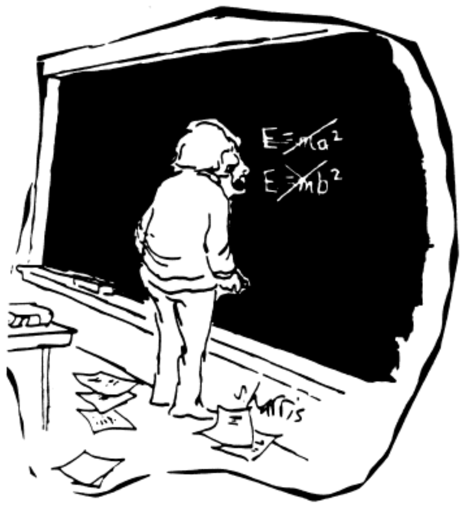
APCTP Winter School, January 17-18, 2003

Introduction to Numerical Relativity

RIKEN Institute, Hisaaki Shinkai

1. Subjects for Numerical Relativity
   Why Numerical Relativity?
2. The Standard Approach to Numerical Relativity
   The ADM formulation
3. Alternative Approaches to Numerical Relativity
   etc
4. Unsolved problems
   Gravitational Wave Physics and related problems
   Conjecture Hunting -- reported and unreported issues --



신카이 히사아키

Numerical Relativity – open issues

Box 1.2

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? $\Rightarrow$ see e.g. gr-qc/0209111 Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

Several known theorems on Black Holes

BH Uniqueness Theorem — Israel (1967), Robinson (1977), Carter (1971)

Any static solution of Einstein's vacuum equations satisfying conditions (1)-(3) is spherically symmetric and coincides with the Schwarzschild metric.
(1) it is asymptotically flat,
(2) it has an event horizon, and
(3) it has no singularities on or outside the event horizon.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

Similarly, for stationary configuration, the Kerr metric.

$$ds^2 \;=\; -\frac{\Delta}{\Sigma}[dt - a\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)d\phi - adt]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \tag{2}$$

where

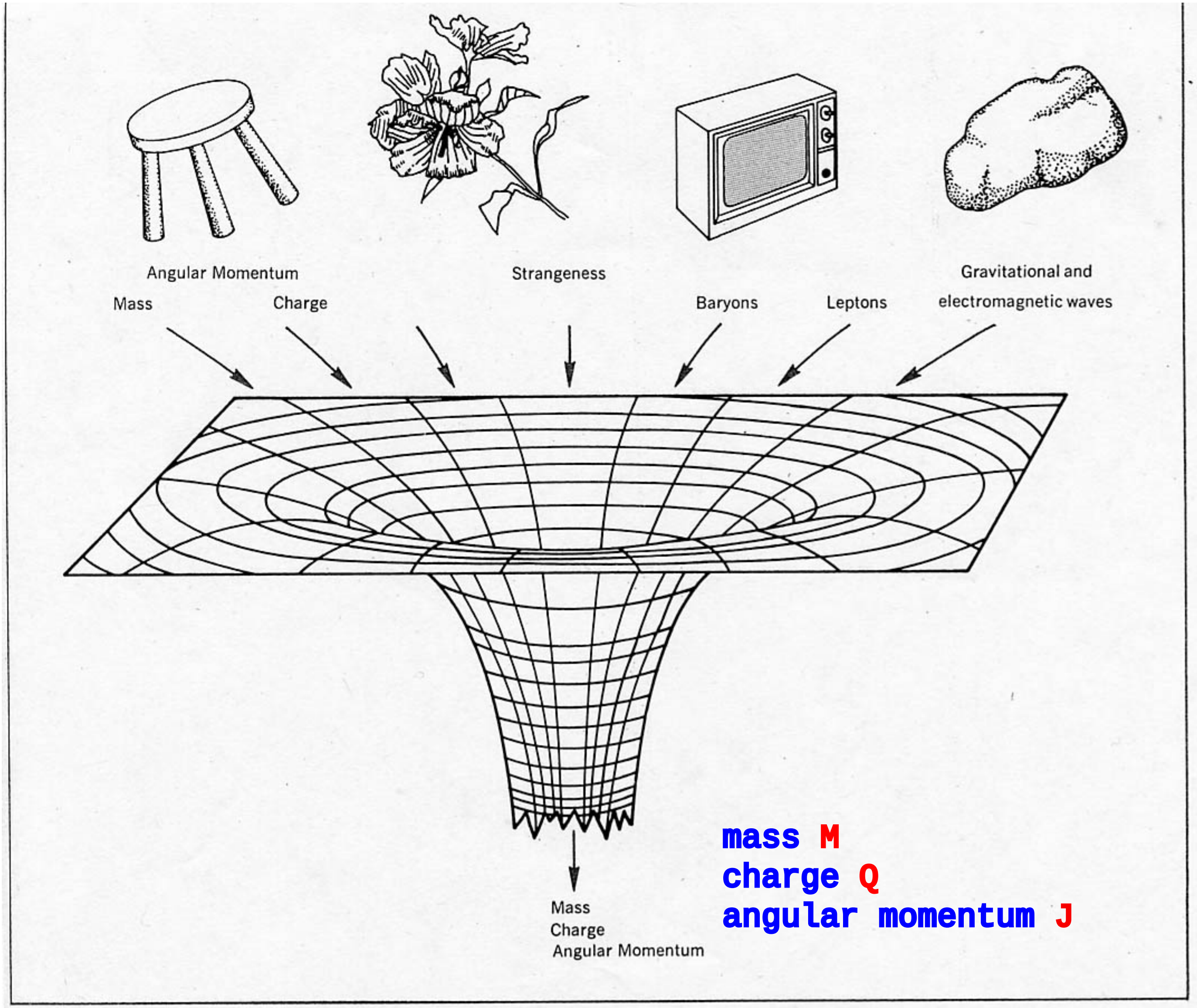
$$\Delta = r^2 - 2mr + a^2(+q^2), \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

BH No-hair Conjecture — Ruffini-Wheeler (1971)

Regardless of the specific details of the collapse or the structure and properties of the collapsing body, the resulting stationary black hole is described by a geometry specified by the parameters M, J, and Q.

