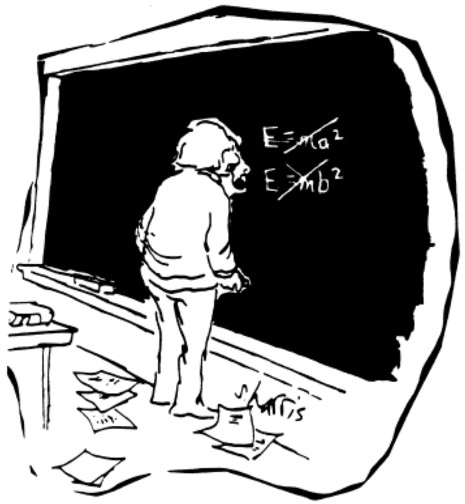
APCTP Winter School, January 17-18, 2003

Introduction to Numerical Relativity

RIKEN Institute, Hisaaki Shinkai

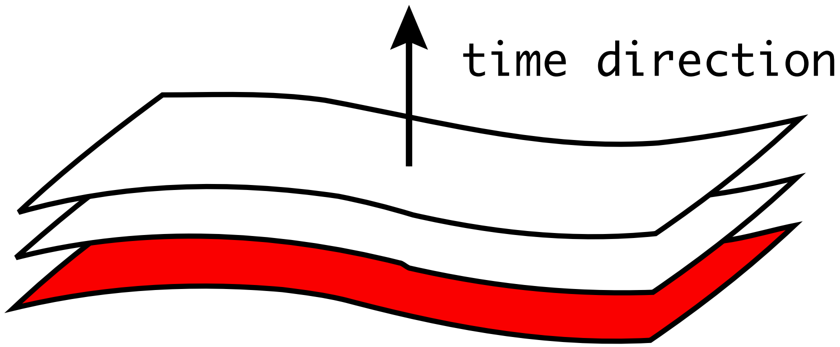


1. Subjects for Numerical Relativity
   - Why Numerical Relativity?
2. The Standard Approach to Numerical Relativity
   - The ADM formulation
3. Alternative Approaches to Numerical Relativity
   - Full numerical, but different foliations
     - Cauchy and characteristic/matching, hyperboloidal foliations
   - Several approximations
     - Cauchy-perturbative, close-limit, quasi-spherical
4. Unsolved problems
   - etc, etc

신카이 히사아키

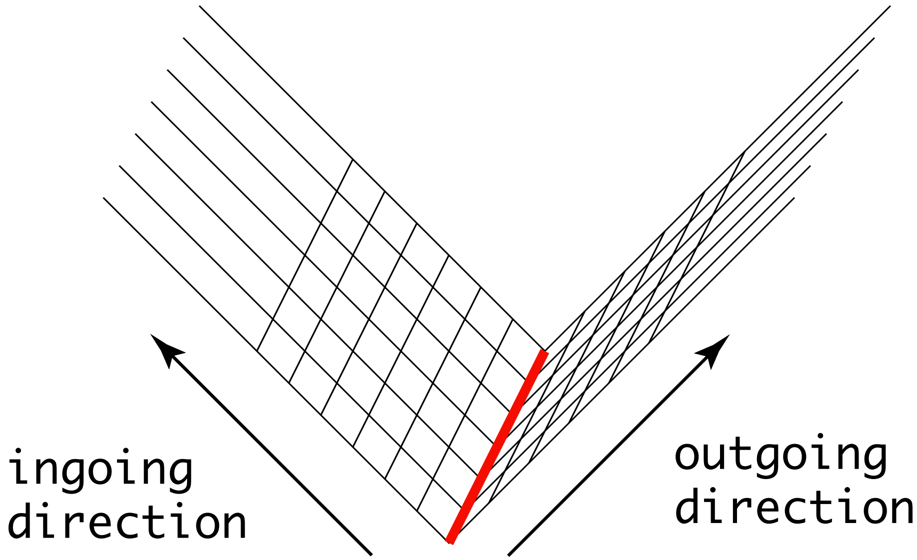
First Question: How to foliate space-time?

**Cauchy approach
or ADM 3+1 formulation**



Σ: Initial 3-dimensional Surface

**Characteristic approach
(if null, dual-null 2+2 formulation)**



S: Initial 2-dimensional Surface

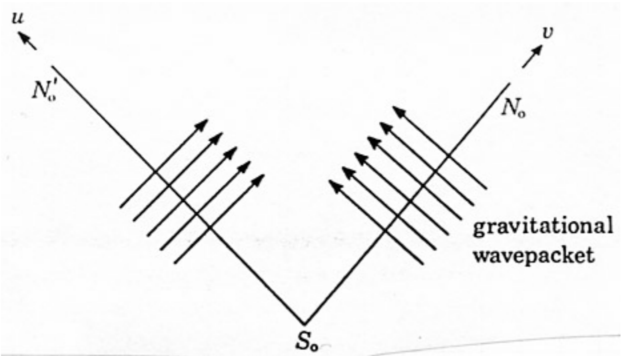
3+1 versus 2+2

	Cauchy (3+1) evolution	Characteristic (2+2) evolution
pioneers	ADM (1961), York-Smarr (1978)	Bondi *et al* (1962), Sachs (1962), Penrose (1963)
variables	easy to understand the concept of time evolution	has geometrical meanings 1 complex function related to 2 GW polarization modes
foliation	has Hamilton structure	allows implementation of Penrose's space-time compactification
initial data	need to solve constraints	no constraints
evolution	PDEs need to avoid constraint violation	ODEs with consistent conditions propagation eqs along the light rays
singularity	need to avoid by some method	can truncate the grid
disadvantages	can not cover space-time globally	difficulty in treating caustics hard to treat matter

Numerical Relativity in Dual-Null Foliation

J.M. Stewart, H.Friedrich, Proc. R. Soc. Lond. A 384, 427 (1982)

R.W. Corkill, J.M. Stewart, Proc. R. Soc. Lond. A 386, 373 (1983)



Colliding Plane-wave

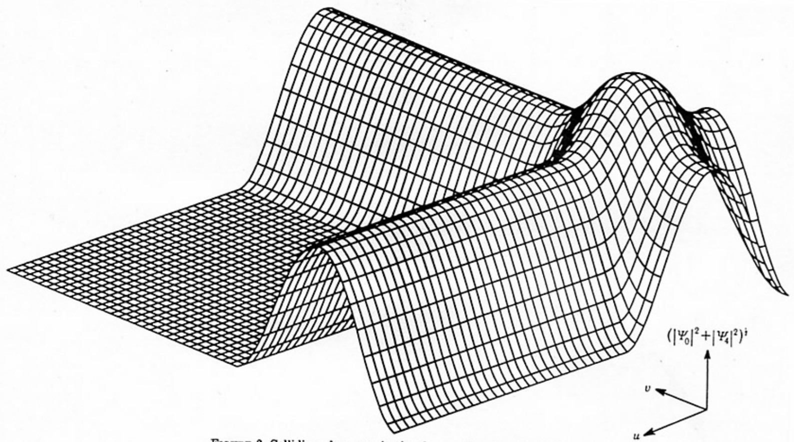


FIGURE 2. Colliding plane gravitational waves in a linearized theory.

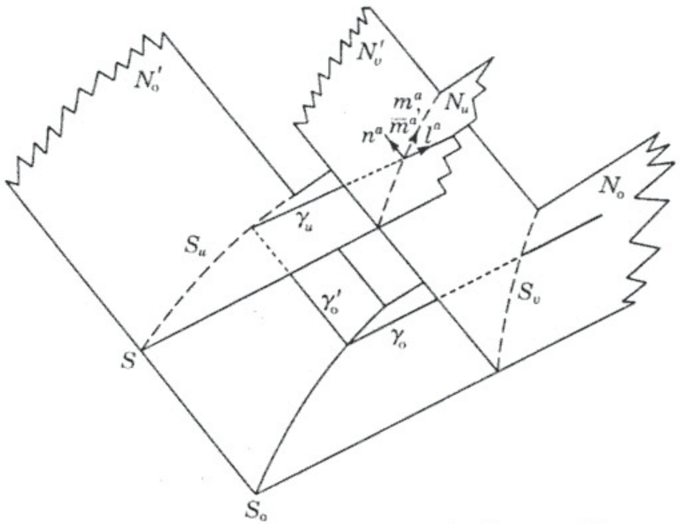


FIGURE 3. The coordinate system on $M.(x^A)$ $(A = 2, 3)$ are arbitrary coordinates on S_o. The coordinate u is an arbitrary parameter along the generators γ_o' of N_o' that vanishes on S_o. S_u is the 2-surface u = constant. We define u in M by requiring u = constant on N_u, the unique null hypersurface (other than N_0') through S_u. The coordinate v is similarly defined. We propagate (x^A) on to N_o' by requiring x^A = constant along the γ_o', and into M by requiring x^A = constant along the generators γ_u of N_u.

