APCTP Winter School, January 17-18, 2003

Introduction to Numerical Relativity

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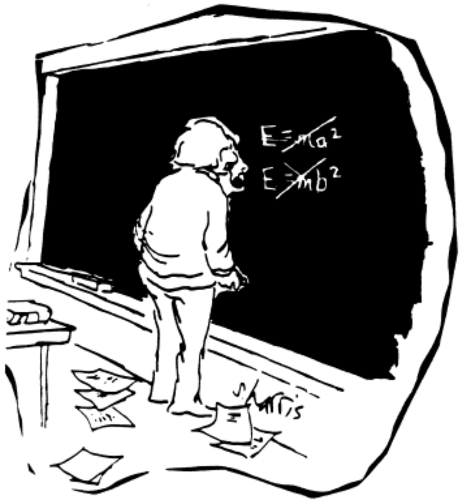
1. **Subjects for Numerical Relativity**
   - Why Numerical Relativity?
2. **The Standard Approach to Numerical Relativity**
   - The ADM formulation
   - How to construct initial data 1: Conformal approach
   - How to construct initial data 2: Thin-Sandwich approach
   - How to choose gauge conditions: slicing conditions
   - How to evolve the system: formulation problems
3. **Alternative Approaches to Numerical Relativity**
   - etc
4. **Unsolved problems**
   - etc, etc



Procedure of the Standard Numerical Relativity

■ 3+1 (ADM) formulation

■ Preparation of the Initial Data

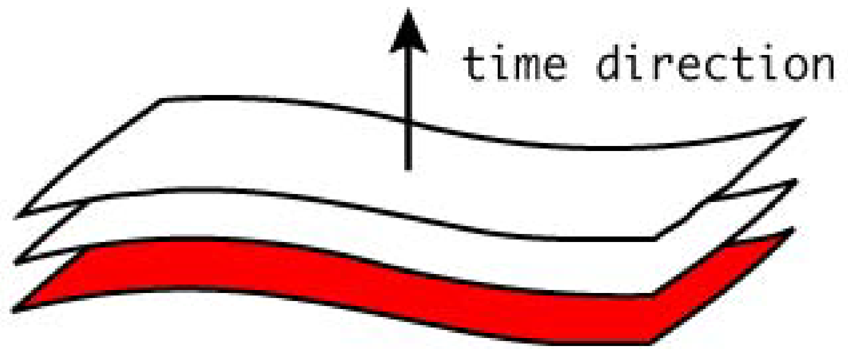
- ◆ Assume the background metric
- ◆ Solve the constraint equations

■ Time Evolution

do time=1, time_end

- ◆ Specify the slicing condition
- ◆ Evolve the variables
- ◆ Check the accuracy
- ◆ Extract physical quantities

end do



Σ: Initial 3-dimensional Surface

The 3+1 decomposition of space-time, The ADM formulation

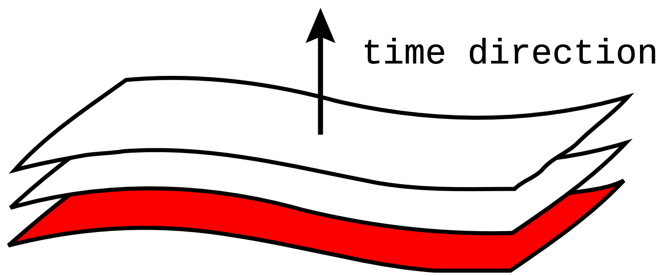
[1] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. by L.Witten, (Wiley, New York, 1962).

[2] J.W. York, Jr. in *Sources of Gravitational Radiation*, (Cambridge, 1979)

Dynamics of Space-time = Foliation of Hypersurface

- Evolution of t =const. hypersurface $\Sigma(t)$.

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3) \\
\text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij}dx^i dx^j, \quad (i, j = 1, 2, 3)
\end{aligned}
$$



- The unit normal vector of the slices, n^μ.

$$
\begin{aligned}
n_\mu &= (-\alpha, 0, 0, 0) \\
n^\mu &= g^{\mu\nu}n_\nu = (1/\alpha, -\beta^i/\alpha)
\end{aligned}
$$

- The lapse function, α. The shift vector, β^i.

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

The decomposed metric:

$$
\begin{aligned}
ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\
&= (-\alpha^2 + \beta_l \beta^l) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j
\end{aligned}
$$

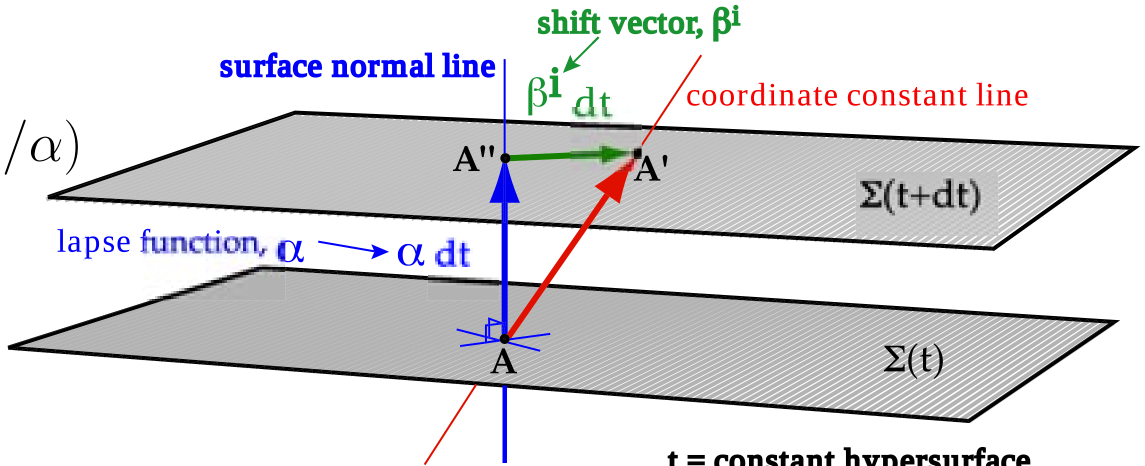
$$
g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l \beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}
$$

where α and β_j are defined as $\alpha \equiv 1/\sqrt{-g^{00}}, \quad \beta_j \equiv g_{0j}$.

- The unit normal vector of the slices, n^μ.

$$
\begin{aligned}
n_\mu &= (-\alpha, 0, 0, 0) \\
n^\mu &= g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha)
\end{aligned}
$$

- The lapse function, α.
- The shift vector, β^i.