Black-Hole No-Hair Conjecture

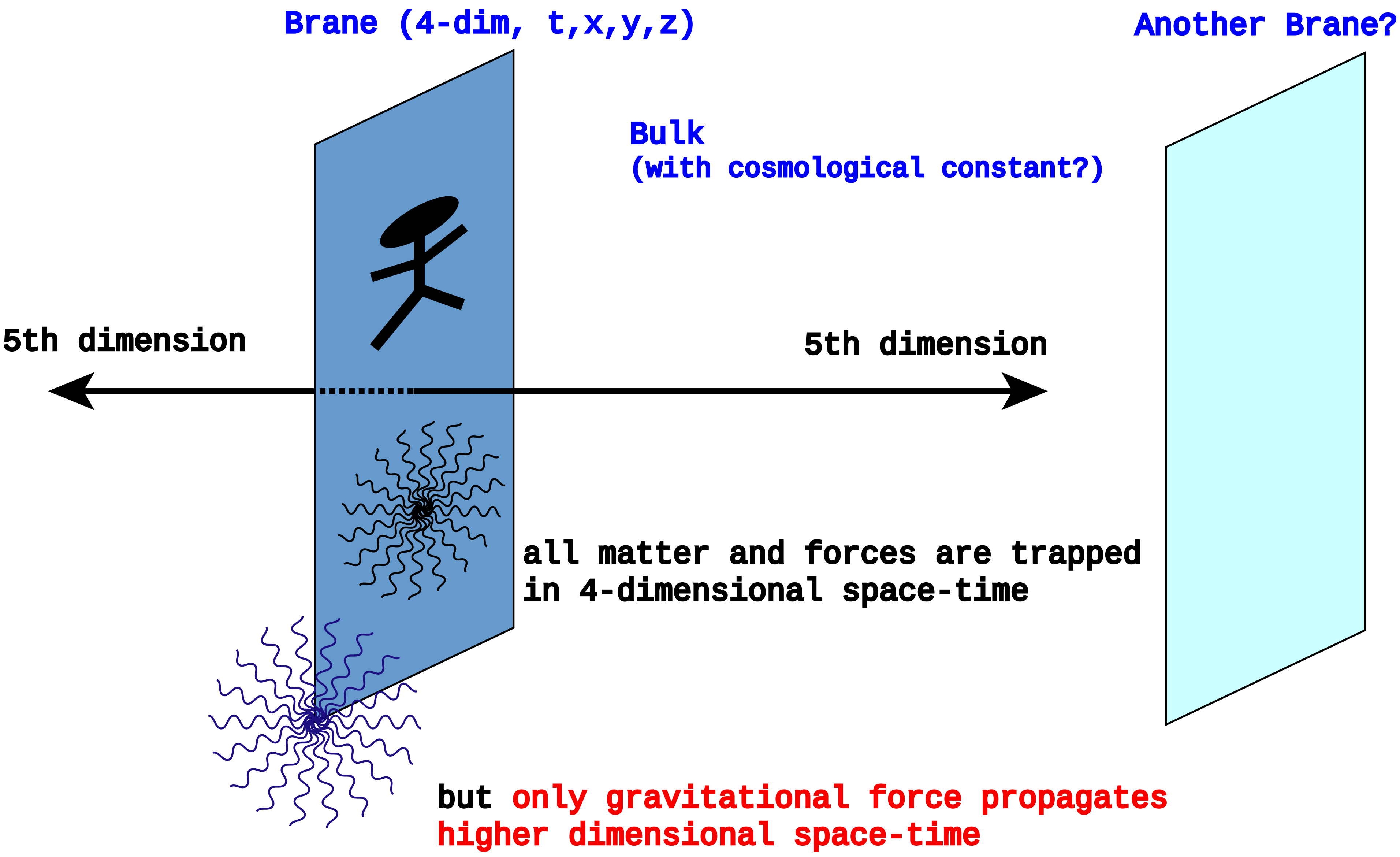
Ruffini-Wheeler (1971)



mass M
charge Q
angular momentum J

Figurative representation of a black hole in action. All details of the infalling matter are washed out. The final configuration is believed to be uniquely determined by mass, electric charge, and angular momentum. Figure 1

Brane-World model



A Rotating Black Ring Solution in Five Dimensions

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(Received 8 November 2001; published 21 February 2002)

The vacuum Einstein equations in five dimensions are shown to admit a solution describing a stationary asymptotically flat spacetime regular on and outside an event horizon of topology $S^1 \times S^2$. It describes a rotating "black ring." This is the first example of a stationary asymptotically flat vacuum solution with an event horizon of nonspherical topology. The existence of this solution implies that the uniqueness theorems valid in four dimensions do not have simple five-dimensional generalizations. It is suggested that increasing the spin of a spherical black hole beyond a critical value results in a transition to a black ring, which can have an arbitrarily large angular momentum for a given mass.

DOI: 10.1103/PhysRevLett.88.101101 PACS numbers: 04.50.+h, 04.20.Jb, 04.70.Bw

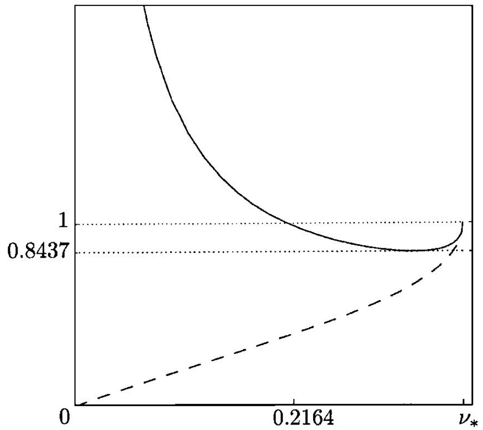


FIG. 2. $(27\pi/32G)J^2/M^3$ as a function of ν. Here and in the following graph, the solid line corresponds to the black ring, the dashed line to the black hole. The two dotted lines delimit the values for which a black hole and two black rings with the same mass and spin can exist.