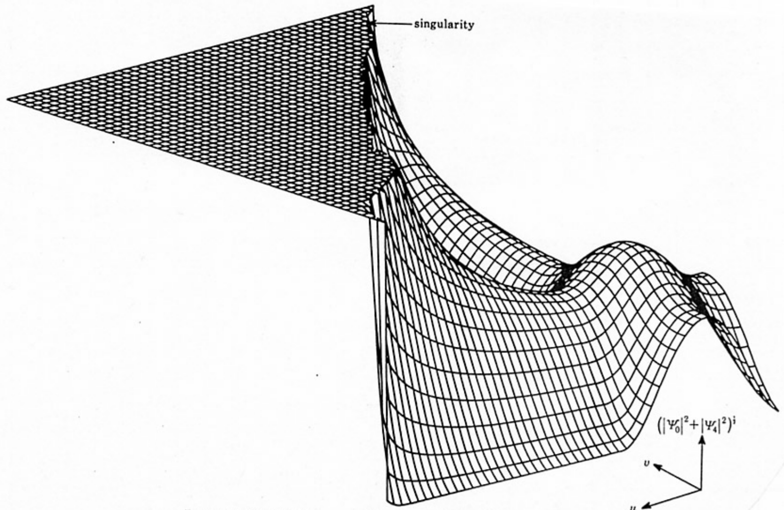
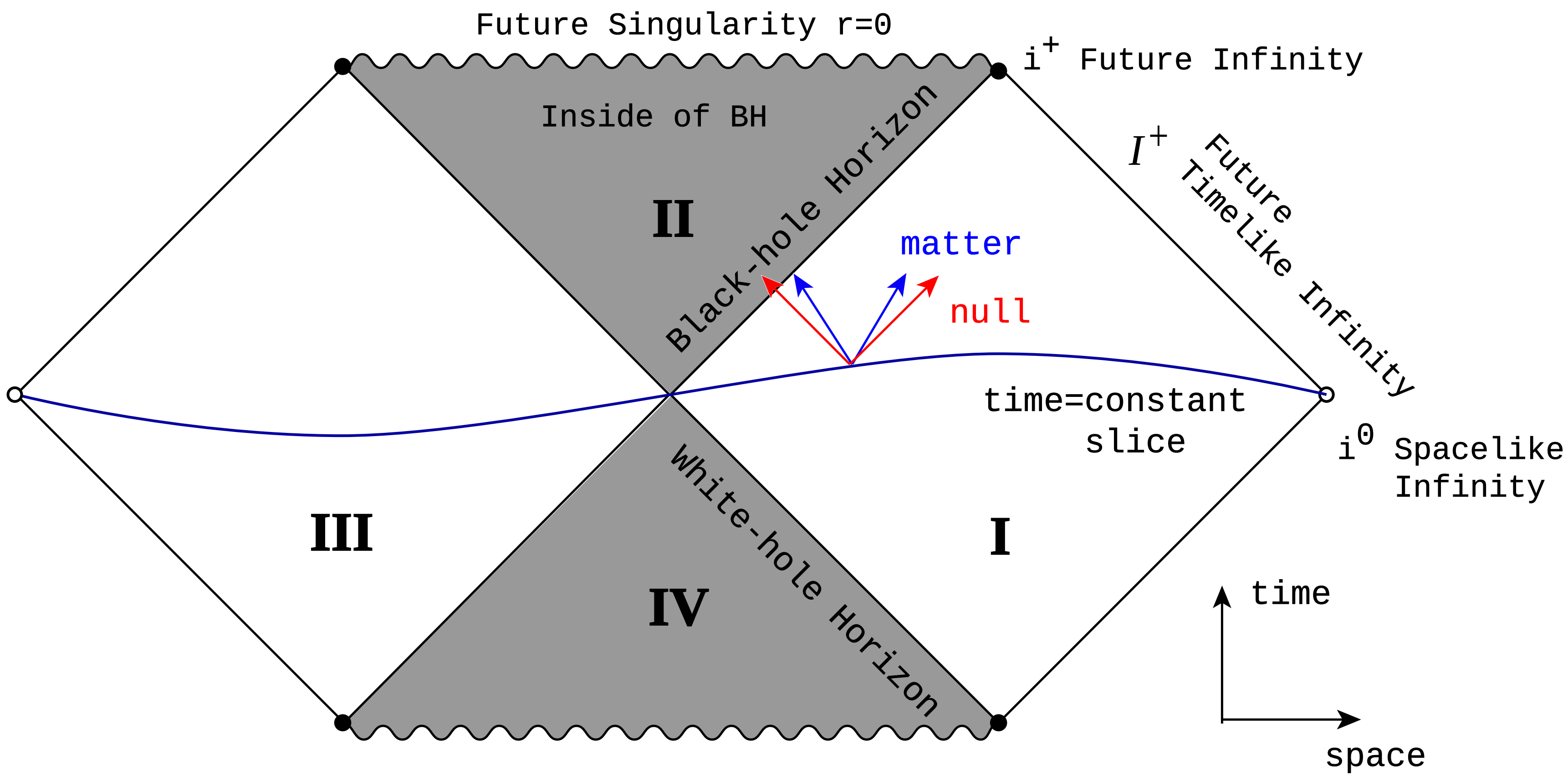


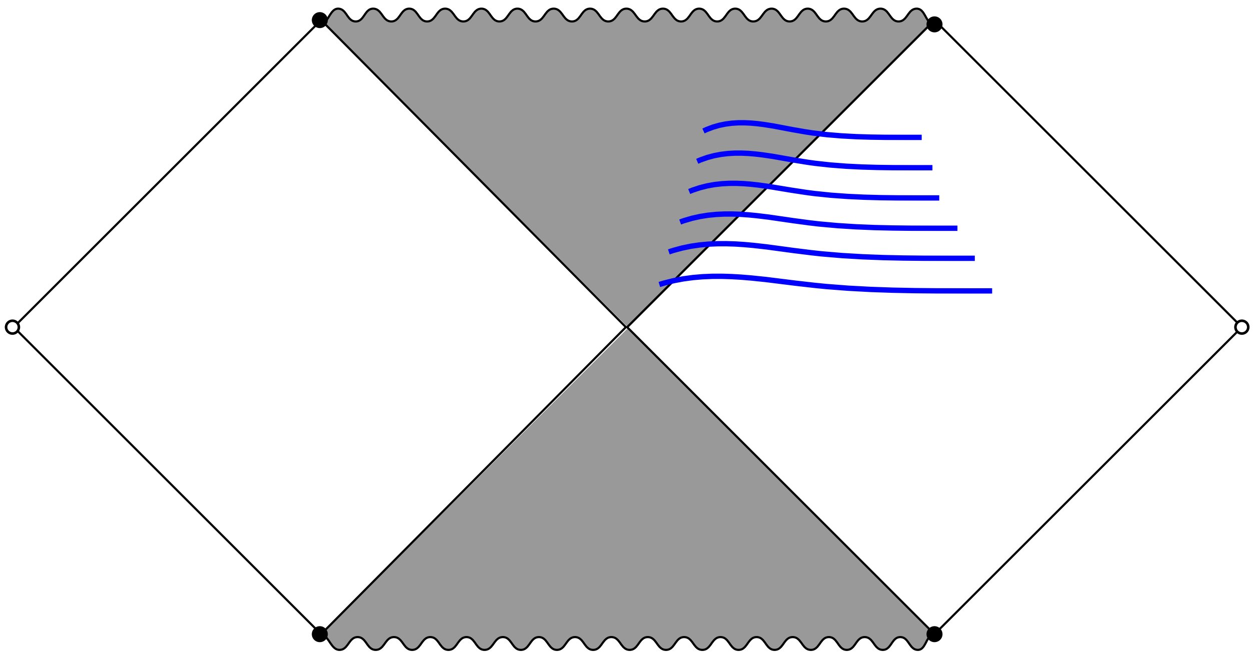
FIGURE 3. Colliding plane gravitational waves in the full nonlinear theory.

Penrose diagram of Schwarzschild Black-hole



Cauchy approach is not enough...

Need outer boundary tricky treatments,
Need better gravitational wave extractions,
Need wider coverage of space-time,



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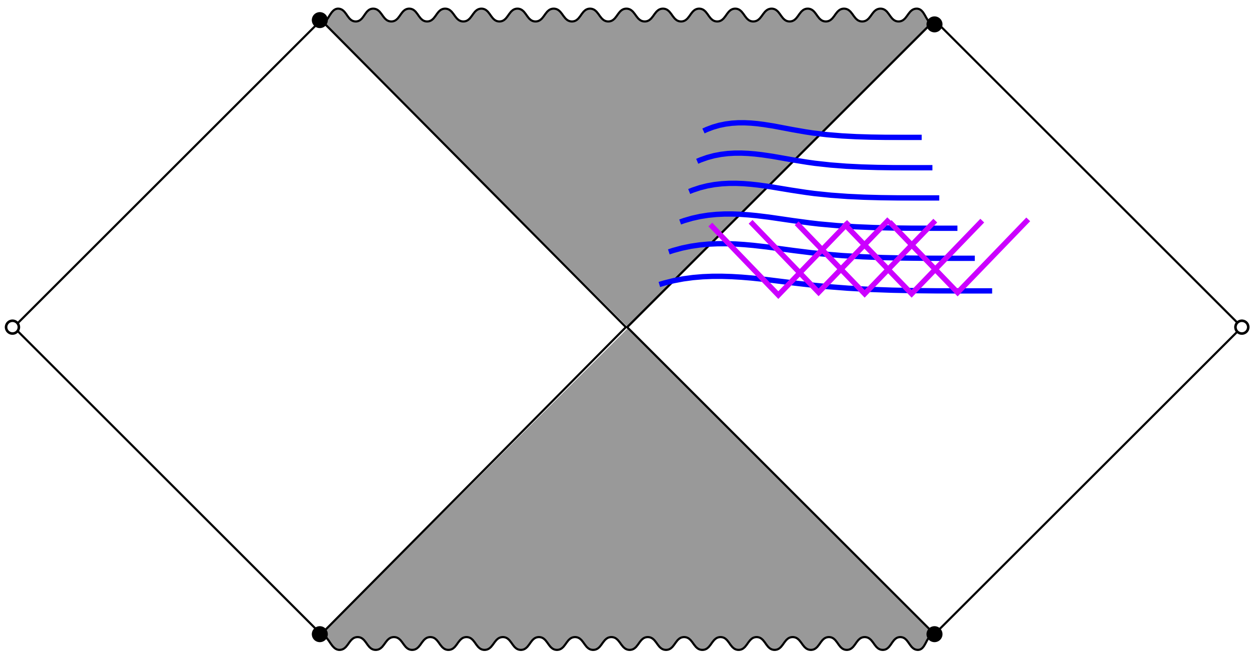
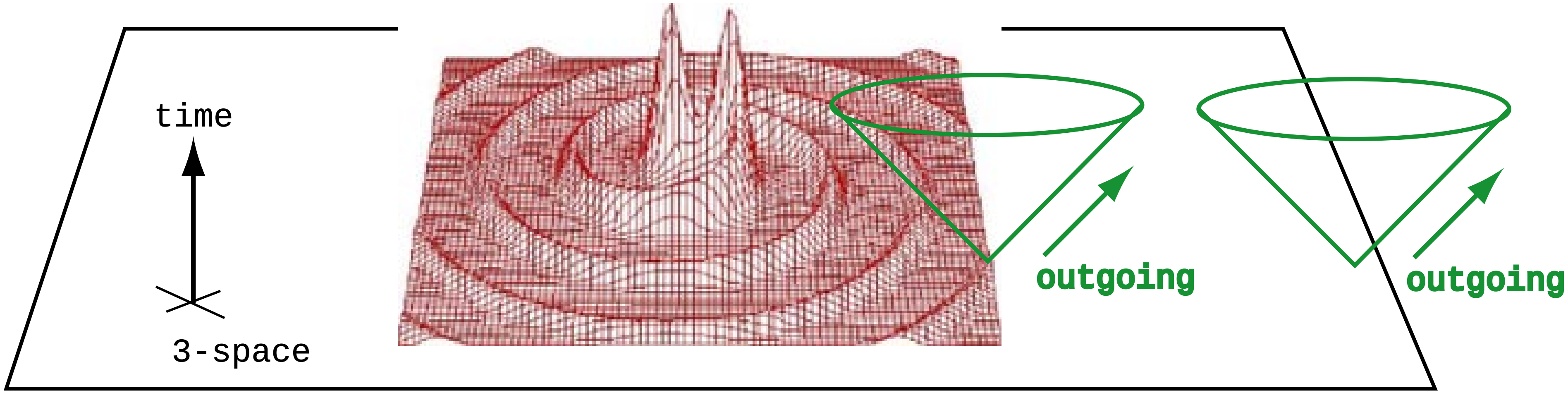


