APCTP Winter School, January 17-18, 2003

Introduction to **Numerical Relativity**

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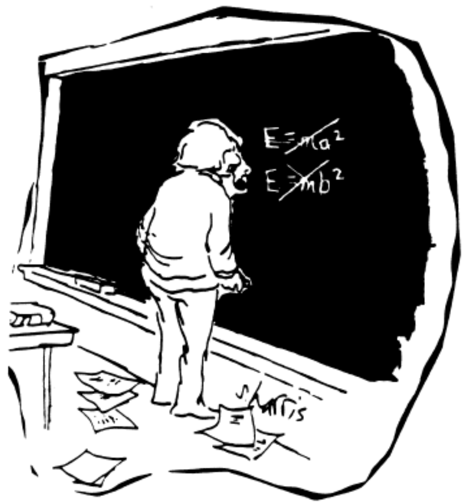
理化学研究所　計算科学技術推進室　真貝寿明

신 카 이　히 사 아 키

1. Subjects for Numerical Relativity
   Why Numerical Relativity?
2. The Standard Approach to Numerical Relativity
   The ADM formulation
3. Alternative Approaches to Numerical Relativity
   etc
4. Unsolved problems
   etc, etc



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Introduction to Numerical Relativity

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1. Subjects for Numerical Relativity
   - Why Numerical Relativity?
   - Overview of Numerical Relativity
   - Gravitational Wave Physics (Why Blackholes/Neutron Stars?)

2. The Standard Approach to Numerical Relativity
   - The ADM formulation
3. Alternative Approaches to Numerical Relativity
   - etc
4. Unsolved problems
   - etc, etc

The Einstein equation

geodesics

spacetime curvature → matter distribution

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein tensor

Energy-Momentum tensor

cosmological constant

Solve for metric
$g_{\mu\nu}$(t,x,y,z)
(10 components)

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu := g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ & g_{xx} & g_{xy} & g_{xz} \\ & & g_{yy} & g_{yz} \\ sym. & & & g_{zz} \end{pmatrix}$$

flat spacetime (Minkowskii spacetime):

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

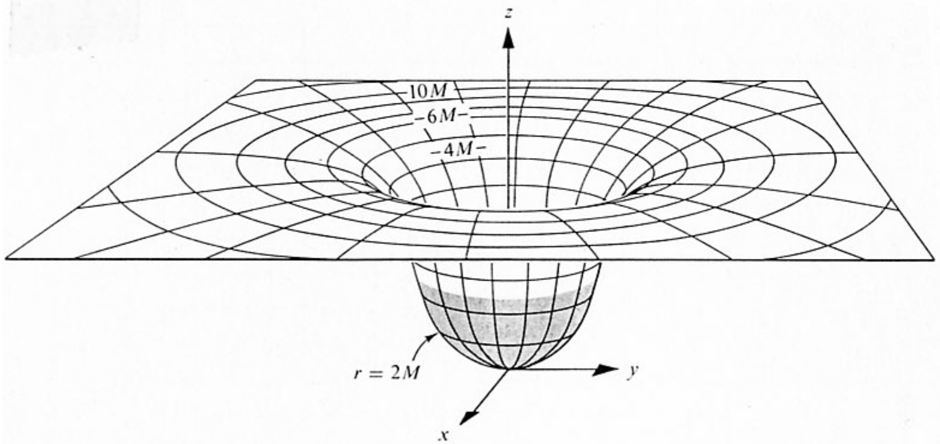


Figure 23.1.
Geometry within (grey) and around (white) a star of radius $R = 2.66M$, schematically displayed. The star is in hydrostatic equilibrium and has zero angular momentum (spherical symmetry). The two-dimensional geometry

$$ds^2 = [1 - 2m(r)/r]^{-1}\, dr^2 + r^2\, d\phi^2$$

of an equatorial slice through the star ($\theta = \pi/2$, $t =$ constant) is represented as embedded in Euclidean 3-space, in such a way that distances between any two nearby points (r, ϕ) and $(r + dr, \phi + d\phi)$ are correctly reproduced. Distances measured off the curved surface have no physical meaning; points off that surface have no physical meaning; and the Euclidean 3-space itself has no physical meaning. Only the curved 2-geometry has meaning. A circle of Schwarzschild coordinate radius r has proper circumference $2\pi r$ (attention limited to equatorial plane of star, $\theta = \pi/2$). Replace this circle by a sphere of proper area $4\pi r^2$, similarly for all the other circles, in order to visualize the entire 3-geometry in and around the star at any chosen moment of Schwarzschild coordinate time t. The factor $[1 - 2m(r)/r]^{-1}$ develops no singularity as r decreases within $r = 2M$, because $m(r)$ decreases sufficiently fast with decreasing r.