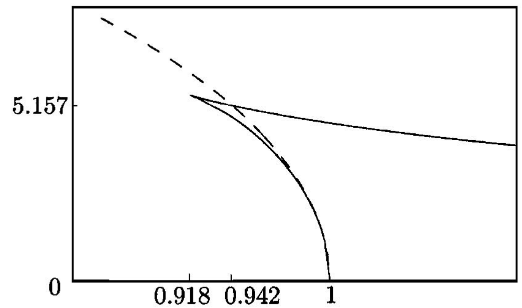
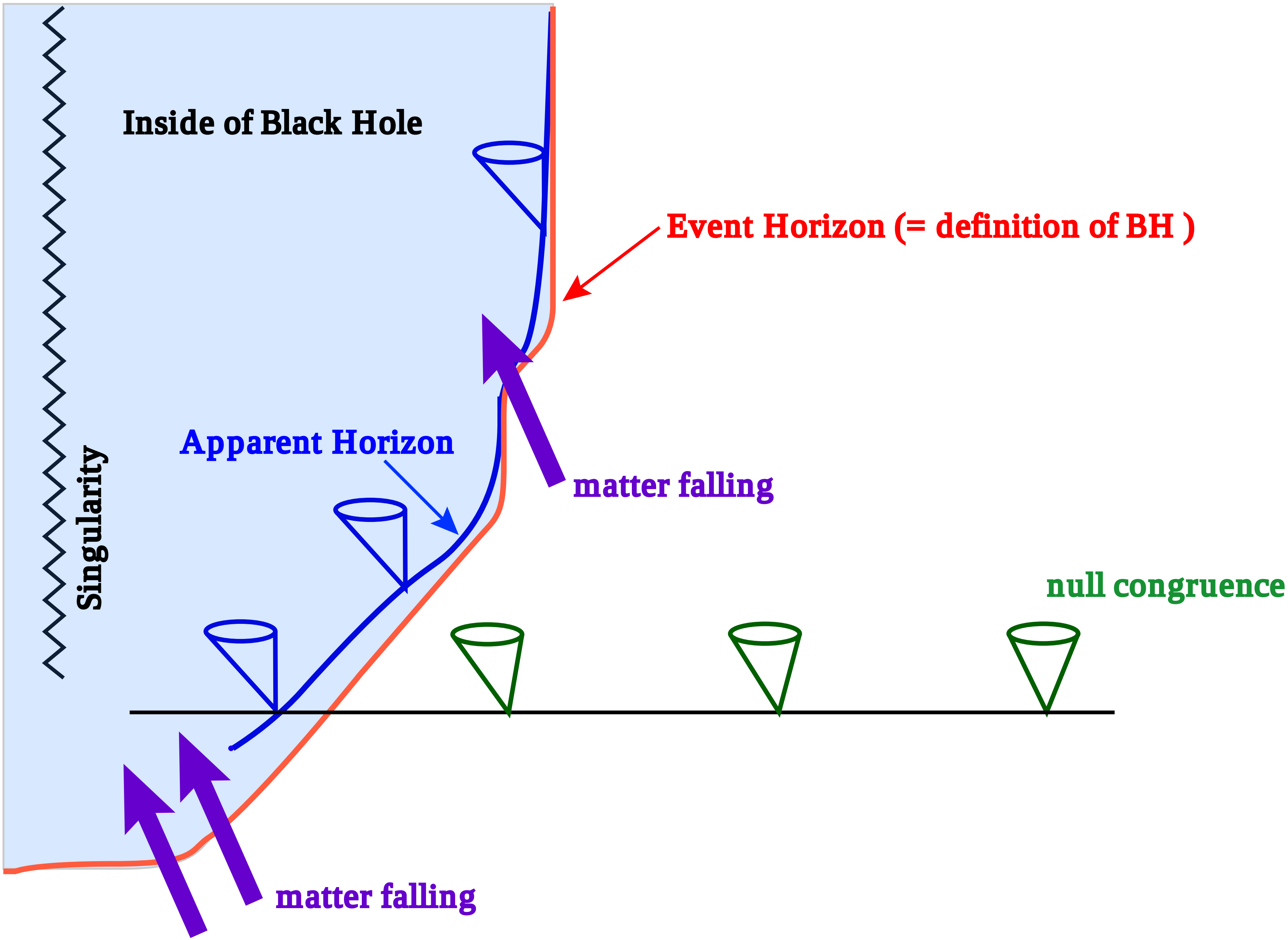
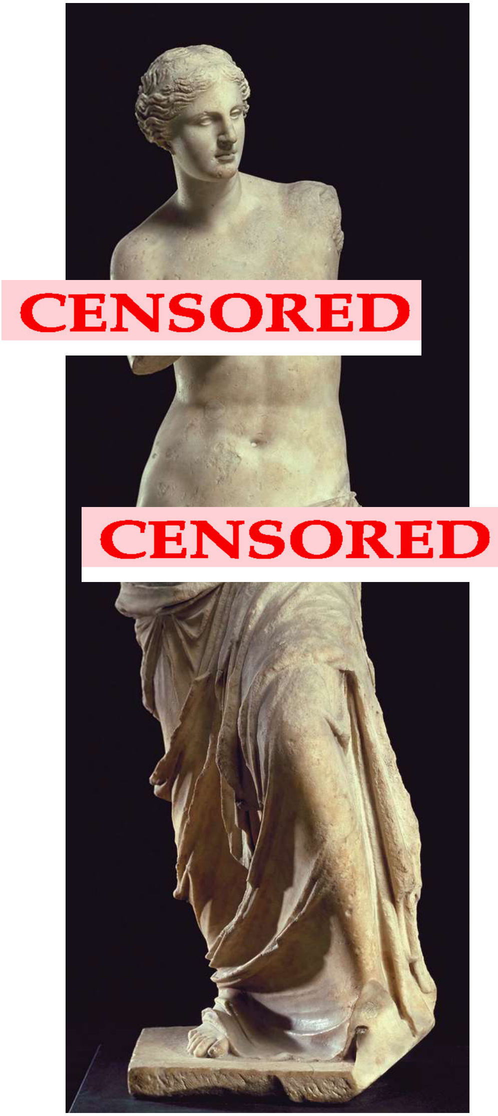


FIG. 3. $\mathcal{A}/(GM)^{3/2}$ against $\sqrt{27\pi/32G}\,J/M^{3/2}$, around the regime in which a black hole and two black rings with the same M and J exist. For $\sqrt{27\pi/32G}\,J/M^{3/2} \approx 0.942$ there exist a black hole and a black ring with the same mass, spin, and area $\mathcal{A} \approx 5.157(GM)^{3/2}$.



Inside of Black Hole
Event Horizon (= definition of BH)
Apparent Horizon
matter falling
Singularity
null congruence
matter falling

Cosmic Censorship Conjecture   Penrose (1969)



Weak version

R. Penrose, Riv. Nuovo Cim. 1 (1969) 252

A naked singularity (i.e. a singularity visible to distant observers) cannot evolve from a regular initial state of the system under any physically reasonable assumptions concerning the properties of the matter.