transform ADM to Newman-Penrose

Connection Formula from ADM to Newman-Penrose variables

From (γ_{ij}, K_{ij}) to $(\Psi_0, \cdots \Psi_4)$ via Weyl tensor's Electric & Magnetic decomposition

Gunnersen-Shinkai-Maeda, Class. Quant. Grav. 12 (1995) 133



Weyl curvature and its electric and a magnetic components:

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3}Rg_{a[c}g_{d]b},$$

$$E_{ab} \equiv - \ C_{ambn}t^m t^n, \quad B_{ab} \equiv - \ {}^*C_{ambn}t^n t^m,$$

where ${}^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd}$ is a dual of the Weyl tensor.
With ADM variables,

$$\begin{aligned} E_{ab} &= R_{ab} - K_a{}^m K_{bm} + KK_{ab} - \frac{2}{3}\Lambda\gamma_{ab}, \\ B_{ab} &= \varepsilon_a{}^{mn} D_m K_{nb}. \end{aligned}$$

By defining $s_{ab} = \gamma_{ab} - \hat{z}_a\hat{z}_b$, we decompose

$$\begin{aligned} E_{ab} &= e\hat{z}_a\hat{z}_b + 2e_{(a}\hat{z}_{b)} + e_{ab} - (1/2)s_{ab}e. \\ B_{ab} &= b\hat{z}_a\hat{z}_b + 2b_{(a}\hat{z}_{b)} + b_{ab} - (1/2)s_{ab}b. \end{aligned}$$

With $l^a = (1/\sqrt{2})(t^a + z^a), n^a = (1/\sqrt{2})(t^a - z^a), \quad m^a = (1/\sqrt{2})(x^a - iy^a)$,
$J_a{}^b \equiv \varepsilon_a{}^{bcd}\hat{z}_c t_d$, and $s = -1$ for $(-,+,+,+)$,

$$\begin{aligned} \Psi_0 &\equiv C_{abcd}l^a m^b l^c m^d &&= -(e_{ab} + sJ_a{}^c b_{bc})m^a m^b, \\ \Psi_1 &\equiv C_{abcd}l^a n^b l^c m^d &&= -(s/\sqrt{2})(e_a + sJ_a{}^c b_c)m^a, \\ \Psi_2 &\equiv C_{abcd}l^a m^b \bar{m}^c n^d &&= -(1/2)(e + ib), \\ \Psi_3 &\equiv C_{abcd}l^a n^b \bar{m}^c n^d &&= -(s/\sqrt{2})(e_a - sJ_a{}^c b_c)\bar{m}^a, \\ \Psi_4 &\equiv C_{abcd}n^a \bar{m}^b n^c \bar{m}^d &&= -(e_{ab} - sJ_a{}^c b_{bc})\bar{m}^a \bar{m}^b. \end{aligned}$$

Note that

$$\begin{aligned} C_{abcd}C^{abcd} &= \Psi_4\Psi_0 - 4\Psi_1\Psi_3 + 3\Psi_2^2. \\ R_{abcd}R^{abcd} &= C_{abcd}C^{abcd} + 2R_{ab}R^{ab} - (1/3)R^2. \end{aligned}$$

... applied to many groups' numerical codes.

Head-on Collision of 2 Black-Holes (Misner initial data)

NCSA group 1995

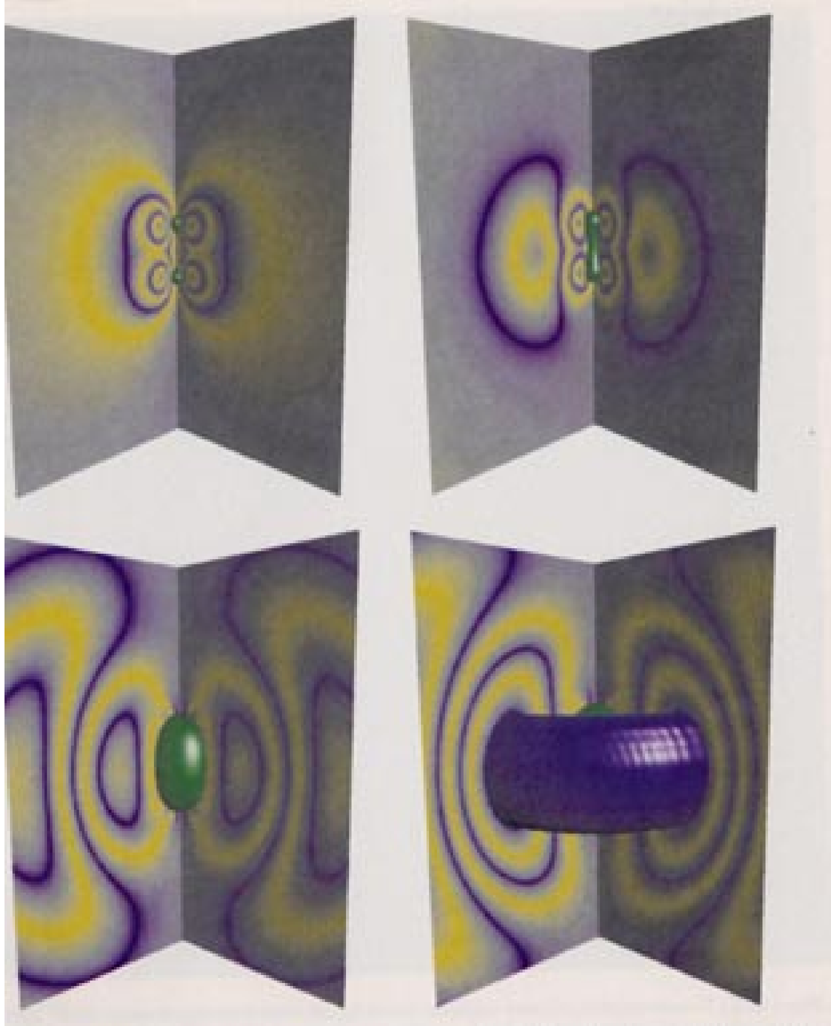


Fig. C.6. 3D evolution of the radiation field Ψ_4 of the head-on collision of two equal-mass black holes shown as a blue and yellow color-map

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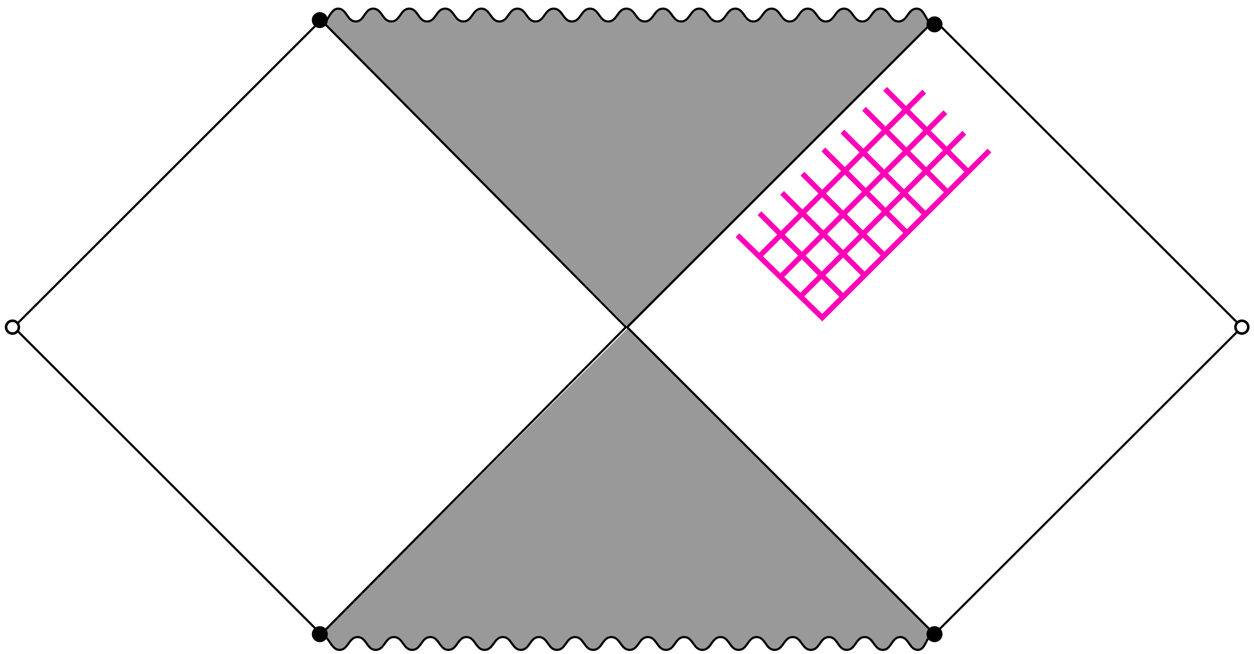