Projection of the Einstein equation:

- Projection operator (or intrinsic 3-metric) to $\Sigma(t)$,

$$\begin{aligned}\gamma_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu \\ \gamma^\mu_\nu &= \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu\end{aligned}$$

- Define the extrinsic curvature K_{ij},

$$\begin{aligned}K_{ij} &\equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} \\ &= -(\delta^\mu_i + n^\mu n_i)(\delta^\nu_j + n^\nu n_j) n_{\mu;\nu} \\ &= -n_{i;j} \\ &= \Gamma^\alpha_{ij} n_\alpha = \cdots = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + \beta_{i|j} + \beta_{j|i}\right).\end{aligned}$$

- Projection of the Einstein equation:

$$\begin{aligned}G_{\mu\nu}\, n^\mu\, n^\nu &= 8\pi G\, T_{\mu\nu}\, n^\mu\, n^\nu \equiv 8\pi \rho_H &&\Rightarrow \text{ the Hamiltonian constraint eq.} \\ G_{\mu\nu}\, n^\mu \perp^\nu_i &= 8\pi G\, T_{\mu\nu}\, n^\mu \perp^\nu_i \equiv -8\pi J_i &&\Rightarrow \text{ the momentum constraint eqs.} \\ G_{\mu\nu} \perp^\mu_i \perp^\nu_j &= 8\pi G\, T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv 8\pi S_{ij} &&\Rightarrow \text{ the evolution eqs.}\end{aligned}$$

The Standard ADM formulation (aka York 1978):

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
\partial_t K_{ij} &= \alpha\ {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha \\
&\quad +(D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} \\
&\quad -8\pi G\alpha\{S_{ij} + (1/2)\gamma_{ij}(\rho_H - \mathrm{tr}S)\},
\end{aligned}
$$

where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$
\begin{aligned}
&\text{Hamiltonian constr.} \quad \mathcal{H}^{ADM} := {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, \\
&\text{momentum constr.} \quad \mathcal{M}_i^{ADM} := D_j K^j{}_i - D_i K \approx 0,
\end{aligned}
$$

where ${}^{(3)}R = {}^{(3)}R^i{}_i$.

Original ADM The original construction by ADM uses the pair of (h_{ij}, π^{ij}).

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} = \frac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2\frac{N}{\sqrt{h}}(\pi_{ij} - \frac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\frac{\delta \mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N(^{(3)}R^{ij} - \frac{1}{2}{}^{(3)}Rh^{ij}) + \frac{1}{2}\frac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2) - 2\frac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \frac{1}{2}\pi\pi^{ij}) \\ \qquad + \sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

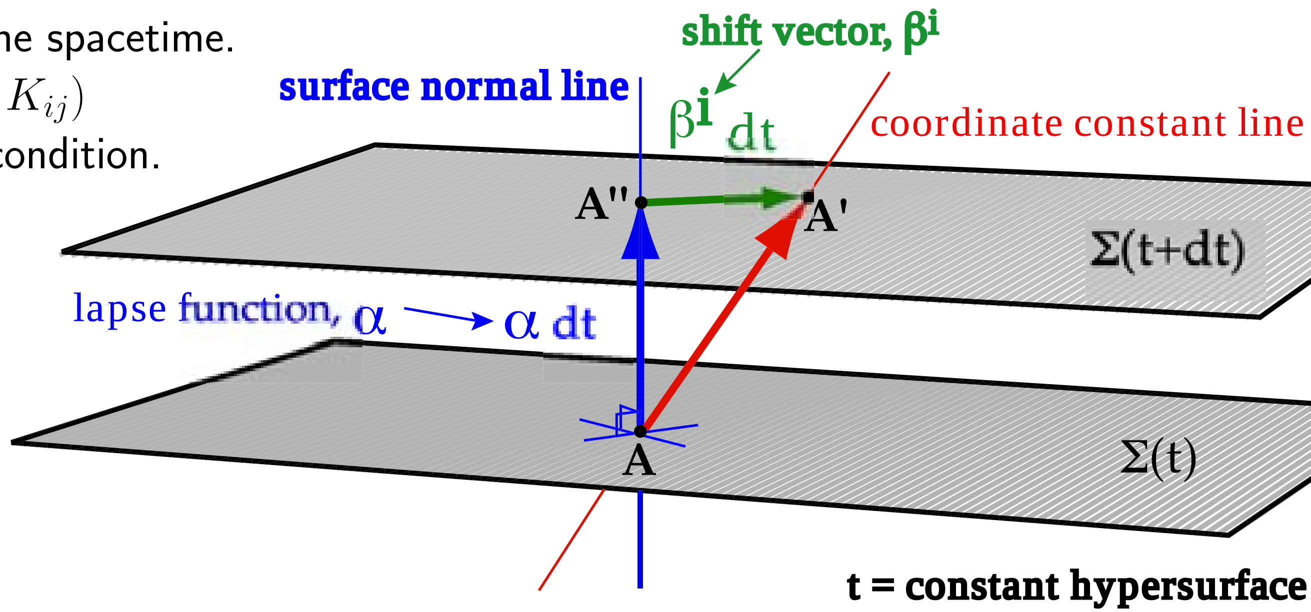
Standard ADM (by York) NRists refer ADM as the one by York with a pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} =$ rot $\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -$rot $\mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

The Constraint Propagations of the Standard ADM:

$$
\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}
$$

From these equations, we know that

if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

But this is not true in numerics....

Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- **Preparation of the Initial Data**
  - Assume the background metric
  - **Solve the constraint equations**

Need to solve elliptic PDEs
-- Conformal approach
-- Thin-Sandwich approach

- Time Evolution

```
do time=1, time_end
    ◆ Specify the slicing condition
    ◆ Evolve the variables
    ◆ Check the accuracy
    ◆ Extract physical quantities
end do
```

Initial Data Construction Problem

Prepare all metric and matter components by solving the two constraints:

- The Hamiltonian constraint equation

$$ {}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho + 2\Lambda \quad (1) $$

- The momentum constraint equations

$$ D_j(K^{ij} - \gamma^{ij}\mathrm{tr}K) = \kappa J^i \quad (2) $$

We have 12 variables (γ_{ij}, K_{ij}) to fix, but only 4 constraints. ... How?