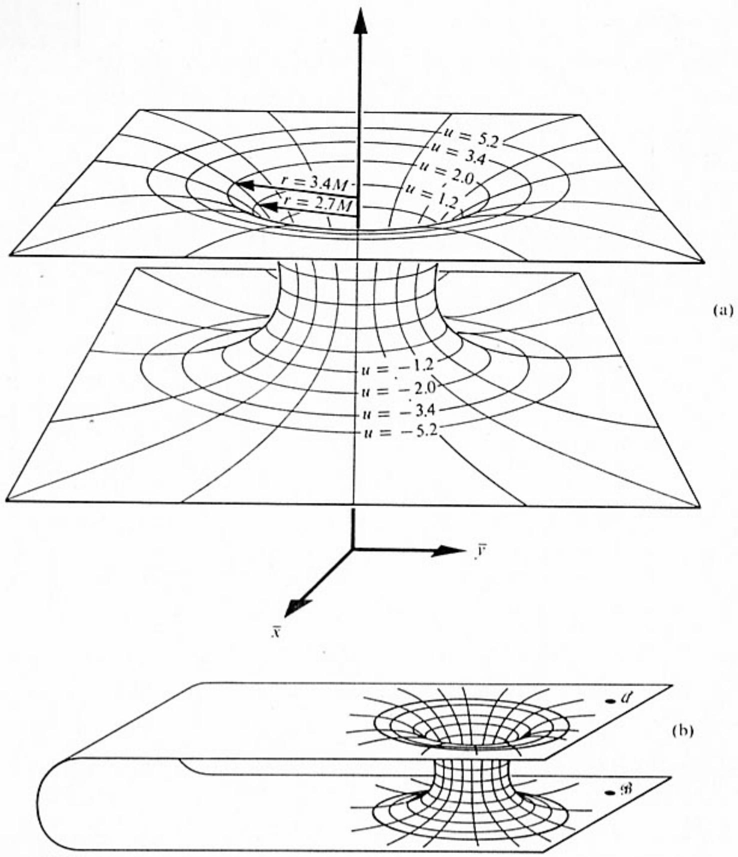


Figure 31.5.
(a) The Schwarzschild space geometry at the "moment of time" $t = v = 0$, with one degree of rotational freedom suppressed ($\theta = \pi/2$). To restore that rotational freedom and obtain the full Schwarzschild 3-geometry, one mentally replaces the circles of constant $\bar{r} = (\bar{x}^2 + \bar{y}^2)^{1/2}$ with spherical surfaces of area $4\pi\bar{r}^2$. Note that the resultant 3-geometry becomes flat (Euclidean) far from the throat of the bridge in both directions (both "universes").

(b) An embedding of the Schwarzschild space geometry at "time" $t = v = 0$, which is geometrically identical to the embedding (a), but which is topologically different. Einstein's field equations fix the local geometry of spacetime, but they do not fix its topology; see the discussion at end of Box 27.2. Here the Schwarzschild "wormhole" connects two distant regions of a single, asymptotically flat universe. For a discussion of issues of causality associated with this choice of topology, see Fuller and Wheeler (1962).

The Einstein equation:

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1)$$

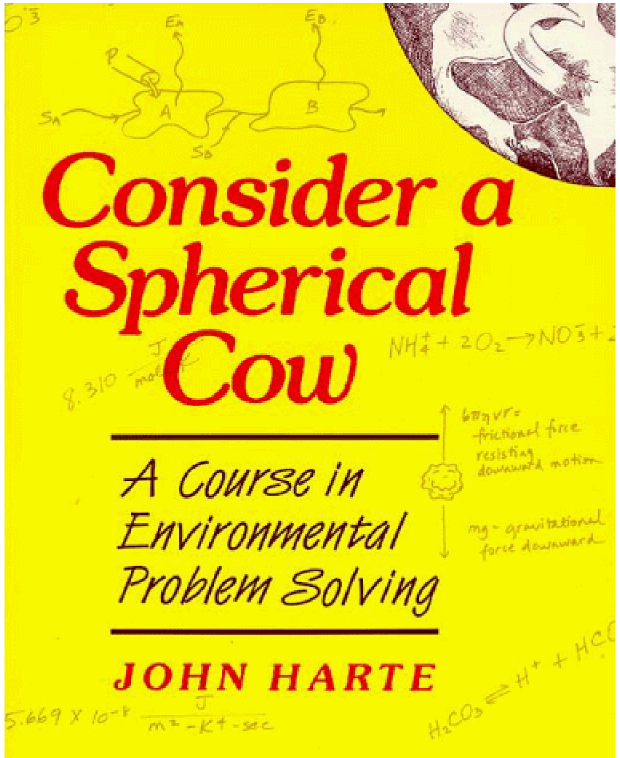
What are the difficulties? (# 1)