A naked singularity cannot be visible.
==> if true, the outside the BH is globally hyperbolic.

Strong version

R. Penrose, in Hawking-Israel (Cambridge 1979)

In general, the singularities produced by gravitational collapse are spacelke so that no observer can see them until he falls into them.

Formation of Naked Singularities: The Violation of Cosmic Censorship

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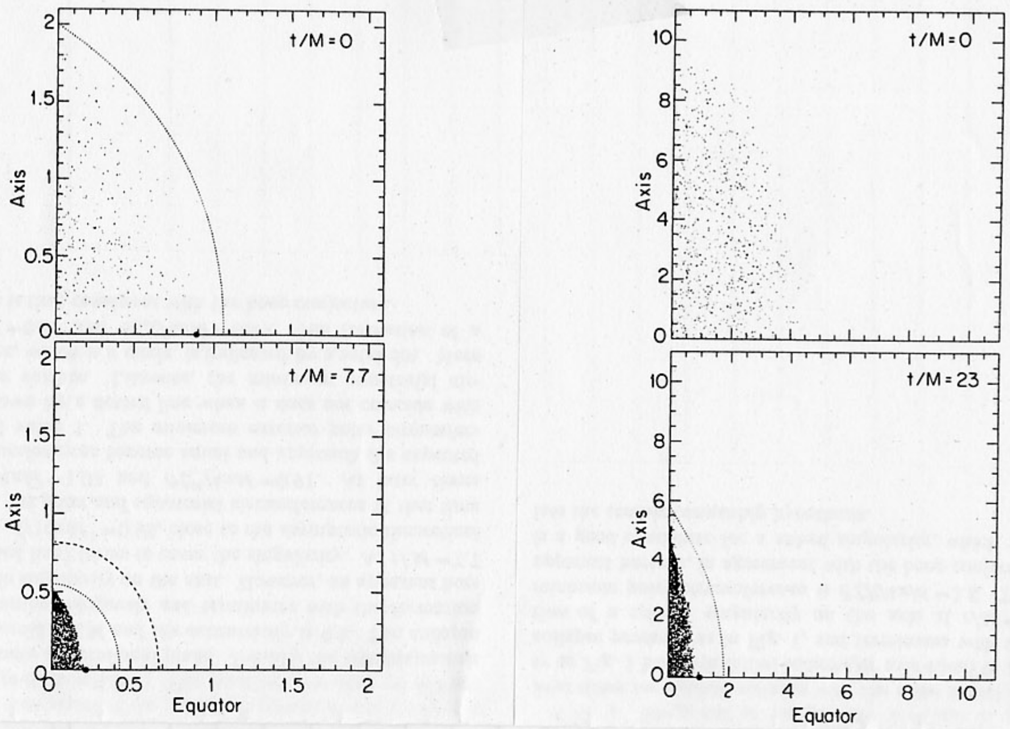
(Received 7 September 1990)

We use a new numerical code to evolve collisionless gas spheroids in full general relativity. In all cases the spheroids collapse to singularities. When the spheroids are sufficiently compact, the singularities are hidden inside black holes. However, when the spheroids are sufficiently large, there are no apparent horizons. These results lend support to the hoop conjecture and appear to demonstrate that naked singularities can form in asymptotically flat spacetimes.



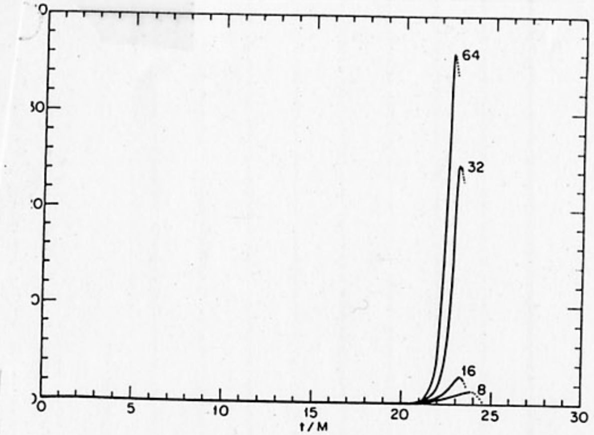


FIG. 3. Growth of the Riemann invariant I (in units of M^{-4}) vs time for the collapse shown in Fig. 2. The simulation was repeated with various angular grid resolutions. Each curve is labeled by the number of angular zones used. We use dots to show where the singularity has caused the code to become inaccurate.

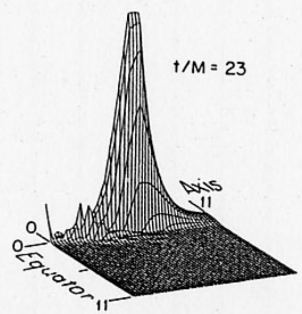
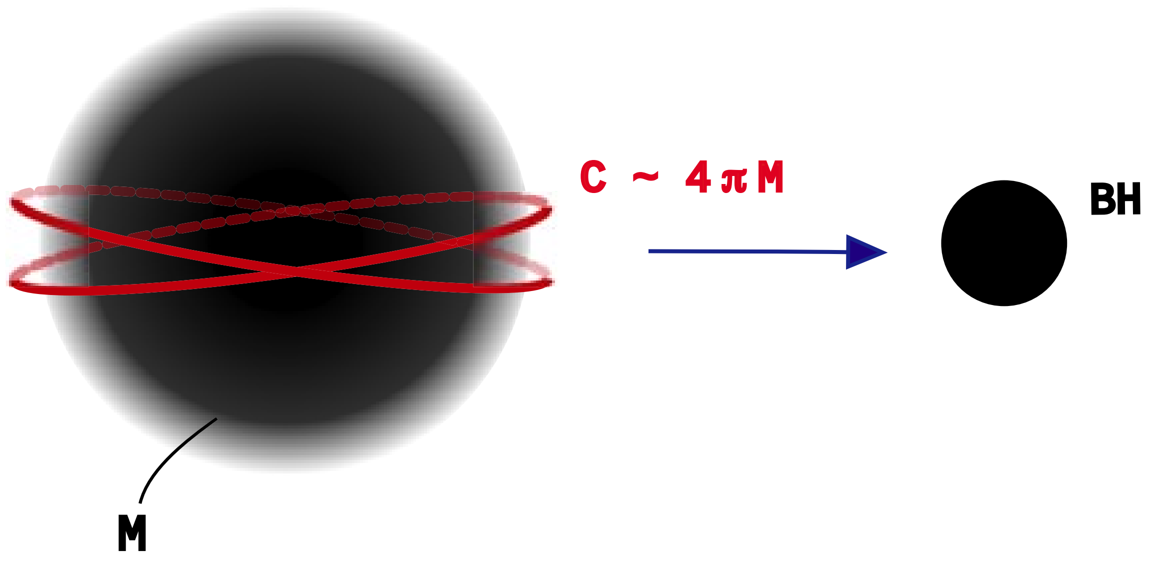


FIG. 4. Profile of I in a meridional plane for the collapse shown in Fig. 2. For the case of 32 angular zones shown here, the peak value of I is $24/M^4$ and occurs on the axis just outside the matter.