1st method **Conformal Approach – York-ÓMurchadha (1974)**

N.ÓMurchadha and J.W.York Jr., Phys. Rev. **D 10**, 428 (1974)

The key idea is **solution $\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}$ trial metric.**

- the decomposition of K_{ij},

$$K_{ij} \Rightarrow \begin{cases} \mathrm{tr}K = \gamma^{ij} K_{ij} & \text{trace part} \\ A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}K & \text{trace-free part} \end{cases}$$

- conformal transformations:

$$\begin{aligned} \gamma_{ij} &= \psi^4 \hat{\gamma}_{ij}, & \gamma^{ij} &= \psi^{-4} \hat{\gamma}^{ij}, \\ A^{ij} &= \psi^{-10} \hat{A}^{ij}, & A_{ij} &= \psi^{-2} \hat{A}_{ij}, \\ \rho &= \psi^{-n} \hat{\rho}, & J^i &= \psi^{-10} \hat{J}^i, \end{aligned}$$

- we suppose

$$\mathrm{tr}K = \hat{\mathrm{tr}}\hat{K}, \quad \mathrm{tr}A = \hat{\mathrm{tr}}\hat{A} = 0.$$

- we then get

$$\Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk} + 2\psi^{-1}(\delta^i{}_j \hat{D}_k\psi + \delta^i{}_k \hat{D}_j\psi - \hat{\gamma}_{jk}\hat{\gamma}^{im}\hat{D}_m\psi),$$
$$R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi.$$

where $\hat{\Delta} = \hat{\gamma}^{jk}\hat{D}_j\hat{D}_k$ and $\hat{R} = R(\hat{\gamma})$, and also $D_jA^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}$.

- decompose $\hat{A}^{ij}$ to transverse-traceless (TT) part and longitudinal part:

$$\hat{A}^{ij} = \underbrace{\hat{A}^{ij}_{TT}}_{\text{divergence-free}} + \underbrace{(\hat{\mathbb{l}}W)^{ij}}_{\text{longitudinal}},$$

$$\hat{D}_j\hat{A}^{ij}_{TT} = 0, \quad \hat{\text{tr}}\hat{A}_{TT} = 0, \quad \text{and} \quad (\hat{\mathbb{l}}W)^{ij} = \hat{D}^iW^j + \hat{D}^jW^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_kW^k.$$

- Using these terms, we can write

$$\hat{D}_j\hat{A}^{ij} = \hat{D}_j(\hat{\mathbb{l}}W)^{ij} \equiv (\hat{\Delta}_{\mathbb{l}}W)^i = (\hat{\Delta}W)^i + (1/3)\hat{D}^i(\hat{D}_jW^j) + \hat{R}^i{}_jW^j.$$

With above transformation, the two constraints becomes

- The Hamiltonian constraint equation

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\text{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n}$$

- The momentum constraint equations

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_kW^k + \hat{R}^i{}_kW^k = \frac{2}{3}\psi^6\hat{D}^i\text{tr}K + 8\pi G\hat{J}^i$$

Conformal approach (York-ÓMurchadha, 1974)

One way to set up the metric and matter components $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints:

1. Specify metric components $\hat{\gamma}_{ij}$, tr$K$, $\hat{A}^{TT}_{ij}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.

2. Solve the next equations for $(\psi,\ W^i)$

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [(2/3)(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \quad (1)$$

$$\hat{\Delta}W^i + (1/3)\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = (2/3)\psi^6\hat{D}^i \mathrm{tr}K + 8\pi G\hat{J}^i \quad (2)$$

   where $\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{D}^i W^j + \hat{D}^j W^i - (2/3)\hat{\gamma}^{ij}\hat{D}_k W^k$.

3. Apply the inverse conformal transformation and get the metric and matter components $\gamma_{ij},\ K_{ij},\ \rho,\ J^i$ in the physical frame:

$$\begin{aligned}
\gamma_{ij} &= \psi^4\hat{\gamma}_{ij}, \\
K_{ij} &= \psi^{-2}[\hat{A}^{TT}_{ij} + (\hat{\mathbf{l}}W)_{ij}] + (1/3)\psi^4\hat{\gamma}_{ij}\mathrm{tr}K, \\
\rho &= \psi^{-n}\hat{\rho}, \\
J^i &= \psi^{-10}\hat{J}^i
\end{aligned}$$

Comments

- Using the idea of conformal rescaling, we have a way to fix 12 components of (γ_{ij}, K_{ij}) that satisfy 4 constraints.
- The Hamiltonian constraint, (3), is a non-linear elliptic equation for ψ, so that we have to solve it by an iterative method.
- The momentum constraints, (3), are PDEs for W^i and coupled with (3). If we assume tr$K = 0$, then two constraints are decoupled. Normally people assume $\boxed{\mathrm{tr}K = 0}$ (maximal slicing condition) or $(\mathrm{tr}K)$ =const. (constant mean curvature slicing) for this purpose.
- For simplicity, people assume the background metric $\hat{\gamma}_{ij}$ is conformally flat $\boxed{\hat{\gamma}_{ij} = \delta_{ij}}$. The physical appropriateness of conformal flatness is often debatable.
- Two freedom of $\hat{A}^{TT}_{ij}$ corresponds to the one of gravitational wave. However, there have been no systematic discussion how to specify them, except applying tensor harmonics in a linearized situation.

Numerical procedures – Several tips

Solving the Hamiltonian constraint

$$8\hat{\Delta}\psi = \hat{R}\,\psi - \hat{K}_{ij}^{TF}\hat{K}^{ij}_{TF}\,\psi^{-7} + \frac{2}{3}\hat{K}^2\,\psi^5 - 16\pi G\,\hat{\rho}\,\psi^{5-n}$$

1. Solve the non-linear equation directly.
2. Solve the linearized equation $\psi = \psi_0 + \delta\psi$ iteratively

$$\begin{aligned} 8\hat{\Delta}\psi &= E\,\psi + F\,\psi^{-7} + G\,\psi^5 + H\,\psi^{-3} + I\,\psi^{-1} \\ &= [E - 7F\psi_0^{-8} + 5G\psi_0^4 - 3H\psi_0^{-4} - 2I\psi_0^{-2}]\,\psi + [8F\psi_0^{-7} - 4G\psi_0^5 + 4H\psi_0^{-3} + 2I\psi_0^{-1}] \end{aligned}$$

Under an appropriate boundary condition, such as Robin BC $\psi = 1 + const./r$, or Dirichlet BC $\psi = 1 + M_{total}/2r$.