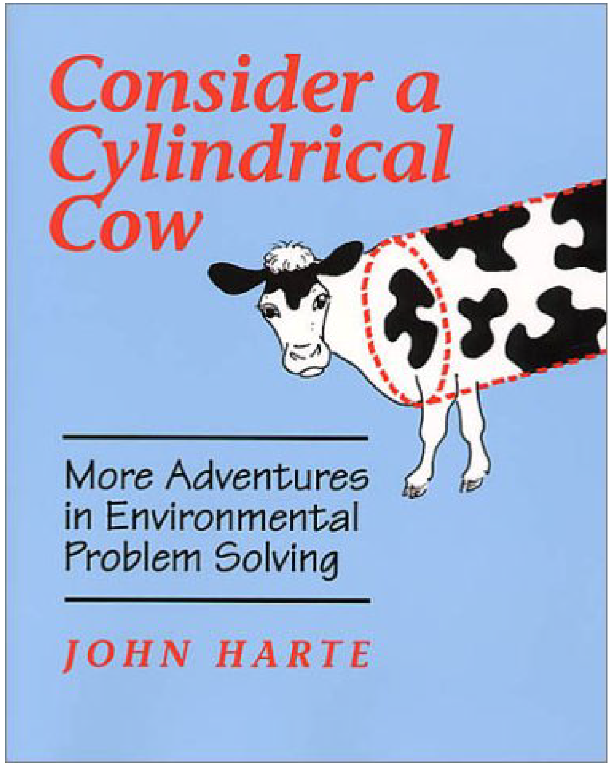
- for 10-component metric, highly nonlinear partial differential equations.
- completely free to choose cooordinates, gauge conditions, and even for decomposition of the space-time.
- mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- has singularity in its nature.

How to solve it?

"First, we assume a spherical cow..."

There is an old joke about a theoretical physicist who was charged with figuring out how to increase the milk production of cows. Although many farmers, biologists, and psychologists had tried and failed to solve the problem before him, the physicist had no trouble coming up with a solution on the spot. "First," he began "we assume a spherical cow..."





How to solve the Einstein eq?

- find exact solutions
  - assume symmetry in space-time, and decomposition of space-time
    spherically symmetric, cylindrical symmetric, ...
  - assume simple situation and matter
    time-dependency, homogeneity, algebraic speciality, ...

We know many exact solutions ($O(100)$) by this "Spherical Cow" approach.

- approximations
  - weak-field limit, linearization, perturbation, ...

We know correct prediction in the solar-system, binary neutron stars, ...
We know post-Newtonian behavior, first-order correction, BH stability, ...

Black-holes, Cosmology, weak-field limit, ... but more interesting issues?

Why don't we solve it using computers?

- dynamical behavior, no symmetry in space, ...
- strong gravitational field, gravitational wave! ...

Numerical Relativity **Box 1.1**

= Solve the Einstein equations numerically.

= Necessary for unveiling the nature of strong gravity.

For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

The most robust way to study the strong gravitational field. Great.

1.2 Overview of Numerical Relativity

Several milestones of NR
New proposals, developments, physical results.

1960s	Hahn-Lindquist	2 BH head-on collision	AnaPhys29(1964)304
	May-White	spherical grav. collapse	PR141(1966)1232
1970s	ÓMurchadha-York	conformal approach to initial data	PRD10(1974)428
	Smarr	3+1 formulation	PhD thesis (1975)
	Smarr-Cades-DeWitt-Eppley	2 BH head-on collision	PRD14(1976)2443
	Smarr-York	gauge conditions	PRD17(1978)2529
	ed. by L.Smarr	"Sources of Grav. Radiation"	Cambridge(1979)
1980s	Nakamura-Maeda-Miyama-Sasaki	axisym. grav. collapse	PTP63(1980)1229
	Miyama	axisym. GW collapse	PTP65(1981)894
	Bardeen-Piran	axisym. grav. collapse	PhysRep96(1983)205
	Stark-Piran	axisym. grav. collapse	unpublished
1990	Shapiro-Teukolsky	naked singularity formation	PRL66(1991)994
	Oohara-Nakamura	3D post-Newtonian NS coalesence	PTP88(1992)307
	Seidel-Suen	BH excision technique	PRL69(1992)1845
	Choptuik	critical behaviour	PRL70(1993)9
	NCSA group	axisym. 2 BH head-on collision	PRL71(1993)2851
	Cook et al	2 BH initial data	PRD47(1993)1471
	Shibata-Nakao-Nakamura	BransDicke GW collapse	PRD50(1994)7304
	Price-Pullin	close limit approach	PRL72(1994)3297
1995	NCSA group	event horizon finder	PRL74(1995)630
	NCSA group	hyperbolic formulation	PRL75(1995)600
	Anninos *et al*	close limit vs full numerical	PRD52(1995)4462
	Scheel-Shapiro-Teukolsky	BransDicke grav. collapse	PRD51(1995)4208
	Shibata-Nakamura	3D grav. wave collapse	PRD52(1995)5428
	Gunnersen-Shinkai-Maeda	ADM to NP	CQG12(1995)133
	Wilson-Mathews	NS binary inspiral, prior collapse?	PRL75(1995)4161
	Pittsburgh group	Cauchy-characteristic approach	PRD54(1996)6153
	Brandt-Brügmann	BH puncture data	PRL78(1997)3606
	Illinois group	synchronized NS binary initial data	PRL79(1997)1182
	Shibata-Baumgarte-Shapiro	2 NS inspiral, PN to GR	PRD58(1998)023002
	BH Grand Challenge Alliance	characteristic matching	PRL80(1998)3915
	Baumgarte-Shapiro	Shibata-Nakamura formulation	PRD59(1998)024007
	Brady-Creighton-Thorne	intermediate binary BH	PRD58(1998)061501
	Meudon group	irrotational NS binary initial data	PRL82(1999)892
	Shibata	2 NS inspiral coalesence	PRD60(1999)104052
	York	conformal thin-sandwich formulation	PRL82(1999)1350
	Brodbeck *et al*	λ-system	JMathPhys40(1999)909
2000	Kidder-Finn	BH, Spectral methods	PRD62(2000)084026
	Shinkai-Yoneda	planar GW, Ashtekar variables	CQG17(2000)4729
	AEI group	full numerical to close limit	CQG17(2000)L149
	AEI group	2 BH grazing collision	PRL87(2001)271103
	Shibata-Uryu	2 NS inspiral coalesence	PTP107(2002)265
	Shinkai-Yoneda	adjusted ADM systems	CQG19(2002)1027
	Meudon group	irrotational BH binary initial data	PRD65(2002)044020
	PennState group	isolated horizon	gr-qc/0206008

