \tag{3.2}$$

$$n^a = \iota^A \iota^{A'} = \frac{1}{\sqrt{2}}(t^{\hat{a}} - z^{\hat{a}}), \qquad \overline{m}^a = \iota^A o^{A'} = \frac{1}{\sqrt{2}}(x^{\hat{a}} + i y^{\hat{a}}), \tag{3.3}$$

where I also put spinor expressions (o^A, ι^A) of those. The indice rules are

$$l_a = g_{ab} l^b, \quad l^a = g^{ab} l_b, \qquad g_{ab} = metric \tag{3.4}$$

$$x_{\hat{a}} = \eta_{\hat{a}\hat{b}} x^{\hat{b}}, \quad x^{\hat{a}} = \eta^{\hat{a}\hat{b}} x_{\hat{b}}, \qquad \eta_{\hat{a}\hat{b}} = (1, -1, -1, -1) \tag{3.5}$$

Metric g_{ab} will be recovered by

$$g_{ab} = 2 l_{(a} n_{b)} - 2 m_{(a} \overline{m}_{b)}, \qquad g^{ab} = 2 l^{(a} n^{b)} - 2 m^{(a} \overline{m}^{b)} \tag{3.6}$$

$$g_{\hat{a}\hat{b}} = t_{\hat{a}} t_{\hat{b}} - x_{\hat{a}} x_{\hat{b}} - y_{\hat{a}} y_{\hat{b}} - z_{\hat{a}} z_{\hat{b}}. \tag{3.7}$$

The Weyl curvature C_{abcd} is defined as

$$C_{abcd} = R_{abcd} - g_{a[c} R_{d]b} + g_{b[c} R_{d]a} - \frac{1}{3} R g_{a[c} g_{d]b}. \tag{3.8}$$

The 10 components of Weyl curvature are expressed by the following 5 complex scalars [9];

$$\Psi_0 \equiv \psi_{ABCD} o^A o^B o^C o^D = C_{abcd} l^a m^b l^c m^d, \quad n^a\text{-directed transverse component}, \{4, 0\} \tag{3.9}$$

$$\Psi_1 \equiv \psi_{ABCD} o^A o^B o^C \iota^D = C_{abcd} l^a n^b l^c m^d, \quad n^a\text{-directed longitudinal component}\{2, 0\} \tag{3.10}$$

$$\Psi_2 \equiv \psi_{ABCD} o^A o^B \iota^C \iota^D = C_{abcd} l^a m^b \bar{m}^c n^d, \quad \text{'Coulomb' component}, \{0, 0\} \tag{3.11}$$

$$\Psi_3 \equiv \psi_{ABCD} o^A \iota^B \iota^C \iota^D = C_{abcd} l^a n^b \bar{m}^c n^d, \quad l^a\text{-directed longitudinal component}, \{-2, 0\} \tag{3.12}$$

$$\Psi_4 \equiv \psi_{ABCD} \iota^A \iota^B \iota^C \iota^D = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \quad l^a\text{-directed transverse component}, \{-4, 0\} \tag{3.13}$$

where $\{p, q\}$ indicates *spin- and boost-weighted type* and prime-operation will be defined later.

Gunnarsen-Shinkai-Maeda [11] derived a transformation formula of Weyl scalar Ψ_i from ADM variables (γ_{ij}, K_{ij}), motivated by an application to interpret numerically generated space-time. Here, we consider vacuum space-time. Let $(\mathcal{M}, \eta_{ab})$ be real, 4-dimensional Lorentz vector space with volume form ε_{abcd}; $\varepsilon_{abcd}\varepsilon^{abcd} = -4!$. Let (t^a, x^a, y^a, z^a) be orthonormal basis of $(\mathcal{M}, \eta_{ab})$, and define

$$t^a t_a = +s \quad (s = \pm 1), \quad \varepsilon_{abc} = \varepsilon^{abcd} t_d, \tag{3.14}$$

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$ We formulate our equations in the signatures both $(+,-,-,-)$ and $(-,+,+,+)$ by choosing $s = 1$ or -1, respectively[4], because the former notation is common in working with the spinors.