Solving the momentum constraints

$$(\Delta W)^i + \frac{1}{3}D^i D_j W^j + R^i{}_j\, W^j - \frac{2}{3}\,\psi^6\,\hat{D}^i K = 8\pi G\,\hat{J}^i$$

1. Solve the non-linear equations directly
2. Bowen's method for conformally flat case [GRG14(1982)1183]
   Under the ($\nabla^i K = 0$) condition,

$$\Delta W^i + \frac{1}{3}\nabla^i \nabla_j W^j = 8\pi S^i.$$

   By introducing a decomposition of W^i into vector and gradient terms $W^i = V^i - \frac{1}{4}\nabla^i \theta$,

$$\begin{aligned}\Delta V^i &= 8\pi S^i,\\ \Delta\theta &= \nabla_i V^i,\end{aligned}$$

If the source is of finite extent, then the the asymptotic behavior of V^i and θ are given by

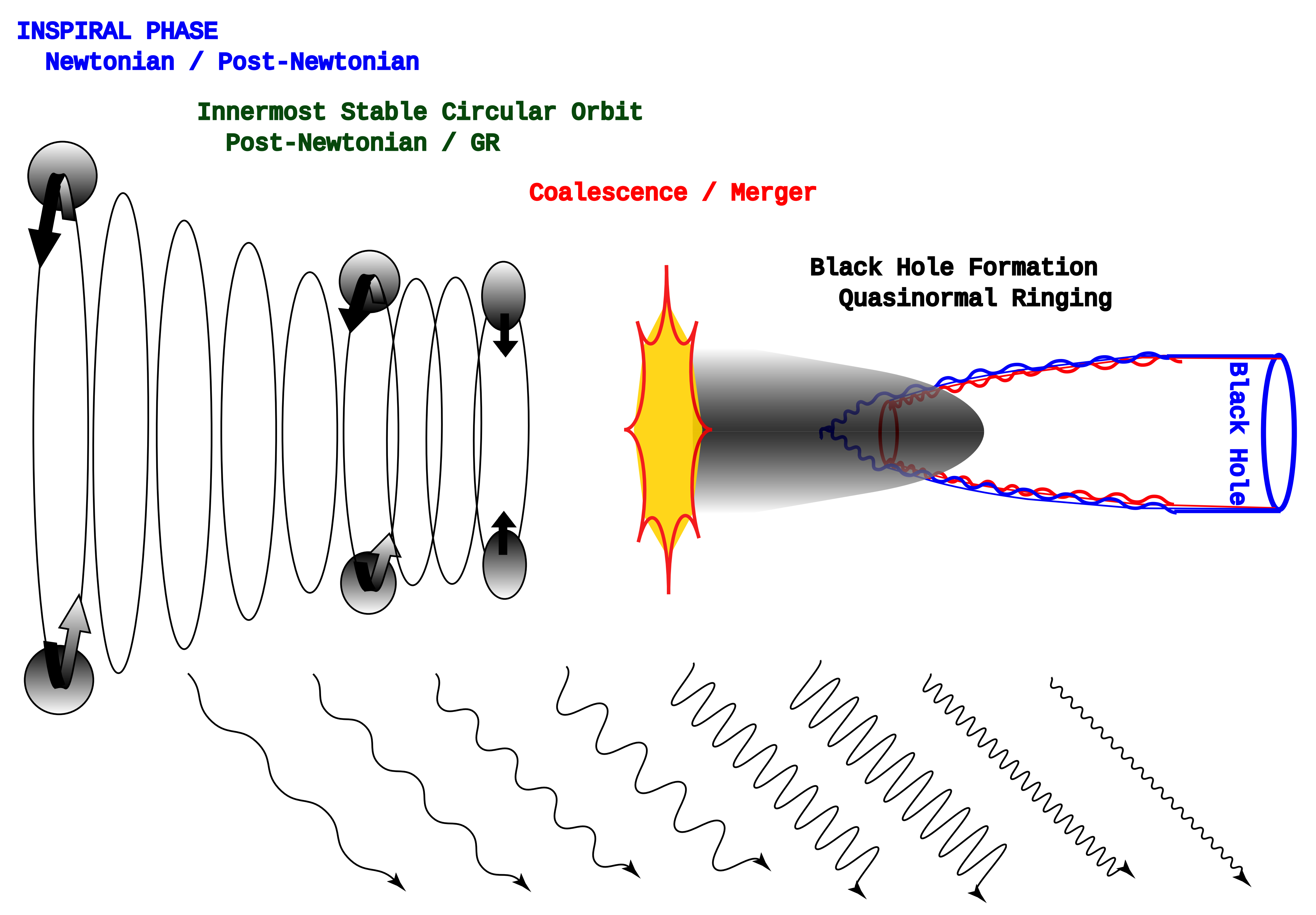
$$\begin{aligned} V^i &= -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}},\\ \theta &= -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} + \sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}} + \sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}} \end{aligned}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles Q and M are defined as

$$\begin{aligned} Q^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int S^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV,\\ M^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int r^2 S^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV, \end{aligned}$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)}$$


INSPIRAL PHASE
Newtonian / Post-Newtonian
Innermost Stable Circular Orbit
Post-Newtonian / GR
Coalescence / Merger
Black Hole Formation
Quasinormal Ringing
Black Hole

1.2 Overview of Numerical Relativity

Several milestones of NR
New proposals, developments, physical results.