Hoop Conjecture

K. Thorne, in "Magic without Magic" ed. by Klauder (1972)

BH with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is bounded by $\mathcal{C} \leq 4\pi M$.



PHYSICAL REVIEW D **66**, 064026 (2002)

Isoperimetric inequality for higher-dimensional black holes

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(Received 3 June 2002; published 30 September 2002)

The initial data sets for the five-dimensional Einstein equation have been examined. The system is designed such that the black hole ($\simeq S^3$) or the black ring ($\simeq S^2 \times S^1$) can be found. We have found that the typical length of the horizon can become arbitrarily large but the area of characteristic closed two-dimensional submanifold of the horizon is bounded above by the typical mass scale. We conjecture that the isoperimetric inequality for black holes in n-dimensional space is given by $V_{n-2} \lesssim GM$, where V_{n-2} denotes the volume of a typical closed $(n-2)$-section of the horizon and M is typical mass scale, rather than $C \lesssim (GM)^{1/(n-2)}$ in terms of the hoop length C, which holds only when $n=3$.

Black-hole Thermodynamics

J.M. Bardeen, B. Carter and S.W. Hawking, Comm. Math. Phys. **31**, 161 (1973).
S.W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

We see the analogous quantities $E \leftrightarrow M$, $T \leftrightarrow \alpha\kappa$, and $S \leftrightarrow (1/8\pi\alpha)A$, where α is a constant.

Law	Thrmodynamics	Black Holes
0th	T constant throughout body in thermal equilibrium	κ constant over horizon of stationary black hole
1st	$dE = TdS+$ work terms	$dM = (1/8\pi)\kappa dA + \Omega_H dJ + \Phi_H dq$
2nd	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
3rd	impossible to achieve $T = 0$ by a physical process (Nernst theorem)	impossible to achieve $\kappa = 0$ by a physical process (censorship conjecture)

Table 1: Black Holes and Thermodynamics.

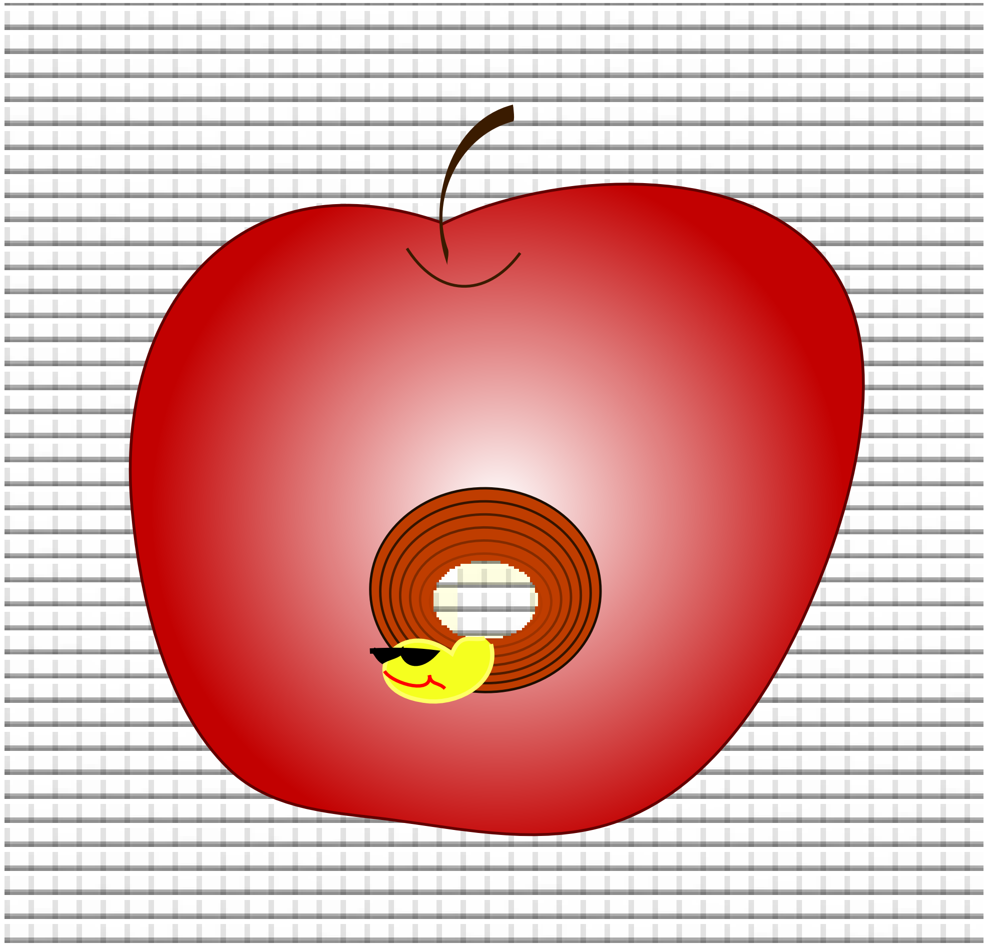
Hawking TemperatureCHawking evapolation

Due to quantum effects, BH has Planck thermal radiation with temperature T, and will evapolate.

For Schwarzschild BH, $T = \dfrac{1}{8\pi M}\left(= \dfrac{hc^3}{8\pi kGM}\right)$. For rotating BH, $T = \dfrac{\kappa}{2\pi}$.

Thus the entropy corresponds as $S = A/4$.