First, we define the Weyl curvature C_{abcd} by (3.8) and decompose those into its electric and a magnetic components,

$$E_{ab} \equiv - \ C_{ambn}t^m t^n, \quad B_{ab} \equiv - \ {}^*C_{ambn}t^n t^m, \tag{3.15}$$

where ${}^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd}$ is a dual of the Weyl tensor. These decomposed elements E_{ab} and B_{ab} are also presented by the 3-metric γ_{ab} and the extrinsic curvature K_{ab} as [12]

$$E_{ab} = R_{ab} - K_a{}^m K_{bm} + KK_{ab} - \frac{2}{3}\Lambda\gamma_{ab}, \tag{3.16}$$

$$B_{ab} = \varepsilon_a{}^{mn} D_m K_{nb}. \tag{3.17}$$

This is why we emphasize that our inputs are '3+1' elements. It follows from two constraint equations that the fields E_{ab}, B_{ab} are both trace-free and symmetric. We can reconstruct the Weyl curvature completely from E_{ab} and B_{ab} by

$$C_{abcd} = 4t_{[a}E_{b][c}t_{d]} + 2\varepsilon_{ab}{}^m B_{m[c}t_{d]} + 2\varepsilon_{cd}{}^m B_{m[a}t_{b]} + \varepsilon_{ab}{}^m \varepsilon_{cd}{}^n E_{mn}. \tag{3.18}$$

The next step is to choose a unit vector field $\hat{z}^a$ on Σ, and to decompose E_{ab}, B_{ab} into components along and perpendicular to $\hat{z}^a$. We set

$$\begin{array}{ll} e = E_{ab}\hat{z}^a\hat{z}^b, & b = B_{ab}\hat{z}^a\hat{z}^b, \\ e_a = E_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), & b_a = B_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), \\ e_{ab} = E_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}es_{ab}, & b_{ab} = B_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}bs_{ab}, \end{array} \tag{3.19}$$

where $s_{ab} = \gamma_{ab} - \hat{z}_a\hat{z}_b$. We note that E_{ab}, B_{ab} is again reconstructed from (3.19)

$$E_{ab} = e\hat{z}_a\hat{z}_b + 2e_{(a}\hat{z}_{b)} + e_{ab} - (1/2)s_{ab}e. \tag{3.20}$$

$$B_{ab} = b\hat{z}_a\hat{z}_b + 2b_{(a}\hat{z}_{b)} + b_{ab} - (1/2)s_{ab}b. \tag{3.21}$$

Such decompositions will be useful to discuss the effects of curvatures on the transversal plane to the $\hat{z}^a$ direction.

We put a rotation operator on the plane spanned by $\hat{x}_a$ and $\hat{y}_a$ as,

$$J_a{}^b \equiv \varepsilon_a{}^{bcd}\hat{z}_c t_d. \tag{3.22}$$

It is easy to check this mapping preserves s_{ab}, and is also easy to check $J_a{}^c J_c{}^b = -(\delta_a{}^b + s\hat{z}_a\hat{z}^b)$, which shows us $J_a{}^b$ has a complex structure, i.e., $J_a{}^b$ lets us define complex multiples of vectors $x^a \in P_z$, according to the formula $(m+in)x^a = mx^a + nJ_b{}^a x^b$. In short, $J_a{}^b$ expresses a rotation by 90 degrees in the plane orthogonal to $\hat{z}^a$.

By substituting (3.18) and (3.2, **??**) into (3.9)-(3.13), we get Ψ_i using (3.19) and (3.22):

$$\Psi_0 = -(e_{ab} + sJ_a{}^c b_{bc})m^a m^b, \tag{3.23}$$

$$\Psi_1 = -(s/\sqrt{2})\,(e_a + sJ_a{}^c b_c)m^a, \tag{3.24}$$

$$\Psi_2 = -(1/2)\,(e + ib), \tag{3.25}$$

$$\Psi_3 = -(s/\sqrt{2})(e_a - sJ_a{}^c b_c)\bar{m}^a, \tag{3.26}$$

$$\Psi_4 = -(e_{ab} - sJ_a{}^c b_{bc})\bar{m}^a\bar{m}^b. \tag{3.27}$$

This relation has been applied to many groups' numerical codes, and helps their simulation's physical understandings. Weyl scalars are also useful for evaluating Riemann (Kretchman) invariant as

$$C_{abcd}C^{abcd} = \Psi_4\Psi_0 - 4\Psi_1\Psi_3 + 3\Psi_2^2. \tag{3.28}$$

Note that

$$R_{abcd}R^{abcd} = C_{abcd}C^{abcd} + 2R_{ab}R^{ab} - (1/3)R^2. \tag{3.29}$$

[4] That is, the metric is $\eta_{ab} = s(t_a t_b - x_a x_b - y_a y_b - z_a z_b)$.