"characteristic approach"

Stable Characteristic Evolution of Generic Three-Dimensional Single-Black-Hole Spacetimes

R. Gómez,[1] L. Lehner,[1] R.L. Marsa,[1,2] J. Winicour,[1] A.M. Abrahams,[3] A. Anderson,[4] P. Anninos,[3] T.W. Baumgarte,[3] N.T. Bishop,[5] S.R. Brandt,[3] J.C. Browne,[2] K. Camarda,[6] M.W. Choptuik,[2] G.B. Cook,[7] R. Correll,[2] C.R. Evans,[4] L.S. Finn,[8] G.C. Fox,[9] T. Haupt,[9] M.F. Huq,[2] L.E. Kidder,[8] S.A. Klasky,[9] P. Laguna,[6] W. Landry,[7] J. Lenaghan,[4] J. Masso,[3] R.A. Matzner,[2] S. Mitra,[2] P. Papadopoulos,[6] M. Parashar,[2] L. Rezzolla,[3] M.E. Rupright,[4] F. Saied,[3] P.E. Saylor,[3] M.A. Scheel,[7] E. Seidel,[3] S.L. Shapiro,[3] D. Shoemaker,[2] L. Smarr,[3] B. Szilágyi,[1] S.A. Teukolsky,[7] M.H.P.M. van Putten,[7] P. Walker,[3] and J.W. York, Jr.[4]

(Binary Black Hole Grand Challenge Alliance)

Wobbling BH
Kerr-Schild BH

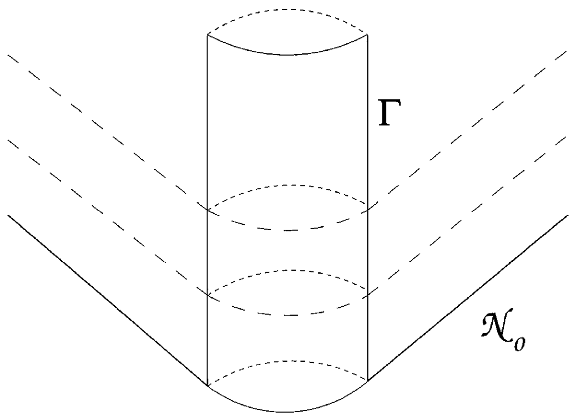

FIG. 1. The outgoing formulation: The exterior of Γ is covered by a sequence of *outgoing* light cones.

outgoing formulation

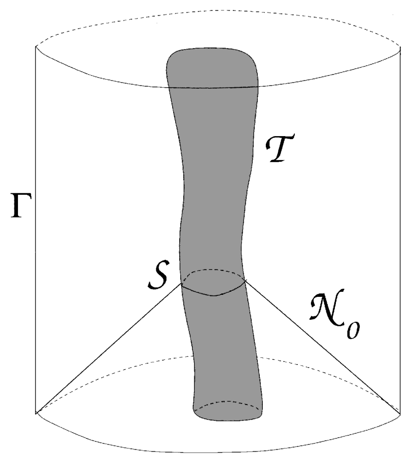

FIG. 2. The ingoing formulation: The interior of Γ is covered by a sequence of *ingoing* light cones. The interior of $\mathcal{T}$ is excised from the evolution.

ingoing formulation

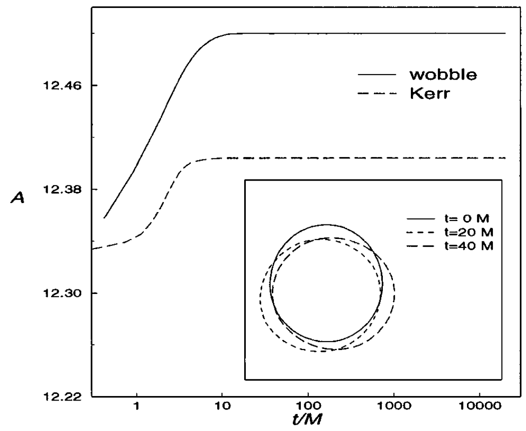

FIG. 3. Surface area vs time for a wobbling hole (with rotation frequency 0.1, offset 0.1, and mass 0.5) and an initially distorted spinning hole (Kerr mass 0.5). The inset shows three different snapshots of the MTS in the case of the "wobble."

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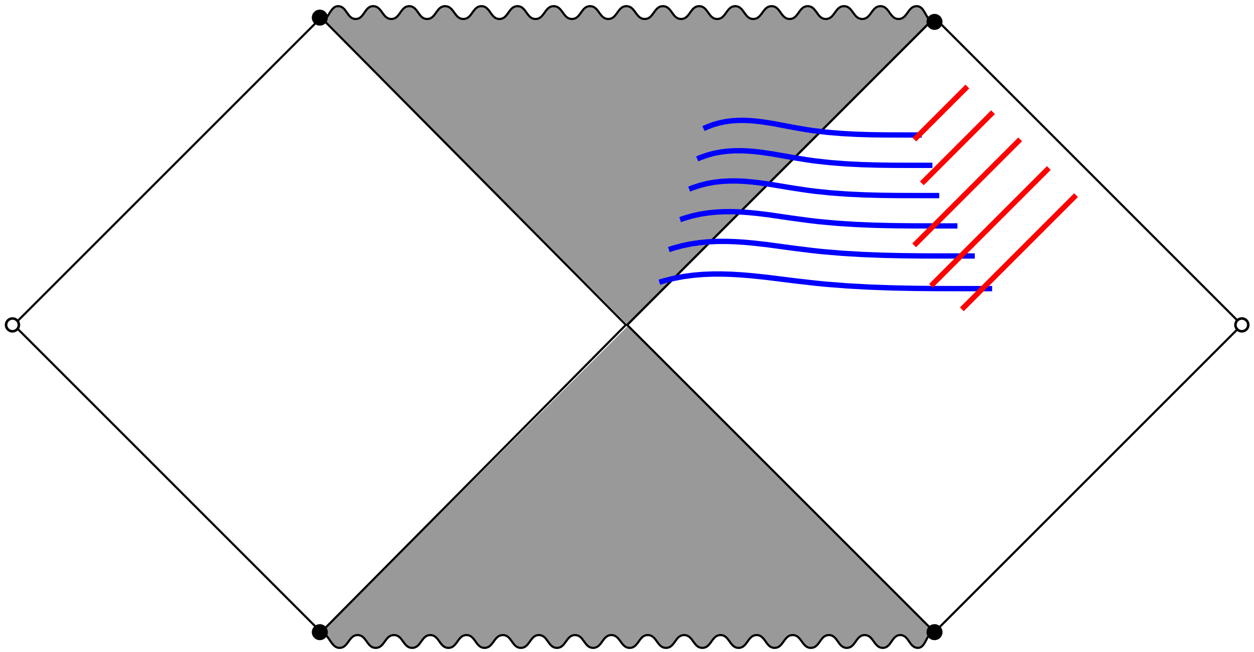

"Cauchy-characteristic matching" technique

GW extraction and outer BCs by Cauchy-characteristic matching

Southampton group (90s), Pittsburgh group (90s)

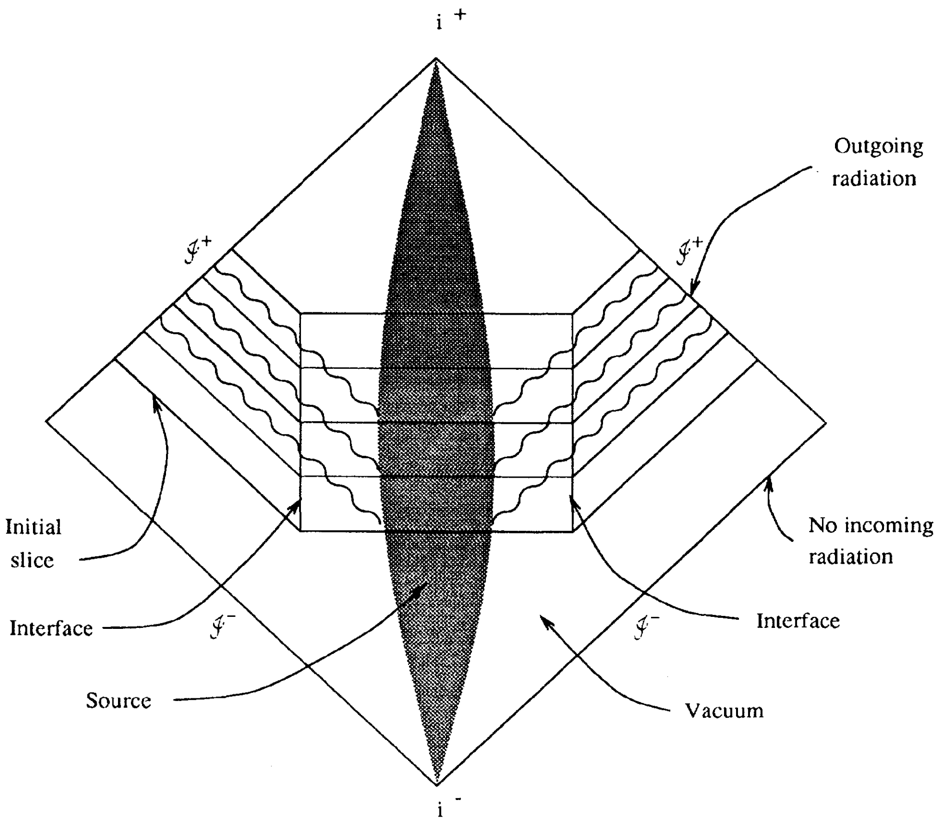

FIG. 1. Combined Cauchy-characteristic surfaces for an isolated radiative system.