Wormhole (realistic)



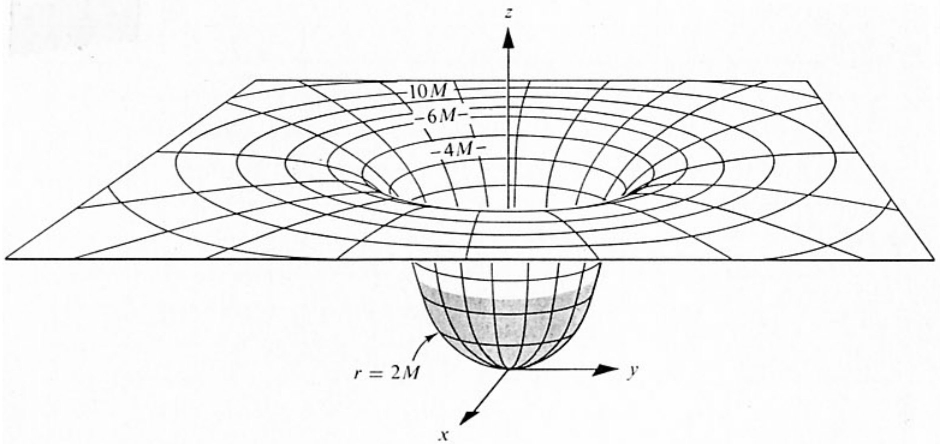
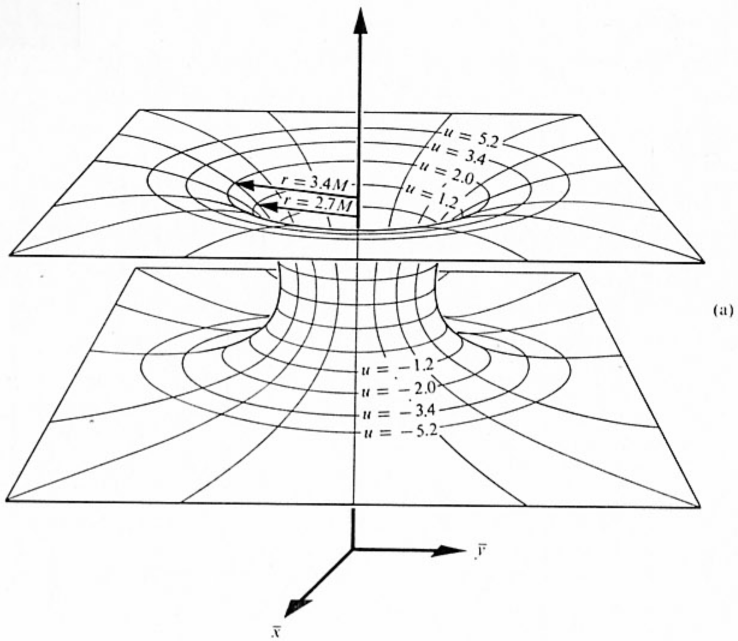


Figure 23.1.
Geometry within (grey) and around (white) a star of radius $R = 2.66M$, schematically displayed. The star is in hydrostatic equilibrium and has zero angular momentum (spherical symmetry). The two-dimensional geometry

$$ds^2 = [1 - 2m(r)/r]^{-1}\,dr^2 + r^2\,d\phi^2$$

of an equatorial slice through the star ($\theta = \pi/2$, t = constant) is represented as embedded in Euclidean 3-space, in such a way that distances between any two nearby points (r, ϕ) and $(r + dr, \phi + d\phi)$ are correctly reproduced. Distances measured off the curved surface have no physical meaning; points off that surface have no physical meaning; and the Euclidean 3-space itself has no physical meaning. Only the curved 2-geometry has meaning. A circle of Schwarzschild coordinate radius r has proper circumference $2\pi r$ (attention limited to equatorial plane of star, $\theta = \pi/2$). Replace this circle by a sphere of proper area $4\pi r^2$, similarly for all the other circles, in order to visualize the entire 3-geometry in and around the star at any chosen moment of Schwarzschild coordinate time t. The factor $[1 - 2m(r)/r]^{-1}$ develops no singularity as r decreases within $r = 2M$, because $m(r)$ decreases sufficiently fast with decreasing r.



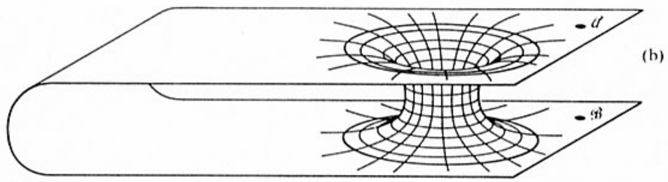


Figure 31.5.
(a) The Schwarzschild space geometry at the "moment of time" $t = v = 0$, with one degree of rotational freedom suppressed ($\theta = \pi/2$). To restore that rotational freedom and obtain the full Schwarzschild 3-geometry, one mentally replaces the circles of constant $\bar{r} = (\bar{x}^2 + \bar{y}^2)^{1/2}$ with spherical surfaces of area $4\pi\bar{r}^2$. Note that the resultant 3-geometry becomes flat (Euclidean) far from the throat of the bridge in both directions (both "universes").

(b) An embedding of the Schwarzschild space geometry at "time" $t = v = 0$, which is geometrically identical to the embedding (a), but which is topologically different. Einstein's field equations fix the local geometry of spacetime, but they do not fix its topology; see the discussion at end of Box 27.2. Here the Schwarzschild "wormhole" connects two distant regions of a single, asymptotically flat universe. For a discussion of issues of causality associated with this choice of topology, see Fuller and Wheeler (1962).