3.1.2 Characteristic approach

As I described in Table 1, the characteristic approach is quite attractive unless the system does not make a caustics in null hypersurface. Since the event horizon of black-hole is itself a characteristic hypersurface, the characteristic technique is powerful tool as a stand-alone. It also allows us to express compactified manifold, so that we can seek gravitational wave dynamics at time infinity.

Although the applicability of characteristic foliation is limited, numerical codes are developed extensively by by Pittsburgh group together with the Binary Black Hole Grand Challenge Alliance (1993-1998), `http://www.npac.syr.edu/projects/bh/`.

They adapted the Bondi-Sachs form of the metric for the null foliation. The coordinate are constructed from a family of outgoing null hypersurfaces, emanating from a worldtube or a timelike geodesic of topology $S^2 \times R$, which is labelled with a parameter u. Each null ray on a specific hypersurface is labelled with x^A where $(A = 2, 3)$, and let r be a surface area distance (i.e. surfaces at r=constant have area $4\pi r^2$). Then, the resulting coordinate is $x^a = (u, r, x^A)$, and the Bondi-Sachs form of the metric takes

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB} U^A U^B\right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B. \tag{3.30}$$

This metric has six real field variables, V, β, U^A, and h_{AB}. V can be understand in the analogue of the Newtonian potential, β represents the expansion of the light rays as they propagate radially. h_{AB} represents the conformal intrinsic geometry, which contains the two degrees of radiation freedom. Note that on the r =const. timelike world tube, the intrinsic metric can be expressed similarly to the Cauchy decomposition,

$${}^{(3)}ds^2 = -e^{2\beta}\frac{V}{r} du^2 + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du). \tag{3.31}$$

For example, a Schwarzschild geometry in outgoing Eddington-Finkelstein coordinates is given by the choice $\beta = 0, V = r - 2m, U^A = 0$ and h_{AB} be a unit sphere metric.

For a single black hole case the unlimited evolution was reported [13, 14, 15]. For the head-on collision of black-holes, the detail analysis of the dynamics of the event horizon was reported [16]. Recently, the formulation is studied also in the direction of including matter dynamics, such as the implementation of high resolution schemes [17] and the weak pressure fluids [18].

3.1.3 Cauchy-characteristic matching approach

This idea is to combine Cauchy evolution (interior) with characteristic evolution (exterior), in order to supply precise radiated waveform in the binary coalescence problem. Two evolution schemes are matched on the worldtube, and both sides of foliation supply the outer boundary values to the other. Numerical codes were developed independently by Southampton group [19] and Pittsburgh group [20] in the middle 90s. A significant advantage on the treatment of the outer boundary in Cauchy evolution region was reported for the cases of pure gravitational wave problem.

3.1.4 Hyperbolical foliation, conformal field equations

A series of works by Friedrich [21] attempted to construct a $3+1$ formulation with hyperboloidal foliations (i.e. asymptotically null foliations), and with conformal compactification. This is the ultimate plan to remove the outer boundary problem in numerical simulation, and to provide a suitable foliation for gravitational radiation problem. However, the current equations are rather quite complicated. In its metric-based expression [22], the evolution variables are 57; γ_{ij}, K_{ij}, the connection coefficients $\gamma^a{}_{bc}$, projections ${}^{(0,1)}\hat{R}_a = n^b \gamma_a{}^c \hat{R}_{bc}$ and ${}^{(1,1)}\hat{R}_{ab} = \gamma_a{}^c \gamma_b{}^d \hat{R}_{bd}$ of 4-dimensional Ricci tensor $\hat{R}_{ab}$, the electric and magnetic components of the rescaled Weyl tensor $C_{abc}{}^d$, and the conformal factor Ω and its related quantities $\Omega_0 \equiv n^a \nabla_a \Omega, \nabla_a \Omega, \nabla^a \nabla_a \Omega$. By specifying suitable gauge functions (α, β^a, R) where R is the Ricci scalar, then the total system forms a symmetric hyperbolic system. Applications to numerical relativity are in progress, but have not yet reached the stage of applying evolution in a non-trivial metric. For more details, see reviews e.g. by Frauendiener [23] or by Husa [24].