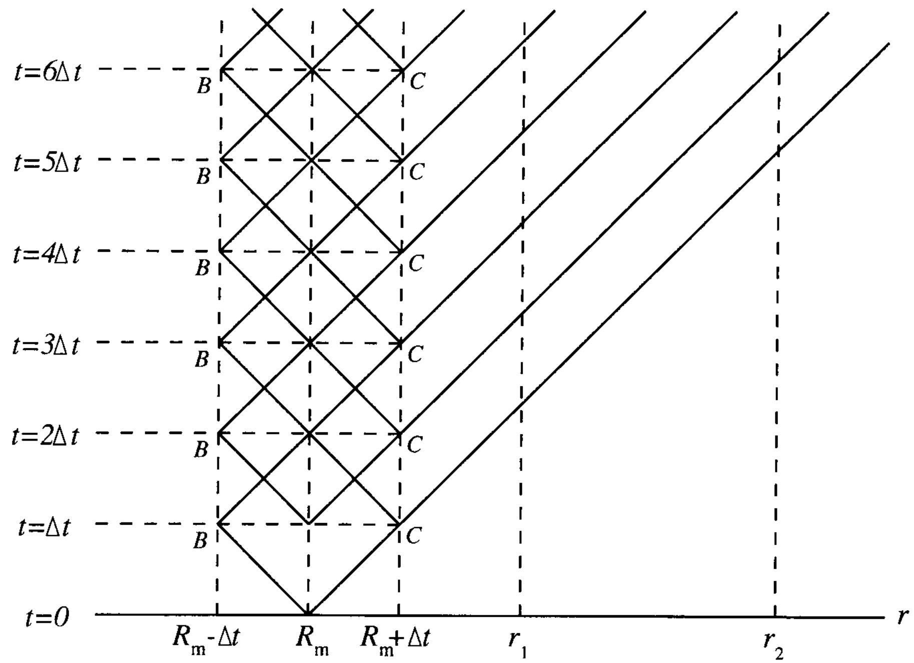

FIGURE 6

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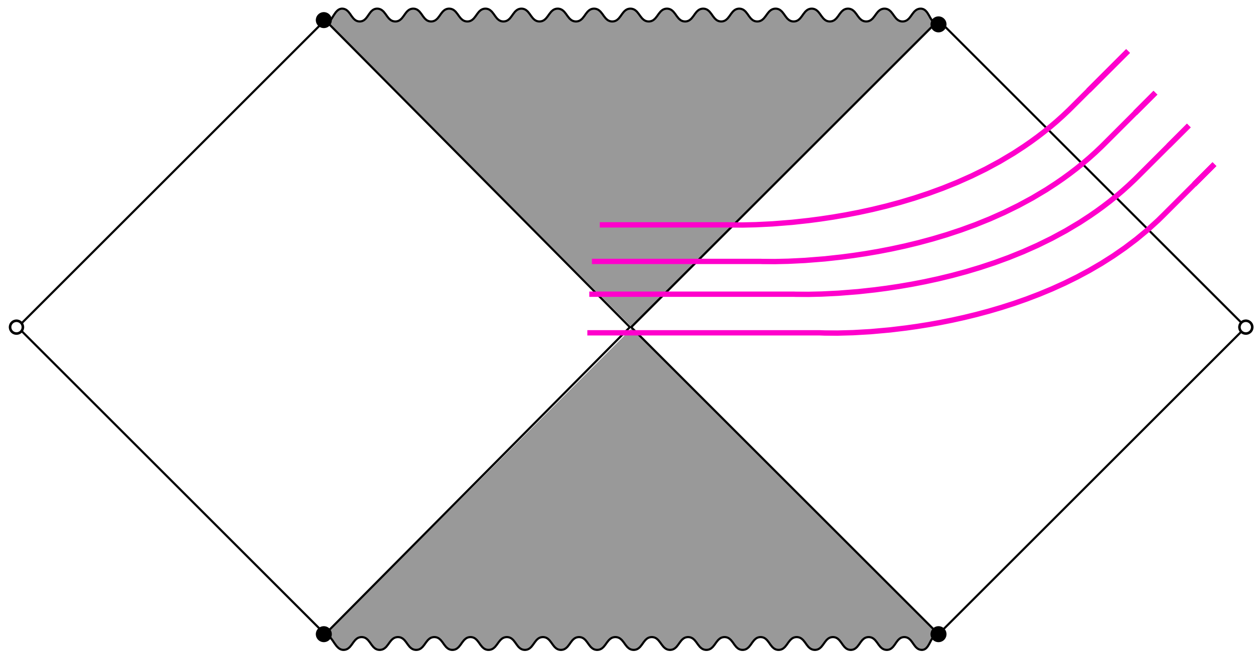

hyperboloical foliation
"conformal Einstein eqs."

Conformal Field Equations, Asymptotically null, Hyperboloidal approach

H. Friedrich, Proc. Roy. Soc. **A375**, 169 (1981); **A378**, 401 (1981).
P. Hübner, Class. Quant. Grav. **16** 2145 (1999); **16** 2823 (1999); **18** 1421 (2001); **18** 1871 (2001).

- A symmetric hyperbolic system.
- 5 gauge functions: (α, β^a, R)
  where R is the Ricci scalar.
- 57 variables:

  γ_{ij}, K_{ij},

  the connection coefficients $\gamma^a{}_{bc}$,

  projections of 4-d $\hat{R}_{ab}$ Ricci tensor

  ${}^{(0,1)}\hat{R}_a = n^b \gamma_a{}^c \hat{R}_{bc} \quad {}^{(1,1)}\hat{R}_{ab} = \gamma_a{}^c \gamma_b{}^d \hat{R}_{bd}$

  the electric and magnetic components of

  the rescaled Weyl tensor $C_{abc}{}^d$,

  the conformal factor Ω

  $\Omega_0 \equiv n^a \nabla_a \Omega, \nabla_a \Omega, \nabla^a \nabla_a \Omega$.

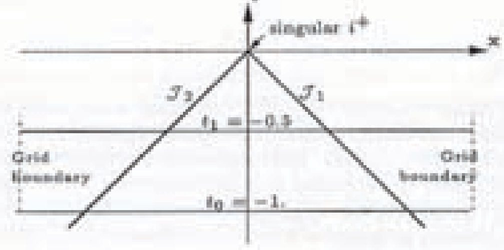

Figure 3. Conformal spacetime for the asymptotically A3 spacetimes with y and z coordinates suppressed.

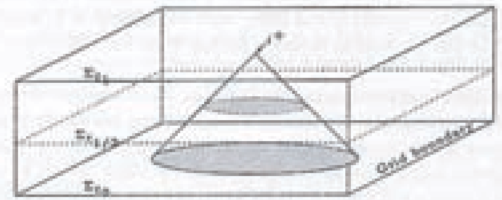

Figure 4. Conformal Minkowski spacetime with z coordinate suppressed.

We start at $t_0 = -1$ and integrate until $t_1 = -\frac{1}{2}$. As long as t_0 and t_1 are negative, their choice is completely arbitrary. The origin (0, 0) is singular, the components of the metric and the curvature become singular there. In principle we could continue the calculation beyond

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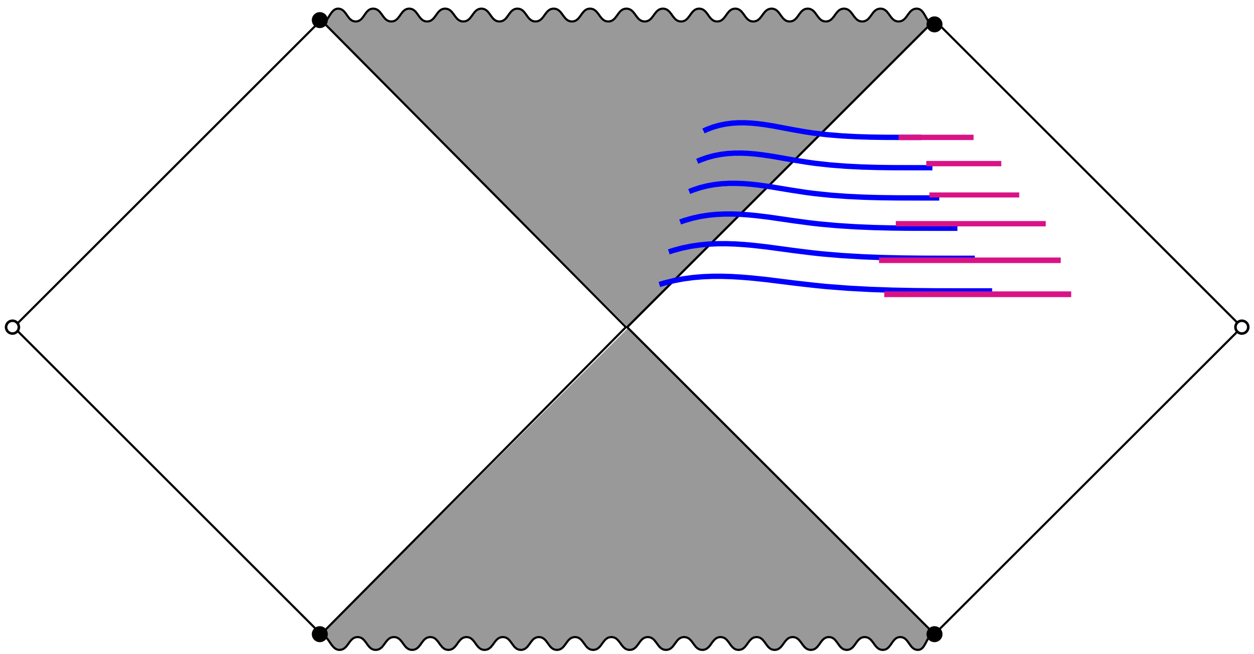