Morris-Thorne's "Traversable" wormhole

M.S. Morris and K.S. Thorne, Am. J. Phys. 56 (1988) 395
M.S. Morris, K.S. Thorne, and U. Yurtsever, PRL 61 (1988) 3182
H.G. Ellis, J. Math. Phys. 14 (1973) 104
(G. Clément, Am. J. Phys. 57 (1989) 967)

Desired properties of traversable WHs

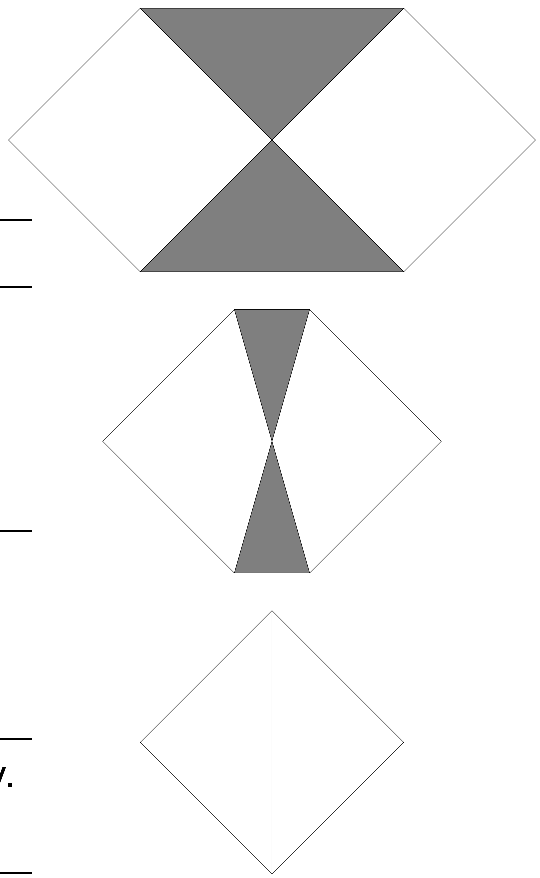
1. Spherically symmetric and Static $\Rightarrow$ M. Visser, PRD 39(89) 3182 & NPB 328 (89) 203
2. Einstein gravity
3. Asymptotically flat
4. No horizon for travel through
5. Tidal gravitational forces should be small for traveler
6. Traveler should cross it in a finite and reasonably small proper time
7. Must have a physically reasonable stress-energy tensor
   $\Rightarrow$ Weak Energy Condition is violated at the WH throat.
   $\Rightarrow$ (Null EC is also violated in general cases.)
8. Should be perturbatively stable
9. Should be possible to assemble

BH and WH are interconvertible ? (New Duality?)

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

- They are very similar – both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)
- Only the causal nature of the THs differs, whether THs evolve in plus / minus density.

	Black Hole	Wormhole
Locally defined by	Achronal(spatial/null) outer TH $\Rightarrow$ 1-way traversable	Temporal (timelike) outer THs $\Rightarrow$ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appearance	occur naturally	Unlikely to occur naturally. but constructible ???



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Fate of the first traversible wormhole: Black-hole collapse or inflationary expansion

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(Received 10 May 2002; published 16 August 2002)

We study numerically the stability of the first Morris-Thorne traversible wormhole, shown previously by Ellis to be a solution for a massless ghost Klein-Gordon field. Our code uses a dual-null formulation for spherically symmetric space-time integration, and the numerical range covers both universes connected by the wormhole. We observe that the wormhole is unstable against Gaussian pulses in either exotic or normal massless Klein-Gordon fields. The wormhole throat suffers a bifurcation of horizons and either explodes to form an inflationary universe or collapses to a black hole if the total input energy, is, respectively, negative or positive. As the perturbations become small in total energy, there is evidence for critical solutions with a certain black-hole mass or Hubble constant. The collapse time is related to the initial energy with an apparently universal critical exponent. For normal matter, such as a traveller traversing the wormhole, collapse to a black hole always results. However, carefully balanced additional ghost radiation can maintain the wormhole for a limited time. The black-hole formation from a traversible wormhole confirms the recently proposed duality between them. The inflationary case provides a mechanism for inflating, to macroscopic size, a Planck-sized wormhole formed in space-time foam.

Ghost pulse input – Bifurcation of the horizons

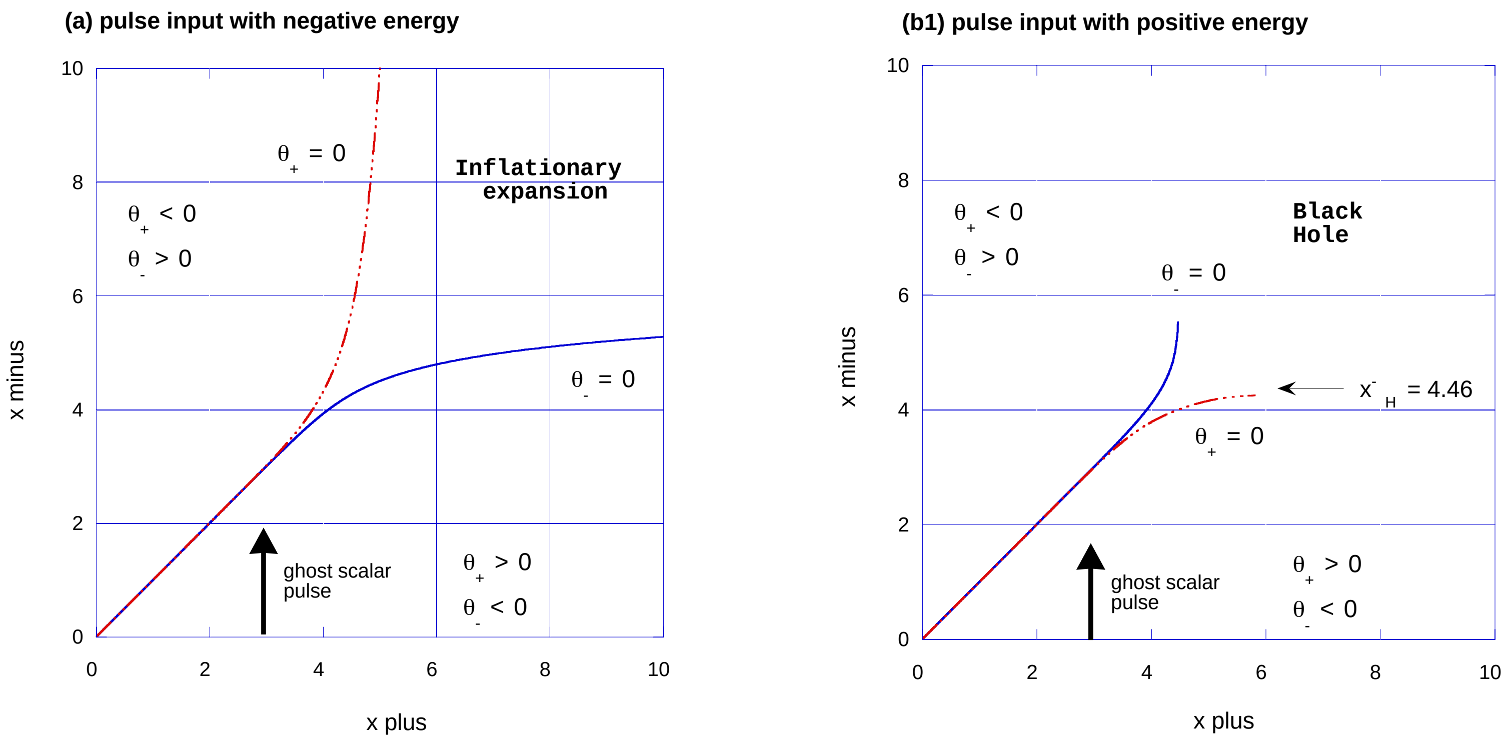
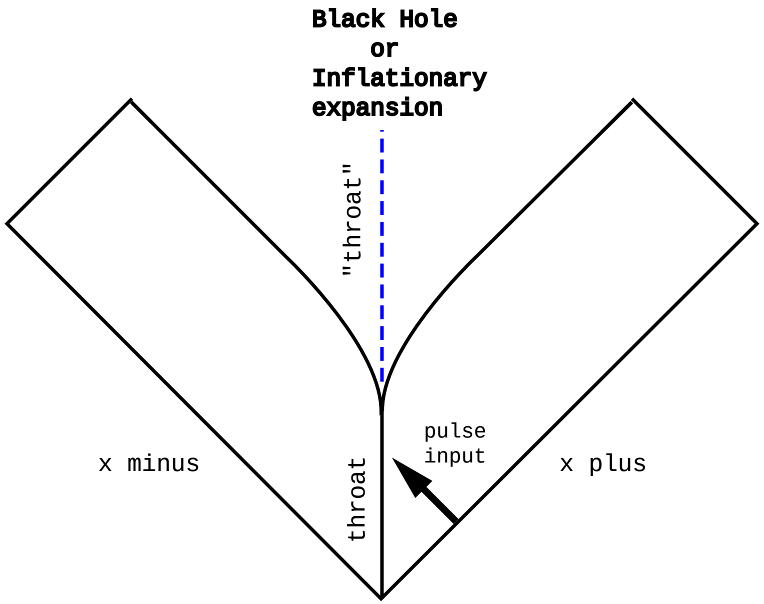