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3.2 Several approximations

3.2.1 Cauchy-perturbative matching: connection in spatial domain

This approximation intends to extract gravitational radiation information. and to provide stable outer boundary conditions for a Cauchy evolution numerical code. The procedure is to match the solution of numerical simulation of non-linear Einstein equations to a set of one-dimensional linear equations obtained through perturbation techniques over a curved background.

In early 90s, 1-dimensional version was implemented and tested 1-d [1]. The Binary Black Hole Grand Challenge Alliance and Rezzolla *et al* developed a 3-dimensional Cauchy-perturbative matching technique [2, 3].

3.2.2 Close-limit approximation: connection in time domain

In binary black-hole problem, the final outcome is a single black-hole which will ring down into equilibrium. Using a perturbation theory, we know black-holes have quasi-normal modes that is expected to be observed. The "close-limit approximation" is an extended idea to apply perturbation theory just after one single common horizon around two black-holes formed [4]. On the validity of this approximation, i.e. from which regime and at what order of perturbation we need, was tested against full numerical simulation using head-on collision of two black-holes [5]. Both radiated energy and waveform agreed quite well even for large values of the momentum. (Numerical outputs are between first and second order perturbation outputs.)

Recent development is one more step advanced. People try to evolve the system with perturbation equations starting from a fully numerically evolved data. This would compensate the current limitation of numerical relativity and derive more astrophysical conclusions. This idea was already tested for the case of collapse of disks using Schwarzschild background [6], and also for the inspiralling binary black-holes using Kerr background [7]. The latter project is named "Lazarus/Zorro".

3.2.3 Quasi-spherical approximation

Hayward [8] proposed a new approximation scheme in a dual-null decomposition of space-time, with the aim of providing a computationally inexpensive estimate of the gravitational waveforms produced by a black-hole or neutron-star collision, given a full numerical simulation up to (or close to) coalescence, or an analytical model thereof. The scheme truncates the Einstein equations by removing second-order terms which would vanish in a spherically symmetric space-time.

Shinkai and Hayward [9] numerically implemented this scheme, testing it against angular momentum by applying it to Kerr black holes. As error measures, we take the conformal strain and specific energy due to spurious gravitational radiation. The strain is found to be monotonic rather than wavelike. The specific energy is found to be at least an order of magnitude smaller than the 1% level expected from typical black-hole collisions, for angular momentum up to at least 70% of the maximum, for an initial surface as close as $r = 3m$.

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4 Unsolved problems

I hope the statements below will inspire our future researches.

4.1 Gravitational Wave Physics and related problems

Please refer also Box 1.2 in §1 for numerical issues.

4.1.1 For extraction of physics

- How binary behaves in the last stage of merger?
- What can we learn from waveform from the final phase of binary merger?
- Can we determine equation of state of neutron star?
- Validity of alternative approaches?
- Validity of new approximations?

4.1.2 From numerical relativitic viewpoint

- Physically reasonable initial data?
- Where to start the simulation? How connect from post-Newtonian evolution?
- How can we evolve the system stably?
- How can we achieve precise numerical simulations of coalescence of binary neutron stars and/or black holes?
- How identify black-hole horizons?
- How to treat black hole singularity if it appears?
- How to extract gravitational wave?
- How can we manage the large-scale simulations?

4.2 Conjecture Hunting – reported and unreported issues –

4.2.1 BH Uniqueness Theorems, No-hair Conjecture

- Are colored BHs realistic?
- In higher dim.?
- Stable configuration of Black String?

4.2.2 Cosmic Censorship Conjecture

- Counter-examples?
- Strong version?

4.2.3 Gravitational Collapse and Hoop Conjecture

- Definition of quasi-local mass?
- Validity?
- In higher dim.?

4.2.4 BH Thermodynamics

- Why area, not volume?
- Under dynamical situation?

4.2.5 Dynamical Wormholes

- Topology change in dynamical transition?
- New critical behavior for forming black-hole mass?
- Time-machine? (closed timelike curve? Chronological protection conjecture?)
- Wormhole thermodynamics?

.... etc etc

4.3 Concluding Remark

There are many unsolved problems in general relativity. All realistic discussion requires numerical simulations. Our understanding for numerical procedures are accumulating, mature now. Computational power is also suitable for actual researches.

Are you ready to go conjecture hunting?

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