"Cauchy-perturbative matching" technique

GW extraction and outer BCs by Cauchy-perturbative matching

Binary BH Grand Challenge Alliance, PRL80(1998)1812

M.E. Rupright et al, PRD57(1998)1084

L. Rezzola et al, PRD58(1998)044005, PRD59(1999)064001

ADM 3D code, Teukolsky wave test

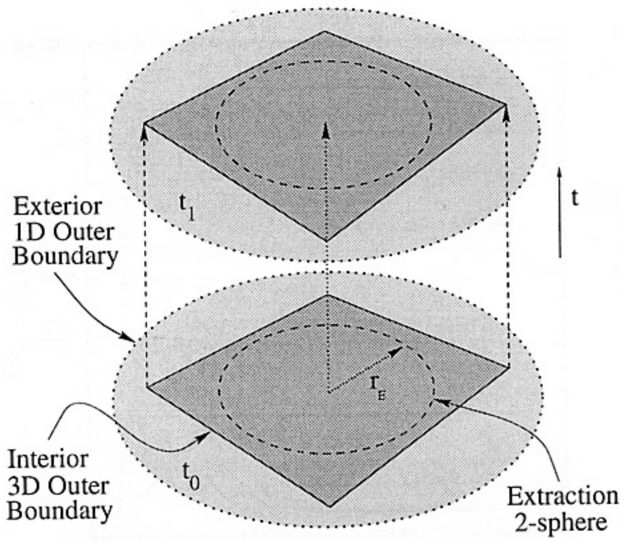

FIG. 1. Location of the different outer boundaries and of the extraction 2-sphere for two successive timeslices. The dark shaded region shows the spatial domain over which the 3D nonlinear equation are solved.

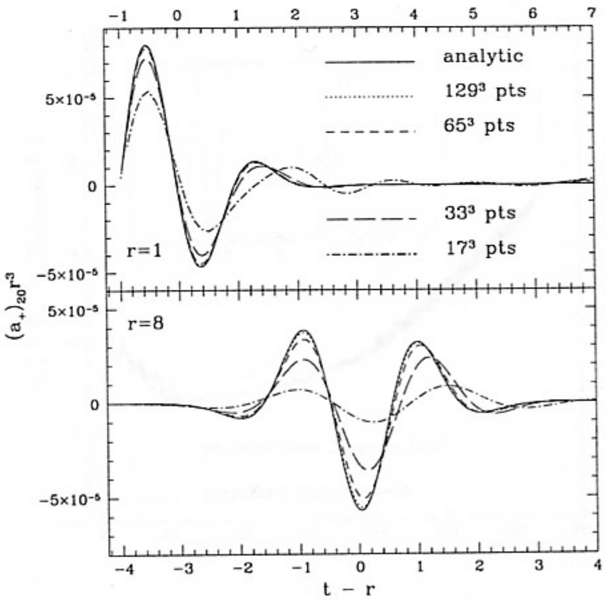

FIG. 2. Convergence to the analytic solution of the extracted ($r = 1$) and evolved ($r = 8$) multipole $(a_+)_{20}$. The amplitude is scaled by r^3 to compensate for the radial fall-off.

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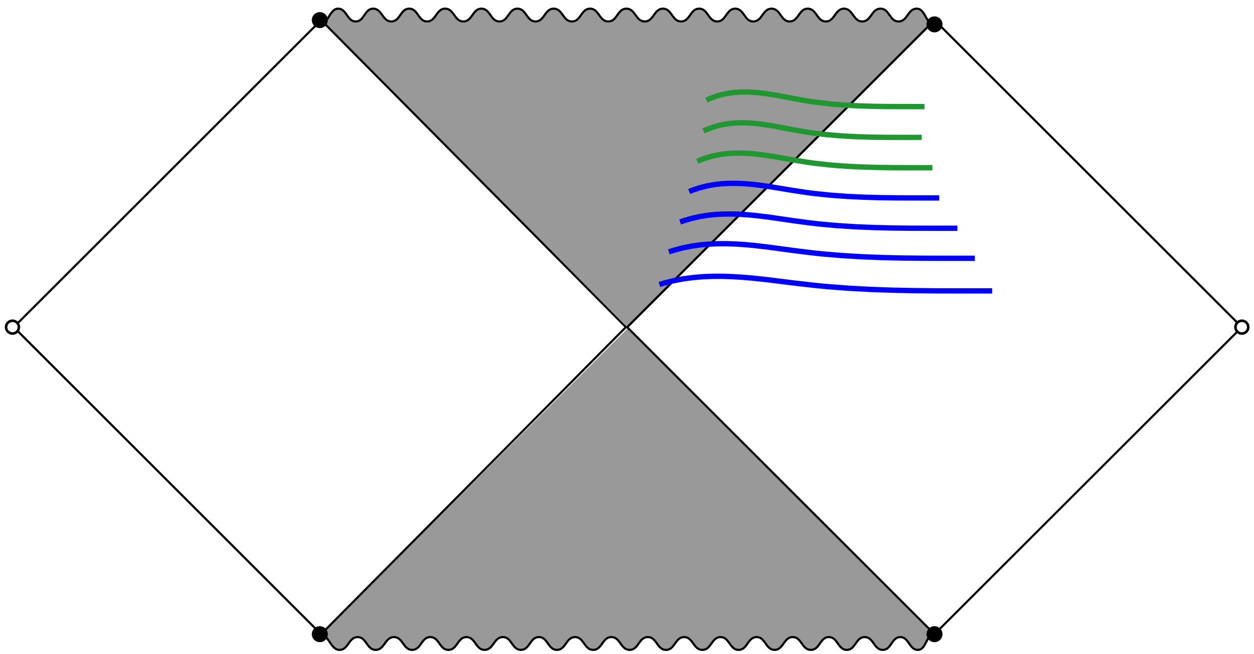


connect to "close-limit approximation"

Close-limit approximation

head-on collision of black-holes (Misner data)

R. Price , J. Pullin, Phys. Rev. Lett. 72, 3297 (1994)

P. Anninos, et al. Phys. Rev. D 52, 4462 (1995)

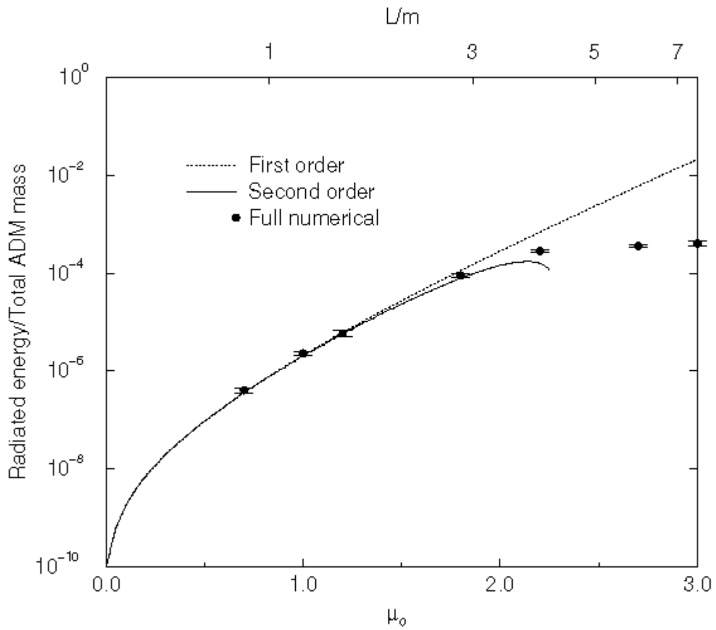


Fig. 3. The radiated energy in a collision of two momentarily-stationary black holes (the Misner problem) compared with the results of full numerical simulations of the NCSA/Potsdam/WashU group. We see that the approximation works well for black holes that are closer than about six times the mass of each hole.