1960s	Hahn-Lindquist	2 BH head-on collision	AnaPhys29(1964)304
	May-White	spherical grav. collapse	PR141(1966)1232
1970s	ÓMurchadha-York	conformal approach to initial data	PRD10(1974)428
	Smarr	3+1 formulation	PhD thesis (1975)
	Smarr-Cades-DeWitt-Eppley	2 BH head-on collision	PRD14(1976)2443
	Smarr-York	gauge conditions	PRD17(1978)2529
	ed. by L.Smarr	"Sources of Grav. Radiation"	Cambridge(1979)
1980s	Nakamura-Maeda-Miyama-Sasaki	axisym. grav. collapse	PTP63(1980)1229
	Miyama	axisym. GW collapse	PTP65(1981)894
	Bardeen-Piran	axisym. grav. collapse	PhysRep96(1983)205
	Stark-Piran	axisym. grav. collapse	unpublished
1990	Shapiro-Teukolsky	naked singularity formation	PRL66(1991)994
	Oohara-Nakamura	3D post-Newtonian NS coalesence	PTP88(1992)307
	Seidel-Suen	BH excision technique	PRL69(1992)1845
	Choptuik	critical behaviour	PRL70(1993)9
	NCSA group	axisym. 2 BH head-on collision	PRL71(1993)2851
	Cook et al	2 BH initial data	PRD47(1993)1471
	Shibata-Nakao-Nakamura	BransDicke GW collapse	PRD50(1994)7304
	Price-Pullin	close limit approach	PRL72(1994)3297
1995	NCSA group	event horizon finder	PRL74(1995)630
	NCSA group	hyperbolic formulation	PRL75(1995)600
	Anninos *et al*	close limit vs full numerical	PRD52(1995)4462
	Scheel-Shapiro-Teukolsky	BransDicke grav. collapse	PRD51(1995)4208
	Shibata-Nakamura	3D grav. wave collapse	PRD52(1995)5428
	Gunnersen-Shinkai-Maeda	ADM to NP	CQG12(1995)133
	Wilson-Mathews	NS binary inspiral, prior collapse?	PRL75(1995)4161
	Pittsburgh group	Cauchy-characteristic approach	PRD54(1996)6153
	Brandt-Brügmann	BH puncture data	PRL78(1997)3606
	Illinois group	synchronized NS binary initial data	PRL79(1997)1182
	Shibata-Baumgarte-Shapiro	2 NS inspiral, PN to GR	PRD58(1998)023002
	BH Grand Challenge Alliance	characteristic matching	PRL80(1998)3915
	Baumgarte-Shapiro	Shibata-Nakamura formulation	PRD59(1998)024007
	Brady-Creighton-Thorne	intermediate binary BH	PRD58(1998)061501
	Meudon group	irrotational NS binary initial data	PRL82(1999)892
	Shibata	2 NS inspiral coalesence	PRD60(1999)104052
	York	conformal thin-sandwich formulation	PRL82(1999)1350
	Brodbeck *et al*	λ-system	JMathPhys40(1999)909
2000	Kidder-Finn	BH, Spectral methods	PRD62(2000)084026
	Shinkai-Yoneda	planar GW, Ashtekar variables	CQG17(2000)4729
	AEI group	full numerical to close limit	CQG17(2000)L149
	AEI group	2 BH grazing collision	PRL87(2001)271103
	Shibata-Uryu	2 NS inspiral coalesence	PTP107(2002)265
	Shinkai-Yoneda	adjusted ADM systems	CQG19(2002)1027
	Meudon group	irrotational BH binary initial data	PRD65(2002)044020
	PennState group	isolated horizon	gr-qc/0206008

2nd method **Thin-Sandwich Approach – York (1999)**

J.W.York Jr., Phys. Rev. Lett. **82**, 1350 (1999)

Benefits:

- The name "sandwich" comes from the proposal that this method prepares two spatial slices at $t = 0$ and $t = \Delta t$.
- The input function is more friendly (3-metric and its time derivative) than the previous conformal approach.
- The input quantity also requires the lapse function, N. (Actually this is the inverse and densitized lapse function.)
- The similar conformal transformation is applied. But the relation $\bar{A}^{ij} = \psi^{-10} A^{ij}$ is *derived* in this version.

Comments:

- The numerical solvability is still debatable.
- Partial applications are seen in constructing quasi-equilibrium binary neutron stars/black-holes.

- introduce the conformal metric g_{ij}, “this world” $\overline{g}_{ij} = \psi^4 g_{ij}$ “that world”

  Also impose $\overline{g}^{ij}\partial_t \overline{g}_{ij} = 0$.

- On the second slice $t = \delta t$, we write the conformal metric

$$g'_{ij} = g_{ij} + u_{ij}\delta t,$$

  where $u_{ij} = \dot{g}_{ij},$ is the velocity tensor (suppose to be a given quantity), and also impose the “weighted” condition, $g^{ij}u_{ij} = 0$ and $g^{ij}\dot{g}_{ij} = 0.$

- By taking the traceless part of the evolution equation,

$$\partial_t \overline{g}_{ij} \equiv \dot{\overline{g}}_{ij} = -2\overline{NK}_{ij} + (\overline{D}_i \overline{\beta}_j + \overline{D}_j \overline{\beta}_i)$$

  we get

$$\dot{\overline{g}}_{ij} - \frac{1}{3}\overline{g}_{ij}\overline{g}^{kl}\dot{\overline{g}}_{kl} \equiv \overline{u}_{ij} = -2\overline{NA}_{ij} + (\overline{L\beta})_{ij}$$

  where $\overline{A}_{ij} \equiv \overline{K}_{ij} - (1/3)\overline{K}\overline{g}_{ij},$ and $(\overline{L\beta})_{ij} \equiv \overline{D}_i \overline{\beta}_j + \overline{D}_j \overline{\beta}_i - (2/3)\overline{g}_{ij}\overline{D}^k \overline{\beta}_k.$

- we then obtain $\overline{u}_{ij} = \psi^4 u_{ij}$. Similarly, we obtain

$$\overline{\beta}^i = \beta^i, \qquad \overline{\beta}_i = \psi^4 \beta_i,$$
$$(\overline{L\beta})_{ij} = \psi^4 (L\beta)_{ij}, \qquad (\overline{L\beta})^{ij} = \psi^{-4}(L\beta)^{ij} \ .$$

- We call the standard $\alpha(t,x) > 0$ slicing function, and define the lapse function $\overline{N}$ as $\boxed{\overline{N} = \overline{g}^{1/2}\alpha.}$. The slicing function is now $\alpha = \overline{g}^{-1/2}\overline{N} = \underset{\sim}{\overline{N}}$, which may be called the inverse densitized lapse.
- Let $\boxed{\overline{\alpha} = \alpha}$. We obtain a new relation $\boxed{\overline{N} = \psi^6 N}$
- We also impose $\boxed{\overline{K} = K}$ as before. Then the next relation is *derived*

$$\begin{aligned}\overline{A}^{ij} &= \psi^{-6}(2N)^{-1}\left[\psi^{-4}(L\beta)^{ij} - \psi^{-4}u^{ij}\right] \\ &= \psi^{-10}\left\{(2N)^{-1}\left[(L\beta)^{ij} - u^{ij}\right]\right\} = \psi^{-10}A^{ij} \qquad \text{that is} \qquad \boxed{\overline{A}^{ij} = \psi^{-10}A^{ij}}\end{aligned}$$

By using above boxed conformal transformations, two constraints become can be transformed as

- The Hamiltonian constraint equation (the same with before)

$$8\Delta_g\psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - [\frac{2}{3}K - 2\Lambda]\psi^5 - 16\pi G\rho\psi^{5-n} = 0 \ ,$$

- The momentum constraint equations

$$D_j\left[(2N)^{-1}(L\beta)^{ij}\right] = D_j\left[(2N)^{-1}u^{ij}\right] + \frac{2}{3}\psi^6 D^i K + 8\pi G J^i$$

Thin-Sandwich approach (York, 1999)

One way to set up the metric, gauge values and matter components $(\overline{g}_{ij}, \overline{K}_{ij}, \overline{N}, \overline{\beta}^i, \overline{\rho}, \overline{J}^i)$ so as to satisfy the constraints is as follows.