Numerical Relativity – open issues **Box 1.2**

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

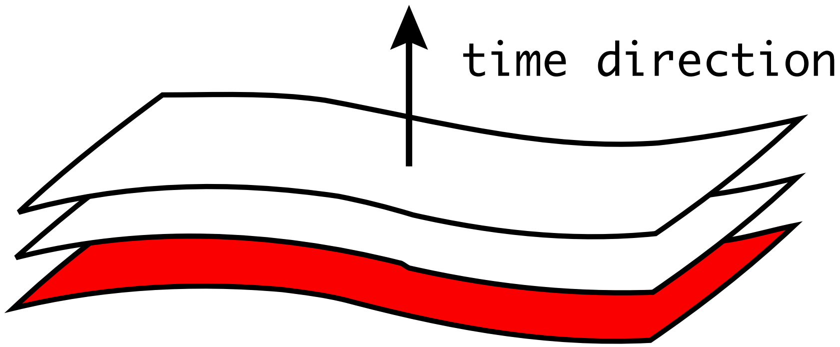
Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? $\Rightarrow$ see e.g. gr-qc/0209111 Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

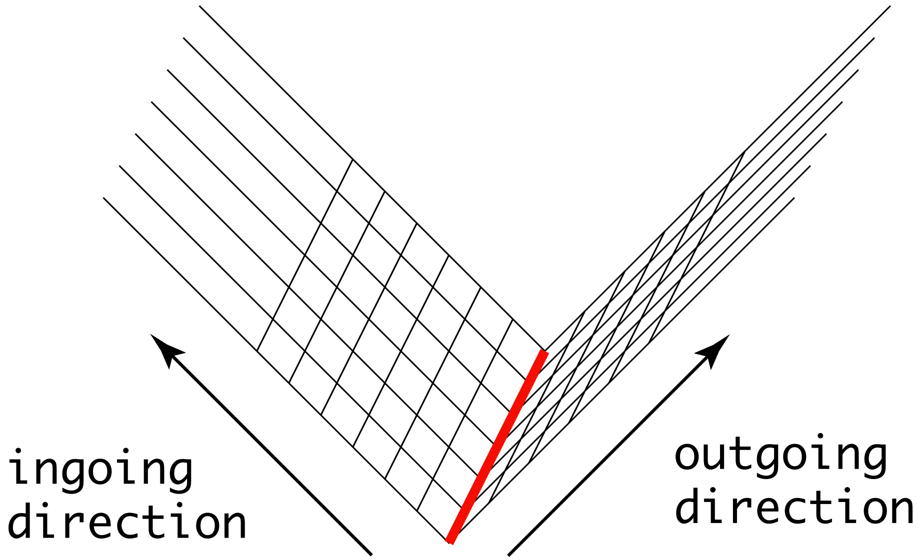
First Question: How to foliate space-time?

Cauchy approach or ADM 3+1 formulation



Σ: Initial 3-dimensional Surface

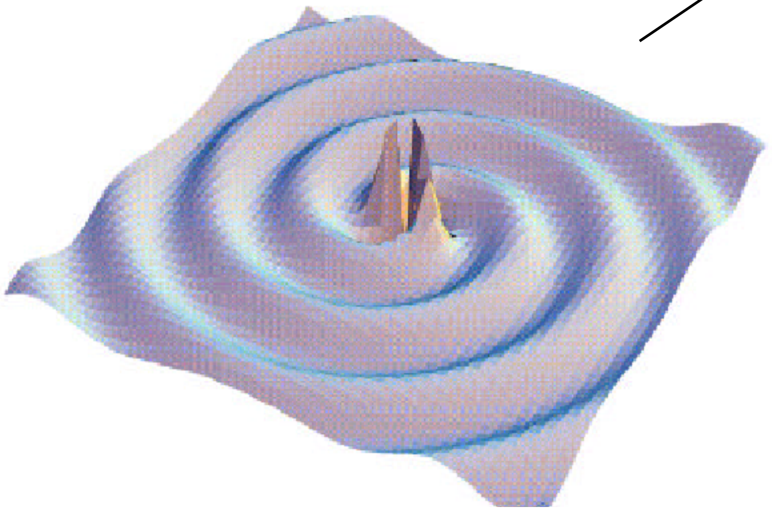
Characteristic approach (if null, dual-null 2+2 formulation)