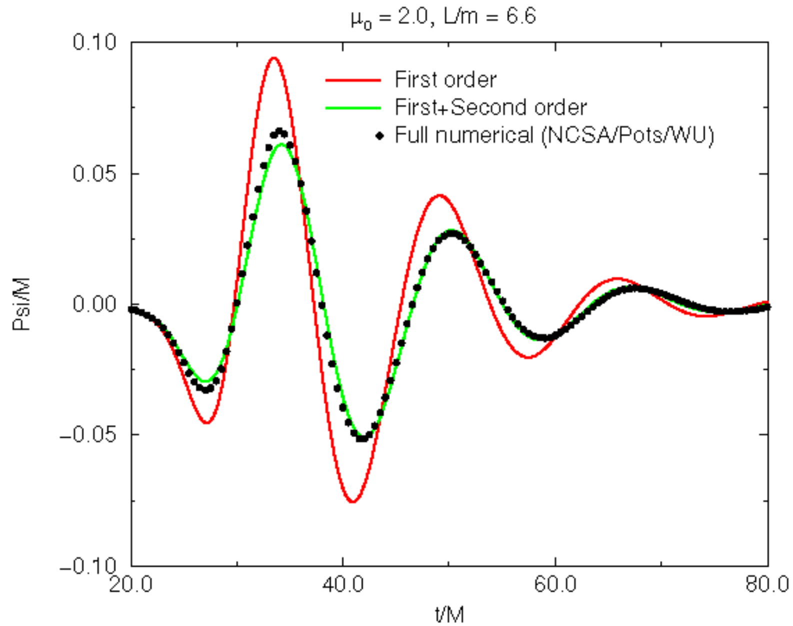
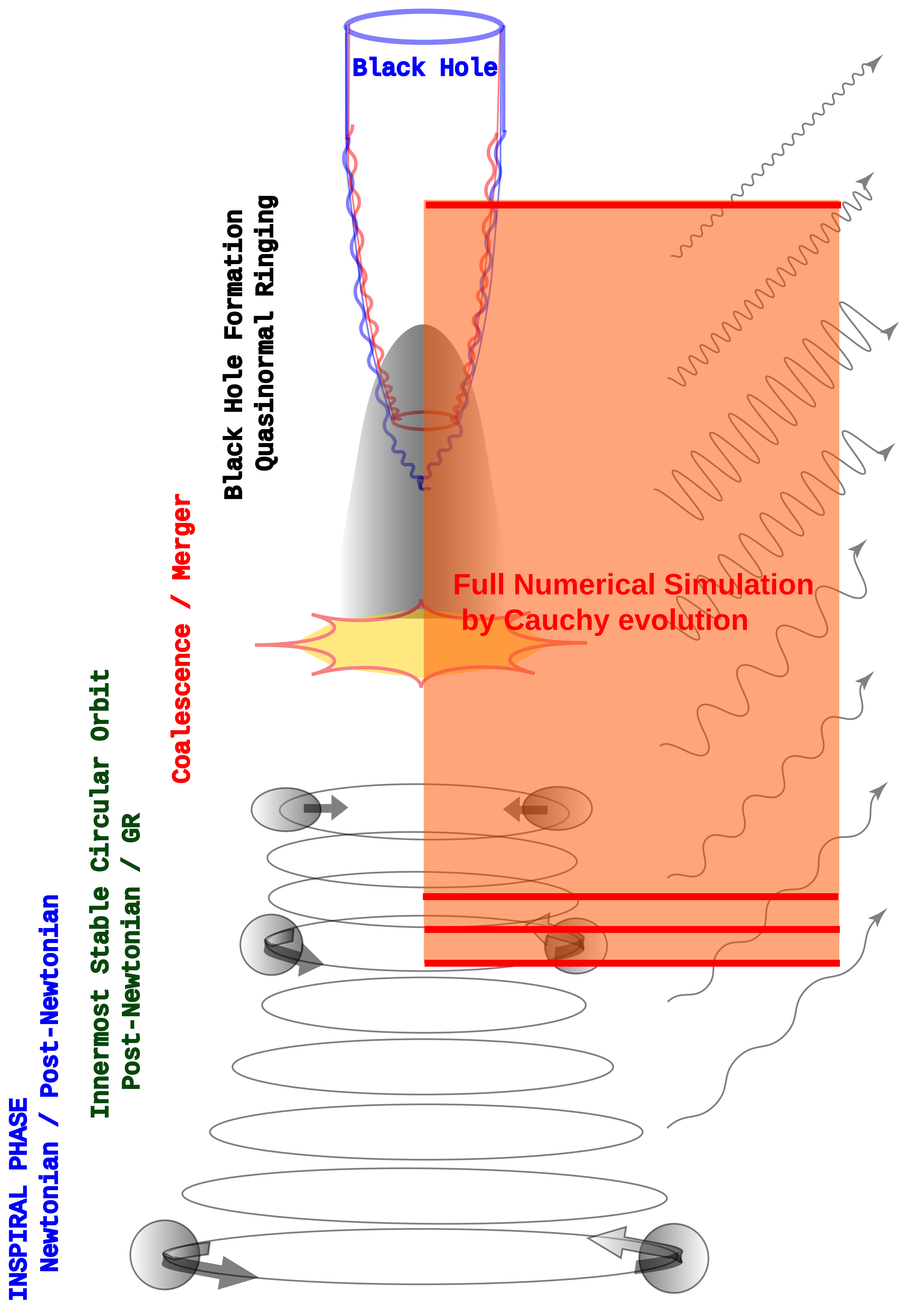
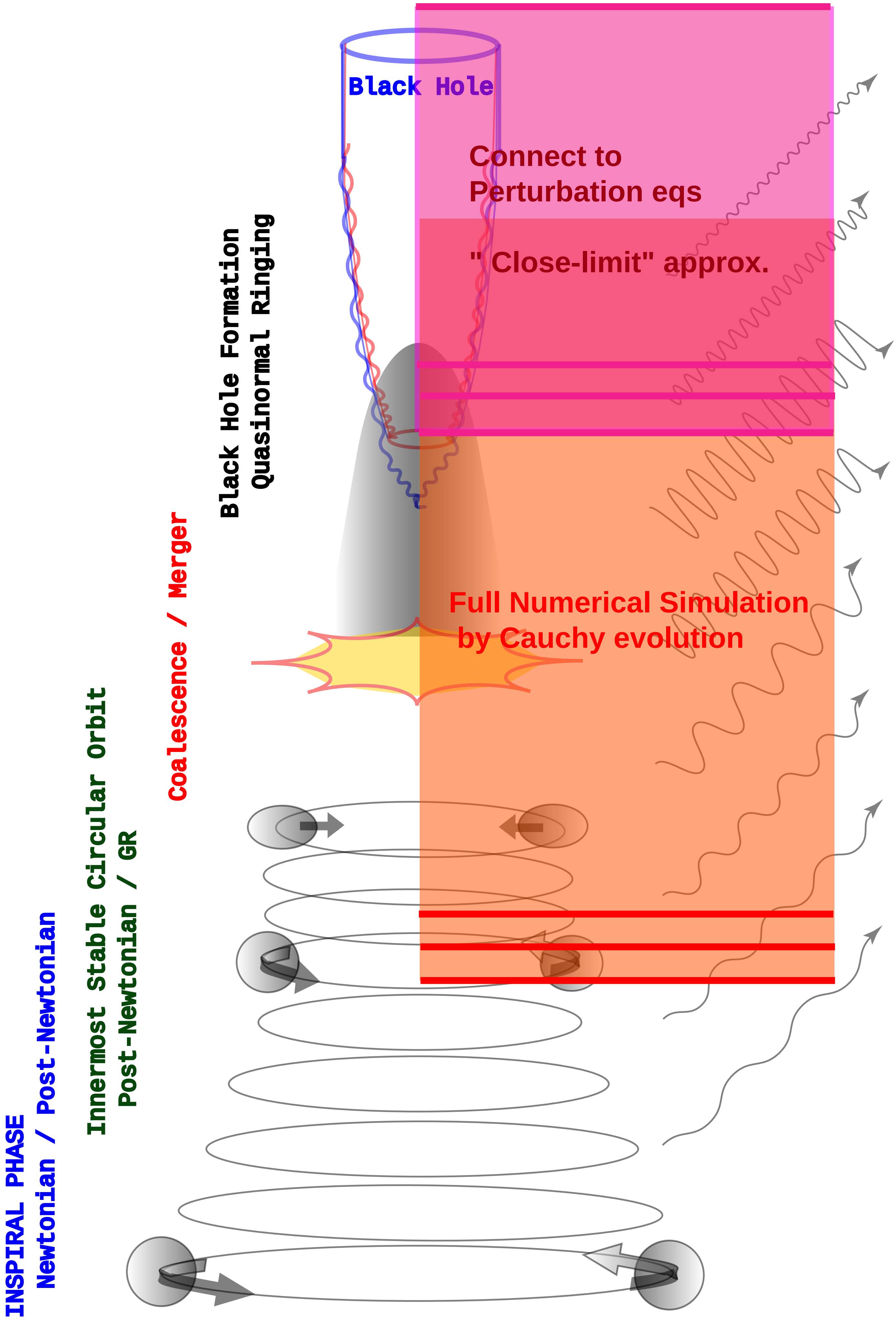


Fig. 4. First and second order waveforms. Because the first and second order Zerilli functions are not the coefficients of an expansion of a function, it makes no sense to compare them. We therefore present the time derivative of the first order Zerilli function and a second order correction to it. These expressions squared are proportional to the radiated power, and convey information about the gravitational waveform.