1. Specify metric components $g_{ij}, u_{ij}(= \dot{g}_{ij}), K,$ the lapse function N, and matter distribution $\rho,\ J^i$ in the conformal frame.

2. Solve the next equations for (ψ, β^i)

$$8\Delta_g\psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - [\frac{2}{3}K - 2\Lambda]\psi^5 - 16\pi G\rho\psi^{5-n} = 0\ , \quad (3)$$

$$D_j\left[(2N)^{-1}(L\beta)^{ij}\right] = D_j\left[(2N)^{-1}u^{ij}\right] + \frac{2}{3}\psi^6 D^i K + 8\pi G J^i, \quad (4)$$

where $A^{ij} = (2N)^{-1}\left[(L\beta)^{ij} - u^{ij}\right]$.

3. Apply the inverse conformal transformation and get the metric and matter components $(\overline{\gamma}_{ij}, \overline{K}_{ij}, \overline{N}, \overline{\beta}^i, \overline{\rho}, \overline{J}^i)$ in the physical frame:

$$\begin{aligned} \overline{N} &= \psi^6 N, \qquad \overline{\beta}^i = \beta^i, \\ \overline{g}_{ij} &= \psi^4 g_{ij}, \quad \overline{K}_{ij} = \psi^{-2}A_{ij} + \frac{1}{3}\psi^4 g_{ij}K, \\ \overline{\rho} &= \psi^{-8}\rho, \quad \overline{J}^i = \psi^{-10}J^i. \end{aligned}$$

Comments

- The two equations, (3) and (4), are coupled, but they will be decoupled if we assume the constant mean curvature condition, $(\text{tr}K) =$const. (This is the same as the conformal approach, but we have to solve the momentum constraints first here.)
- The (general) solvability of (4) is still debatable.

Comparison between two approaches

	conformal approach	thin-sandwich approach
input functions	g_{ij}, K, A_{ij}^{TT} (components: 6, 1, 2) GW components are separated out	g_{ij}, K, u_{ij}, N (comp.: 6, 1, 5, 1) can specify time-derivatives
treatment of gauge functions	lapse and shift are not appearing in the formulation.	lapse is given by the conformal transformation. shift is given by solving the constraints.
usage of the constraints	Hamiltonian constraint is for the conformal factor ψ momentum constraints are for the longitudinal part of A_{ij}.	Hamiltonian constraint is for the conformal factor ψ momentum constraints are for shift function β^i.
counting the freedom	(input 9 functions) plus (3 functions by solving momentum constraints) $=$ 12 $=$ (3-metric) plus (extrinsic curvature).	(input 13 functions) plus (3 functions by solving momentum constraints) $=$ 16 $=$ (3-metric) plus (extrinsic curvature) plus (gauge functions).

Numerical Relativity – open issues **Box 1.2**

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? $\Rightarrow$ see e.g. gr-qc/0209111 Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
  - Solve the constraint equations

Need to solve elliptic PDEs
-- Conformal approach
-- Thin-Sandwich approach

- **Time Evolution**

do time=1, time_end

  - **Specify the slicing conditions**
  - Evolve the variables
  - Check the accuracy
  - Extract physical quantities

end do

singularity avoidance,
simplify the system,
GW extraction, ...

How to choose gauge conditions?

The fundamental guidelines for fixing the lapse function α and the shift vector β_i:

- to avoid hitting the physical and coordinate singularity in its evolution.
- to make the system suitable for physical situation.
- to make the evolution system as simple as possible.
- to enable the gravitational wave extraction easy.

Lapse conditions

geodesic slice	$\alpha = 1$	GOOD	simple, easy to understand
		BAD	no singularity avoidance
harmonic slice	$\nabla_a \nabla^a x^b = 0$	GOOD	simplify eqs.,
		GOOD	easy to compare analytical investigations
		BAD	no singularity avoidance or coordinate pathologies
maximal slice	$K = 0$	GOOD	singularity avoidance
		BAD	have to solve an elliptic eq.
maximal slice (K-driver)	$\partial_t K = -c^2 K$	G&B	same with maximal slice,
		GOOD	easy to maintain $K = 0$
constant mean curvature	$K = const.$	G&B	same with maximal slice,
		GOOD	suitable for cosmological situation
polar slicing	$K_\theta^\theta + K_\varphi^\varphi = 0$, or $K = K_r^r$	GOOD	singularity avoidance in isotropic coord.
		BAD	trouble in Schwarzschild coord.
algebraic	$\alpha \sim \sqrt{\gamma}$, $\alpha \sim 1 + \log \gamma$	GOOD	easy to implement
		BAD	not avoiding singularity

Maximal slicing condition

- A singularity avoiding gauge condition.
- The name of 'maximal' comes from the fact that the deviation of the 3-volume $V = \int \sqrt{\gamma} d^3x$ along to the normal line becomes maximal when we set $K = 0$.
- This is simply written as

$$K = 0 \qquad \text{on} \quad \Sigma(t).$$

  Practically, we solve

$$D^i D_i \alpha = \{ {}^{(3)}R + K^2 + 4\pi G(S - 3\rho_H) - 3\Lambda\}\alpha,$$

  or by using the Hamiltonian constraint further,

$$D^i D_i \alpha = \{K_{ij} K^{ij} + 4\pi G(S + \rho_H) - \Lambda\}\alpha.$$

- This is an elliptic equation. When the curvature is strong (i.e. close to the appearance of a singularity), the RHS of equation become larger, hence the lapse becomes smaller. Therefore the foliation near the singularity evolves slowly.

Maximal Slicing Condition

In Schwarzschild geometry, K=0 slicing conditions allows us to evolve r=1.5M.