S: Initial 2-dimensional Surface

Toward Direct Detection of Gravitational Wave

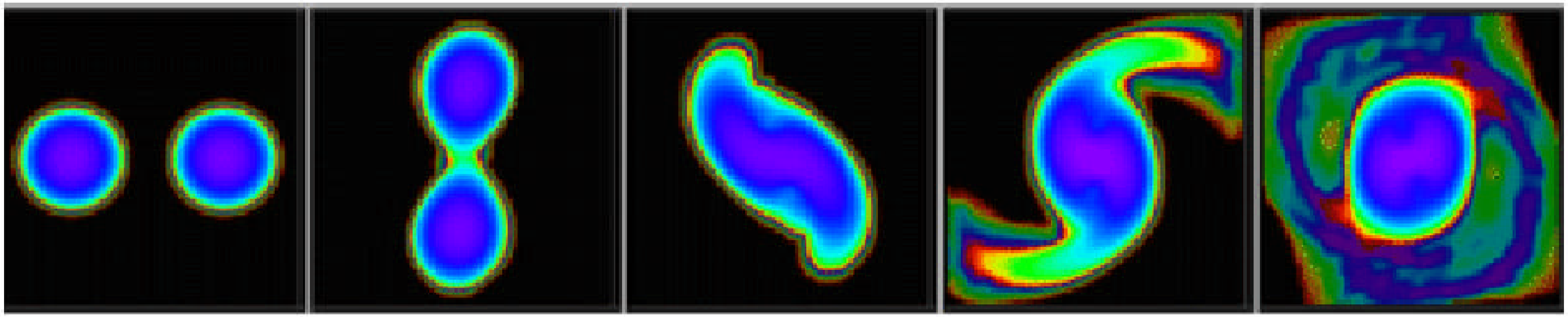
GW is produced by coalescing Black-holes and/or Neutron Stars



Laser Interferometers

JAPAN	300m	2000-
USA	4Km/2Km	2002-
GermanyUK	600m	2002-
ItalyFrance	3Km	2003-

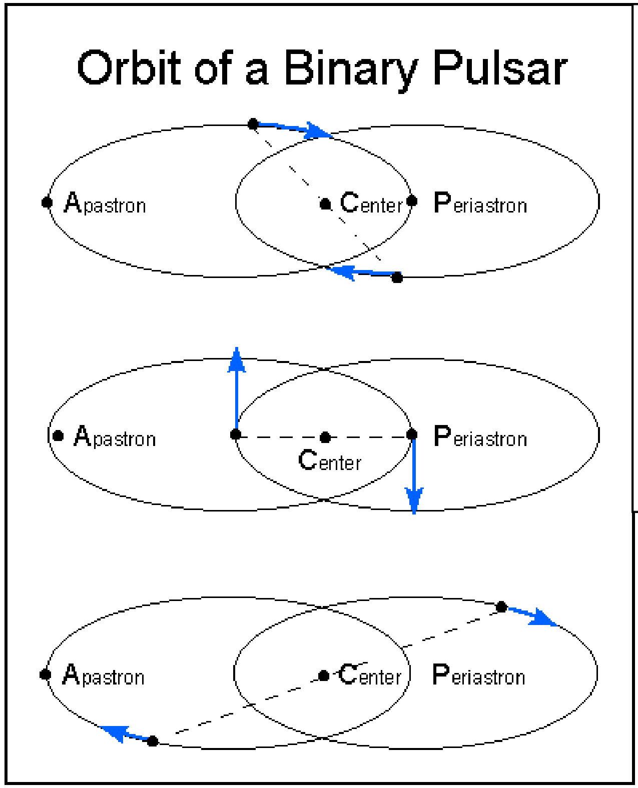
- Neutron star – neutron star (Centrella et al.)