Black Hole
Black Hole Formation
Quasinormal Ringing
Coalescence / Merger
Full Numerical Simulation
by Cauchy evolution
Innermost Stable Circular Orbit
Post-Newtonian / GR
INSPIRAL PHASE
Newtonian / Post-Newtonian


Black Hole
Connect to
Perturbation eqs
" Close-limit" approx.
Full Numerical Simulation
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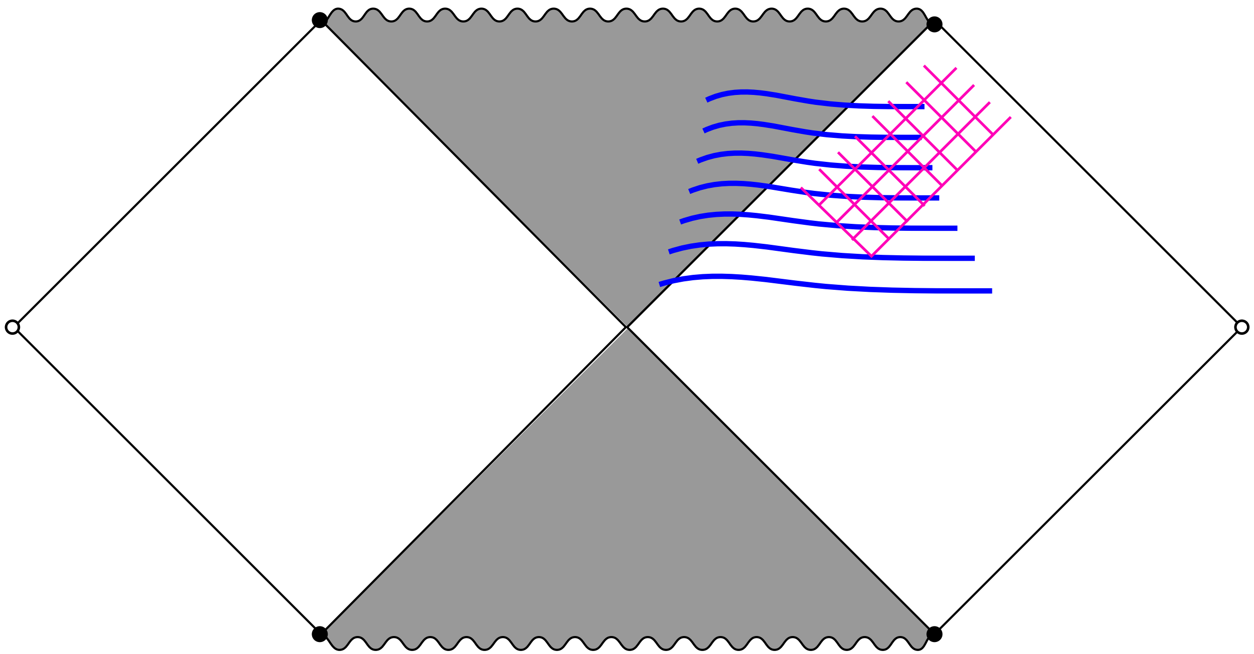
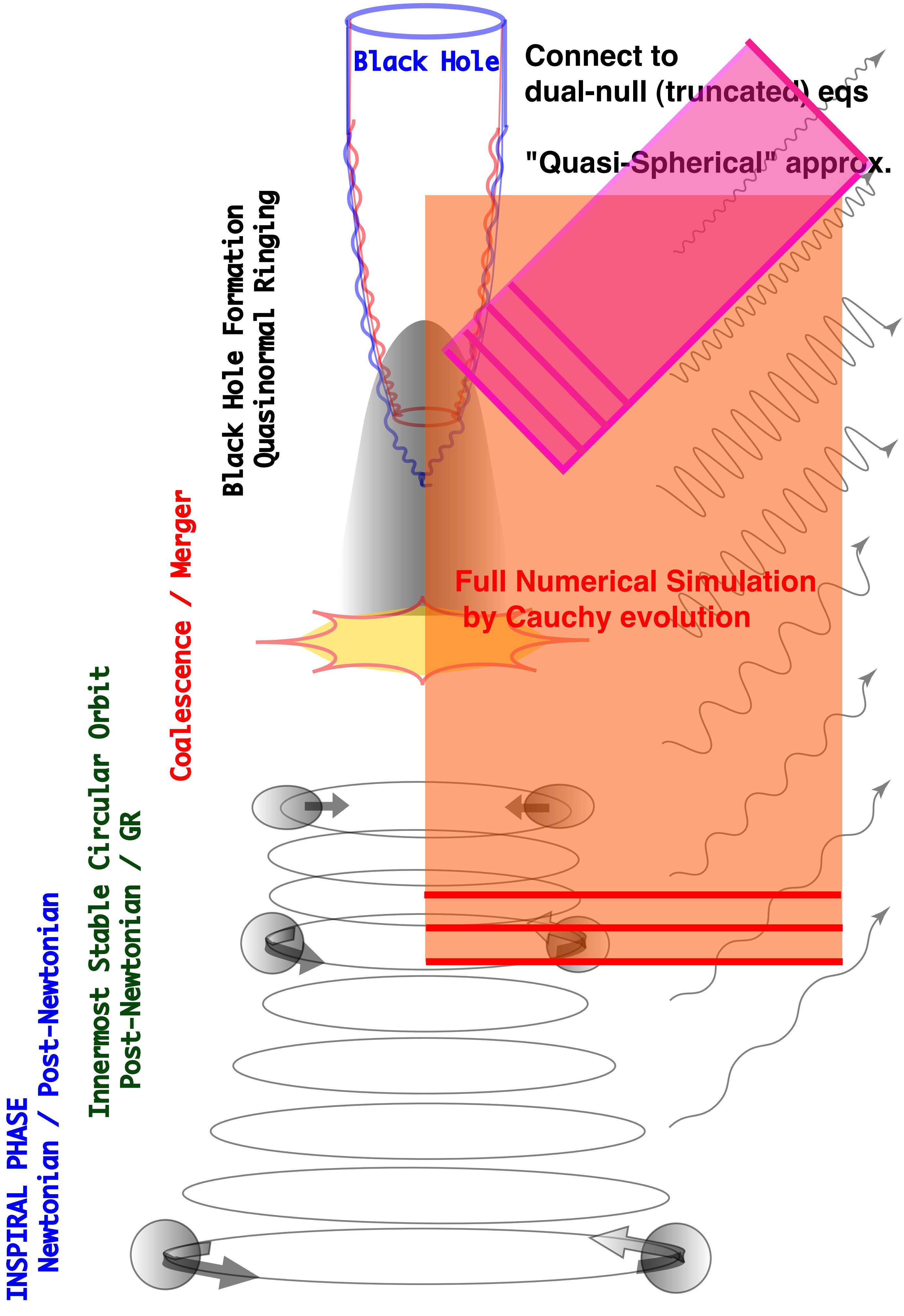


"quasi-spherical approximation"


Black Hole
Connect to
dual-null (truncated) eqs
"Quasi-Spherical" approx.
Black Hole Formation
Quasinormal Ringing
Coalescence / Merger
Full Numerical Simulation
by Cauchy evolution
INSPIRAL PHASE
Newtonian / Post-Newtonian
Innermost Stable Circular Orbit
Post-Newtonian / GR

Quasi-Spherical Approximation scheme

S.A. Hayward, Phys. Rev. **D61**, 101503 (2000).

The scheme truncates the Einstein equations by removing second-order terms which would vanish in a spherically symmetric space-time.

- dual-null formulation is adapted to radiation extraction.
- only ordinary differential equations need be solved
- providing a computationally inexpensive estimate of the gravitational waveforms produced by a black-hole collision, given a full numerical simulation up to (or close to) coalescence, or an analytical model thereof.
- No prescribed background is required and that arbitrarily rapid dynamical processes (close to spherical symmetry) are allowed.

How well the scheme handles deviations from spherical symmetry?

against angular momentum? (cf. coalescing black holes)

PHYSICAL REVIEW D, VOLUME 64, 044002

Quasispherical approximation for rotating black holes

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Centre for Gravitational Physics and Geometry, 104 Davey Laboratory, The Pennsylvania State University, University Park, Pennsylvania 16802-6300
(Received 31 August 2000; published 5 July 2001)

We numerically implement a quasispherical approximation scheme for computing gravitational waveforms for coalescing black holes, testing it against angular momentum by applying it to Kerr black holes. As error measures, we take the conformal strain and specific energy due to spurious gravitational radiation. The strain is found to be monotonic rather than wavelike. The specific energy is found to be at least an order of magnitude smaller than the 1% level expected from typical black-hole collisions, for angular momentum up to at least 70% of the maximum, for an initial surface as close as $r=3m$.

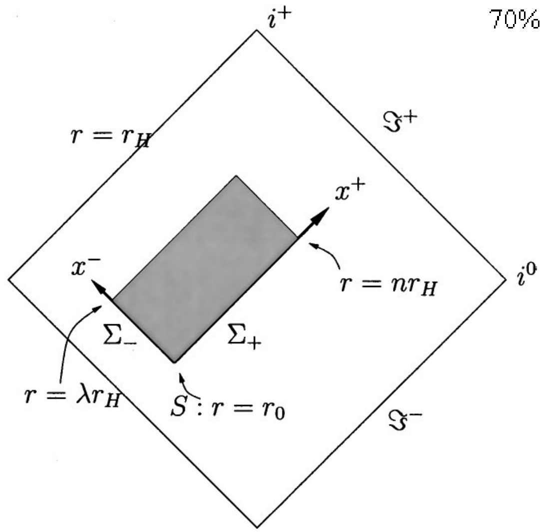

FIG. 1. The region of numerical integration is shown as the shaded region in the picture. Initial data is prescribed on a spatial surface S of constant Boyer-Lindquist $r=r_0$ and t, and the null hypersurfaces $\Sigma_\pm$ generated from it. On Σ_- (Σ_+), the x^- (x^+) coordinate is set so as to cover the region $\lambda r_H \leq r \leq r_0$ ($r_0 \leq r \leq nr_H$), where $1<\lambda\approx 1$ and $n \gg 1$ are constants to be set by hand.

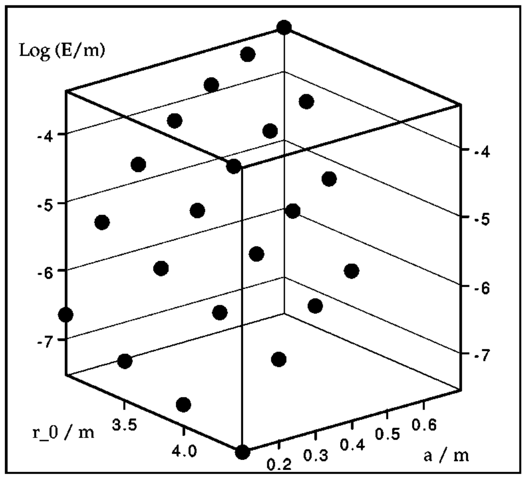

FIG. 6. Logarithmic plot of specific energy E/m due to spurious radiation, as a function of a/m and r_0/m. Energy is measured at $x^+=30$, and the plotted range is $r_0/m \in [3.0,4.5]$ and $a/m \in [0.1,0.7]$.