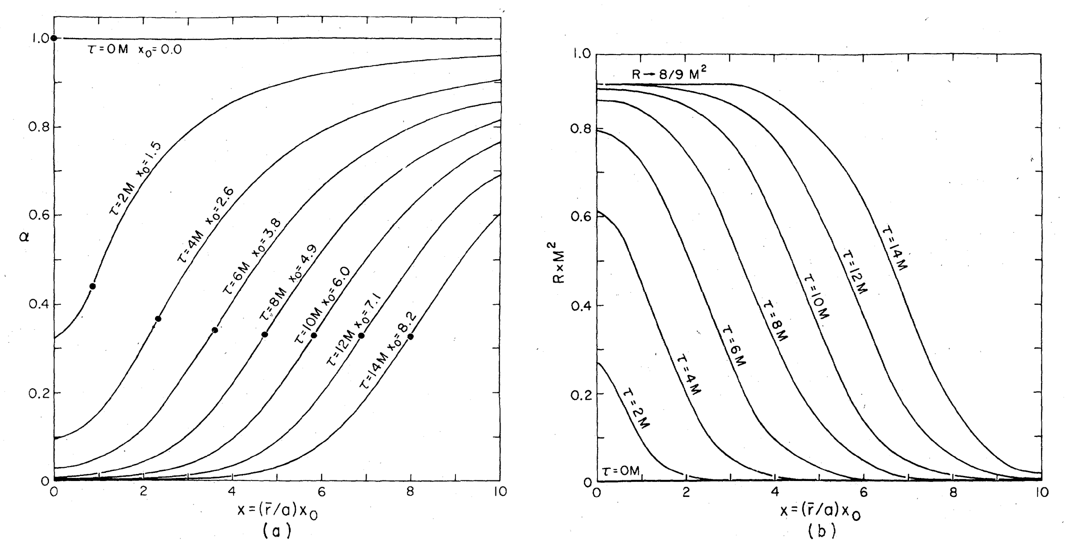


FIG. 3. (a) The spherically symmetric lapse function α is plotted versus the dimensionless radius $x \equiv (\bar{r}/a)x_0$ where $\bar{r}$ is the proper radial distance from the throat and x_0 and a are as defined by Eq. (3.7). These plots of $\alpha(x)$ are given for a series of time slices of the symmetric maximal slices of Schwarzschild-Kruskal spacetime studied by Estabrook *et al.* (Ref. 37). Notice the rapid collapse of the lapse near the throat at late times ($\tau \gtrsim 10M$). Time slices are labeled by the strength parameter x_0 and by proper time at a large finite distance (where we set $\alpha = 1$). The curves rise more rapidly than in Fig. 4 because this distance is not infinite. The location of the event horizon $r_{Sch} = 2M$ is denoted by a dot on each slice (see Eppley, Ref. 1). (b) For the same time slices as in (a), we plot the Ricci scalar $\mathcal{R}(x)$ of the three-metric of the slice. At $\tau = 0$, $\mathcal{R} = 0$ from the constraint equations. It grows in the strong-field region ($x \cong 0$) as time increases. This is what forces the lapses to zero in (a). At late times the central value of $\mathcal{R}$ goes to $8/9M^2$, the value of $\mathcal{R}$ for the hypercylinder $r_{Sch} = 3M/2$. The "effective radius" a [Eq. (3.7)] is found to grow linearly in time and be approximately the proper radial distance from the throat to the horizon in the time slice.

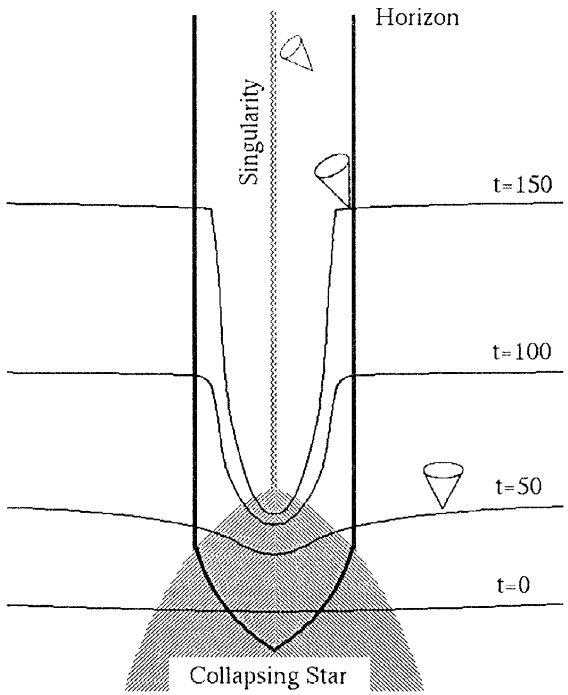


FIG. 1. A black hole spacetime diagram showing various singularity avoiding time slices that wrap up around the singularity inside the horizon. Such slicings allow short-term success in the numerical evolution of black holes, while at the same time causing pathological behavior that eventually dooms the calculation at late times.

Maximal slicing versus Harmonic slicing

A. Geyer and H. Herold, PRD31 (1995) 6182

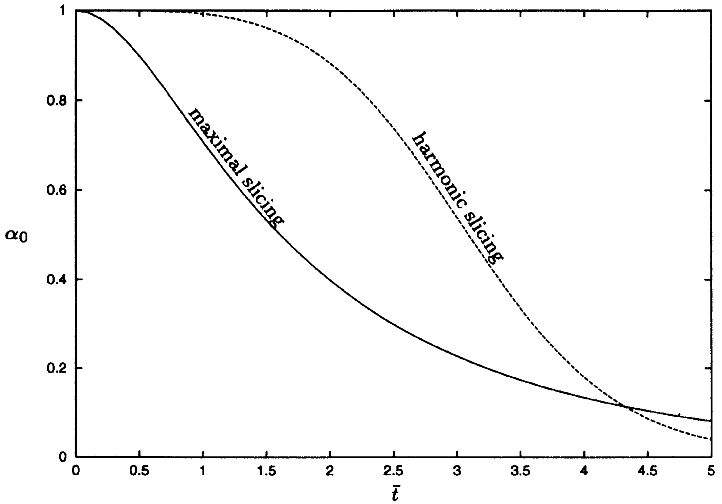


FIG. 2. The lapse on the $u = 0$ axis as a function of $\bar{t}$. For harmonic slicing the "collapse of the lapse" occurs at a later time $\bar{t}$ than in the case of maximal slicing.

Harmonic slicing hits singularity!

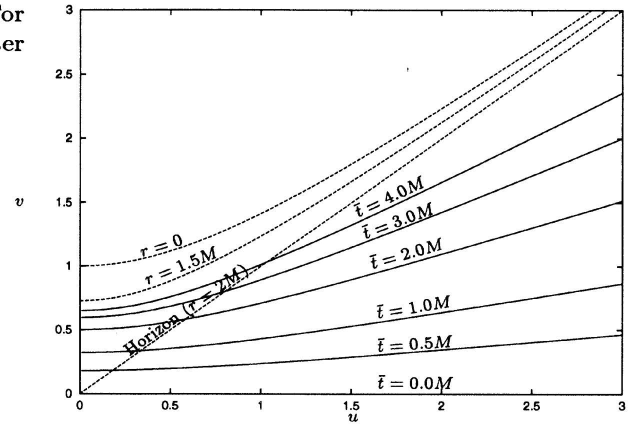


FIG. 1. Foliation of the Schwarzschild spacetime by maximal slices. The picture shows the projection of some $\bar{t} =$ const hypersurfaces in the Kruskal plane (compare [6]).