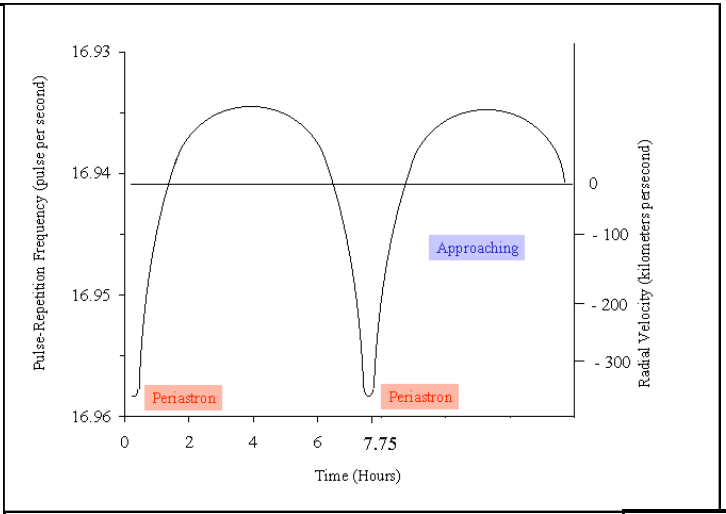


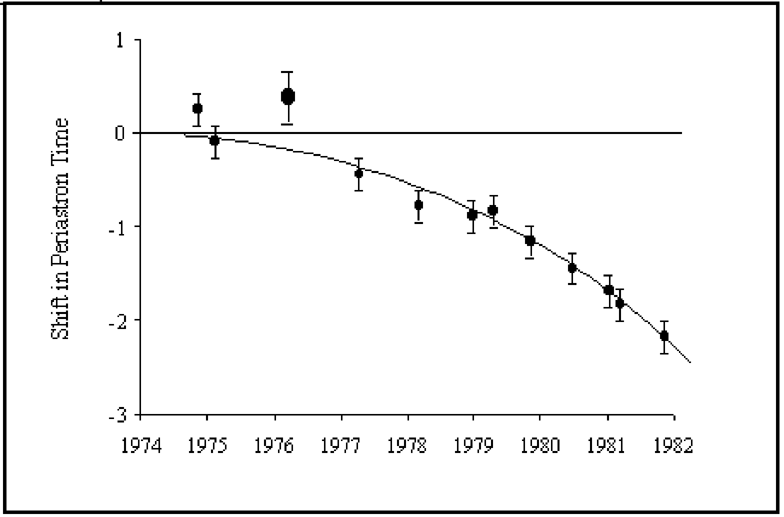
Binary Pulser PSR 1913+16 (Neutron Star *2)

Indirect Proof of Gravitaional Wave emission

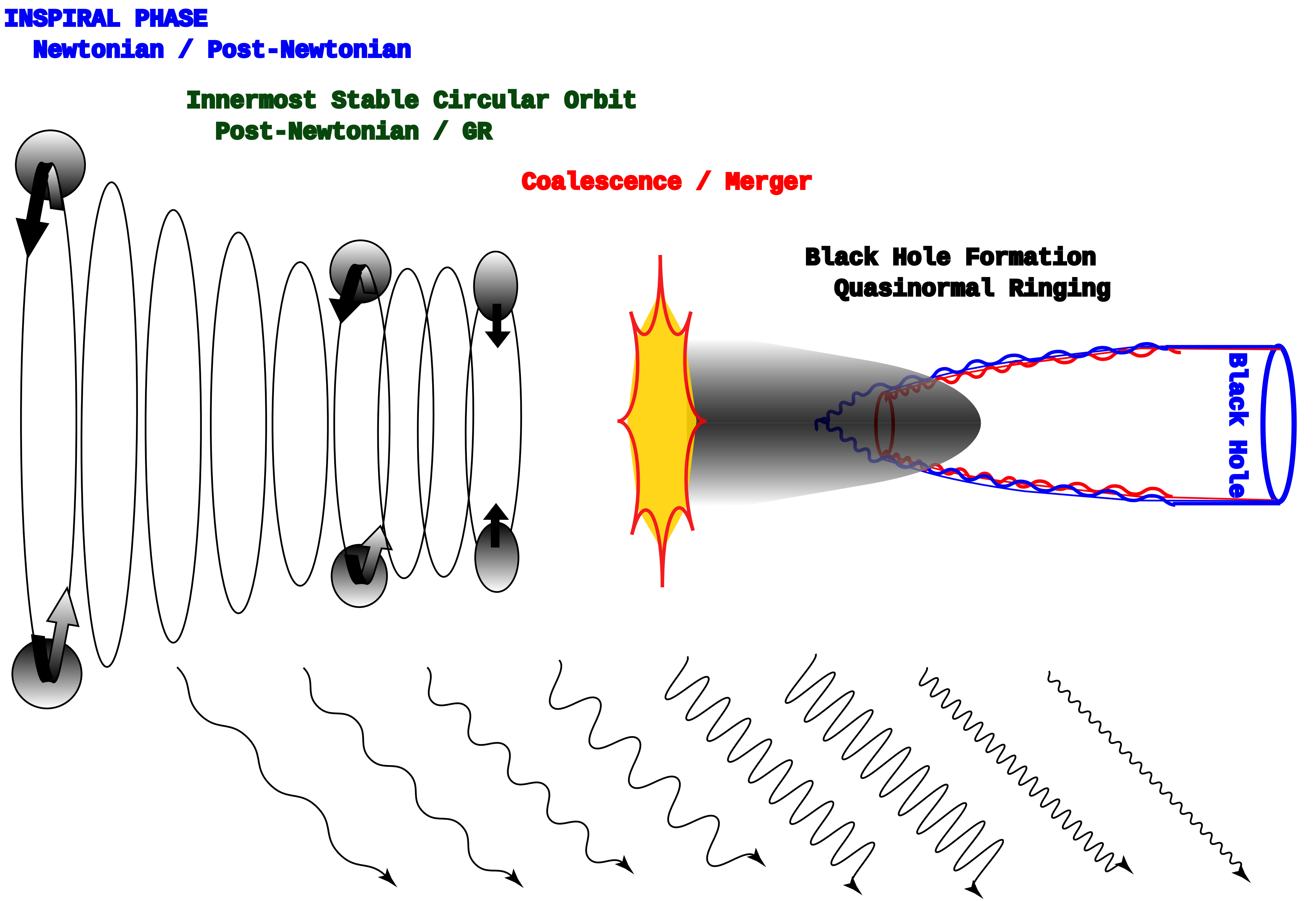
1974, R. Hulse and J. Taylor found by radio ==> 1993 Nobel Prize

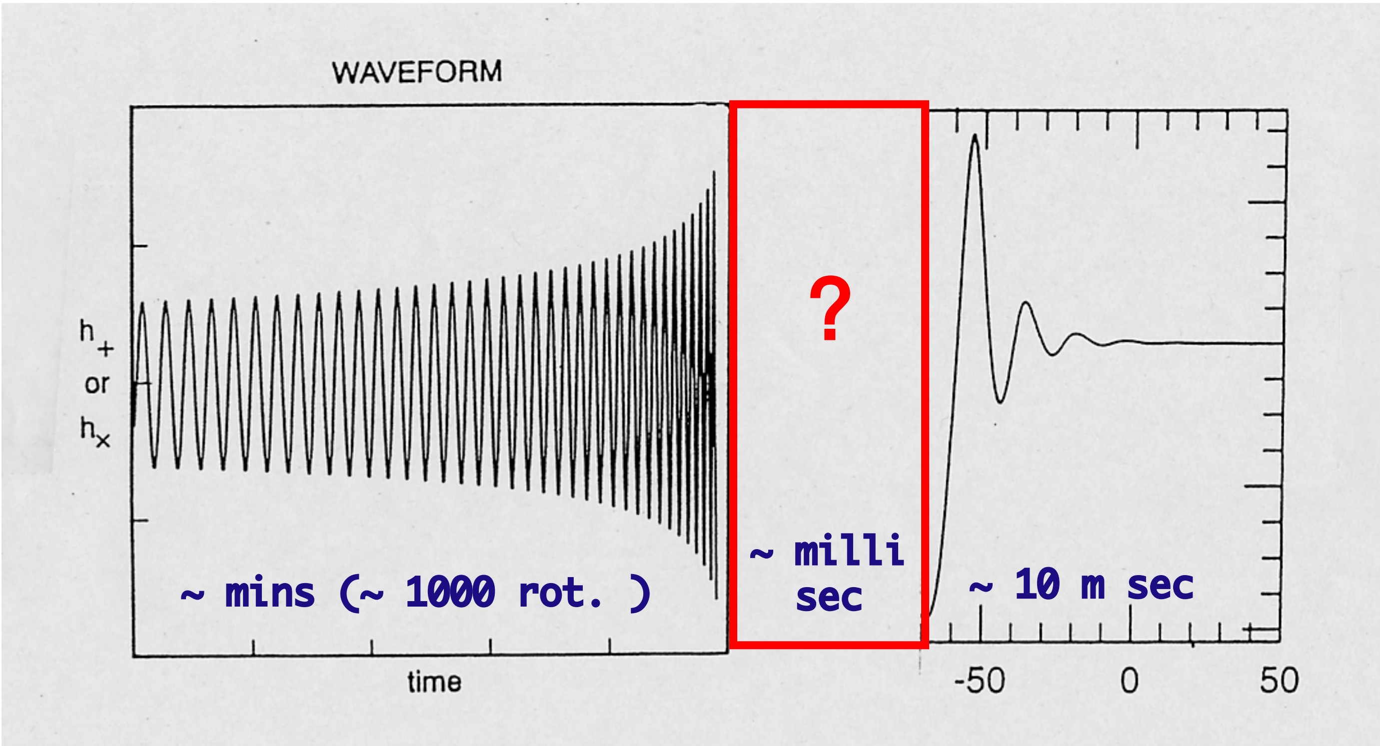






http://astrosun.tn.cornell.edu/courses/astro201/psr1913.htm


INSPIRAL PHASE
Newtonian / Post-Newtonian
Innermost Stable Circular Orbit
Post-Newtonian / GR
Coalescence / Merger
Black Hole Formation
Quasinormal Ringing
Black Hole