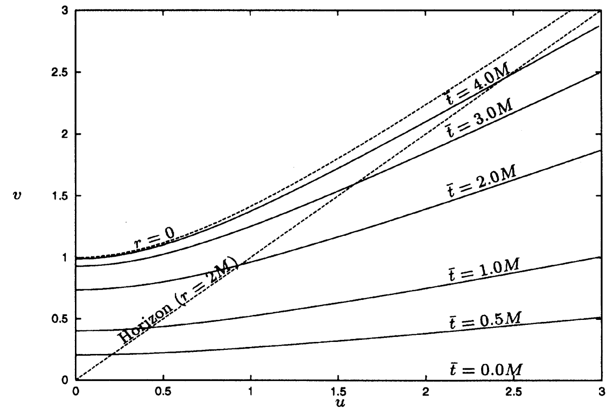


FIG. 3. Harmonic slices in the Schwarzschild spacetime constructed from the initial lapse $\alpha = 1$ on $v = 0$. Note that, in contrast with Fig. 1, the whole spacetime up to the singularity $(r = 0)$ gets covered.

Lapse conditions

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Shift conditions

geodesic slice	$\beta^i = 0$	GOOD	simple, easy to understand
		BAD	too simple
minimal distortion	$\min \Sigma^{ij}\Sigma_{ij}$	GOOD	geometrical meaning
		BAD	elliptic eqs., hard to solve
minimal strain	$\min \Theta^{ij}\Theta_{ij}$	G&B	same with minimal distorsion

Minimal distortion condition, Minimal strain condition

L.Smarr and J.W.York,Jr., Phys. Rev. D **17**, 2529 (1978)

- Against the grid-streching, minimize the distortion in a global sense.
- The expansion tensor $\Theta_{\mu\nu}$: Let the coordinate-constant congruence $t_\mu = \alpha n_\mu + \beta_\mu$. Using the projection operator $\perp^a_b = \delta^a_b + n^a n_b$,

$$\begin{aligned}\Theta_{\mu\nu} &= \perp\nabla_{(\nu}t_{\mu)} \\ &= -\alpha K_{\mu\nu} + \frac{1}{2}D_{(\mu}\beta_{\nu)}\end{aligned}$$

- The distortion tensor Σ_{ij}:

$$\begin{aligned}\Sigma_{ij} &= \Theta_{ij} - \frac{1}{3}\Theta\gamma_{ij} \\ &= -2\alpha\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) + \frac{1}{2}\left(D_{(i}\beta_{j)} - \frac{1}{3}D^k\beta_k\right).\end{aligned}$$

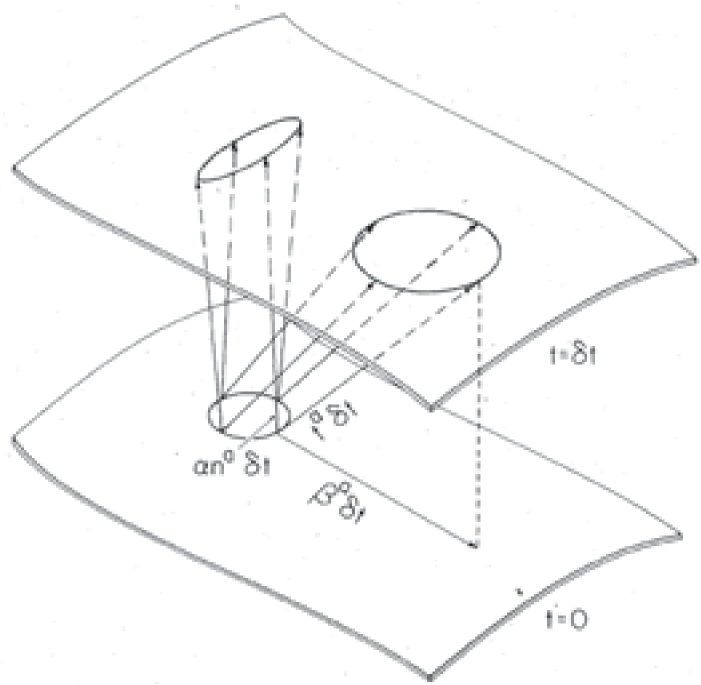

FIG. 7. This schematic diagram illustrates the use of the minimal-distortion shift vector to reduce coordinate shear. If a small sphere (here one spatial dimension is suppressed) is transported along the normal $\hat{n}^a$ to the next slice $\tau = \delta\tau$, it will be sheared into an ellipsoid. If the slicing is maximal, the volume will be preserved to first order. On the other hand, if a shift vector is also used, then some of this coordinate shear can be removed, although with the possible introduction of some change in volume.

The minimal distortion condition

- minimize $\Sigma_{ij}\Sigma^{ij}$

$$\delta S[\beta] = \delta\{\frac{1}{2}\int \Sigma_{ij}\Sigma^{ij} d^3x\} = 0.$$

- This condition can be written as $D^j\Sigma_{ij} = 0$, or

$$D^j D_j \beta_i + D^j D_i \beta_j - \frac{2}{3} D_i D_j \beta^j = D^j \left[2\alpha \left(K_{ij} - \frac{1}{3} trK\gamma_{ij} \right) \right],$$

or

$$\Delta\beta_i + \frac{1}{3} D_i (D^j \beta_j) + R_i^j \beta_j = D^j \left[2\alpha \left(K_{ij} - \frac{1}{3} trK\gamma_{ij} \right) \right],$$

where $\Delta = D^i D_i$.

The minimal strain condition

- minimize $\Theta^{ij}\Theta_{ij}$, similarly.

Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
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Need to solve elliptic PDEs
-- Conformal approach
-- Thin-Sandwich approach

- **Time Evolution**

do time=1, time_end
  - Specify the slicing conditions
  - **Evolve the variables**
  - **Check the accuracy**
  - Extract physical quantities

end do

singularity avoidance,
simplify the system,
GW extraction, ...

Robust formulation ?
-- modified ADM
-- hyperbolization
-- asymptotically constrained