INSPIRAL → COALESCE → BLACKHOLE FORMATION

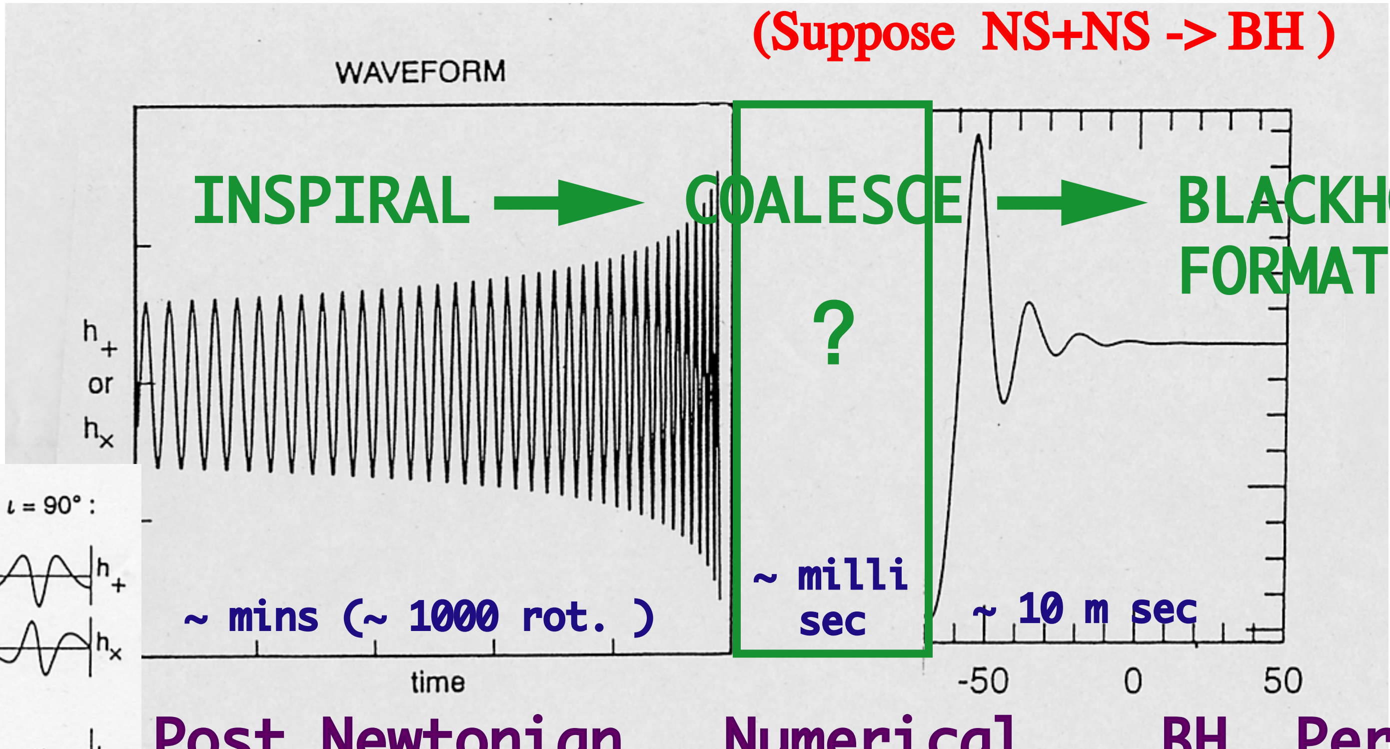
Innermost Stable Circular Orbit?

Post Newtonian Approx.

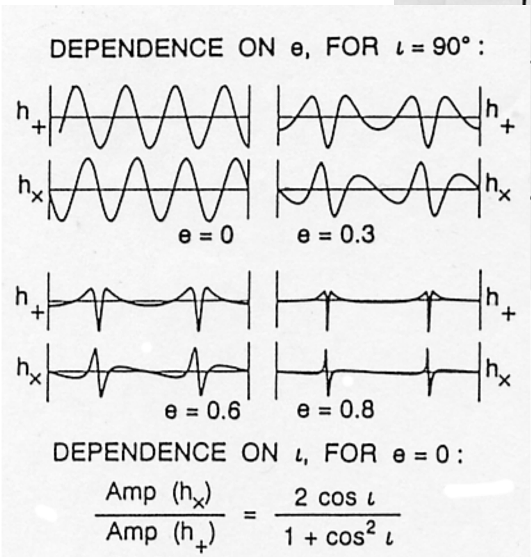
→ Numerical Relativity

→ BH. Perturbation

What can we learn from gravitational waveform?



Post Newtonian Approx. | Numerical Relativity | BH. Perturbation



ISCO freq => EoS of NS,
waveform => Formation of BH or NS,
BH mass,
BH angular momentum,...

"chirps" df/dt => chirp mass, $M_c = (M_1 M_2)^{3/5} / (M_1+M_2)^{1/5}$
amplitude up => M_c, distance
amplitude $h_+/h_\times$ => inclination
waveform => eccentricity
moduration => spin, ...

statistics => cosmological parameters

Requirements for Numerical Relativity

- Where to start the simulation?
- How to construct physically reasonable initial data?
- How can we evolve the system stably?
- How to treat black-hole singularity if it appears?
- How to extract gravitational wave?
- How can we manage the large-scale simulations?