Figure 3: Horizon locations, $\vartheta_\pm = 0$, for perturbed wormhole. Fig.(a) is the case we supplement the ghost field, $c_a = 0.1$, and (b1) and (b2) are where we reduce the field, $c_a = -0.1$ and -0.01. Dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively. In all cases, the pulse hits the wormhole throat at $(x^+, x^-) = (3, 3)$. A 45° counterclockwise rotation of the figure corresponds to a partial Penrose diagram.

Bifurcation of the horizons – go to a Black Hole or Inflationary expansion



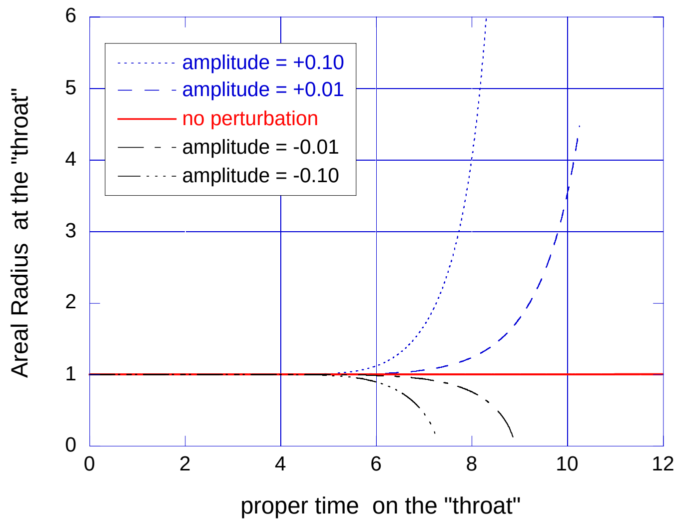


Figure 4: Partial Penrose diagram of the evolved space-time.

Figure 6: Areal radius r of the "throat" $x^+ = x^-$, plotted as a function of proper time. Additional negative energy causes inflationary expansion, while reduced negative energy causes collapse to a black hole and central singularity.

Travel through a Wormhole – with Maintenace Operations!

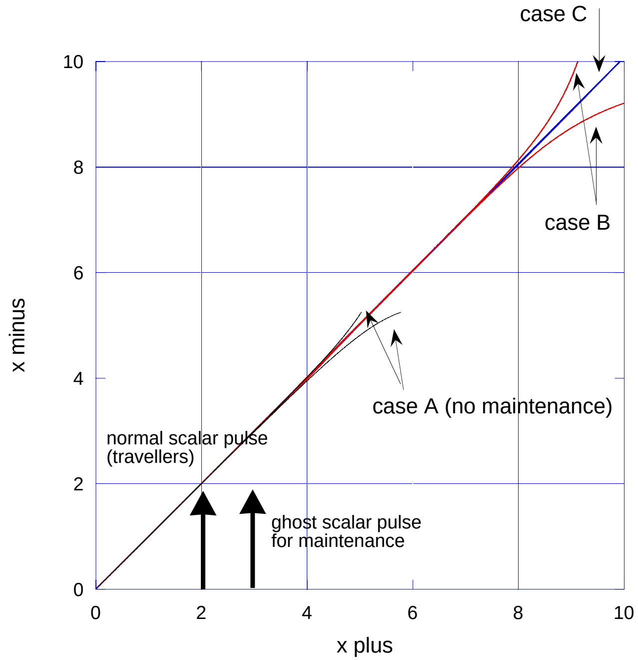


Figure 11: A trial of wormhole maintenance. After a normal scalar pulse, we signalled a ghost scalar pulse to extend the life of wormhole throat. The travellers pulse are commonly expressed with a normal scalar field pulse, $(\tilde{c}_a, \tilde{c}_b, \tilde{c}_c) = (+0.1, 6.0, 2.0)$. Horizon locations $\vartheta_+ = 0$ are plotted for three cases:

(A) no maintenance case (results in a black hole),

(B) with maintenance pulse of $(c_a, c_b, c_c) = (0.02390, 6.0, 3.0)$ (results in an inflationary expansion),

(C) with maintenance pulse of $(c_a, c_b, c_c) = (0.02385, 6.0, 3.0)$ (keep stationary structure upto the end of this range).

Discussion

Dynamics of the Ellis-Morris-Thorne traversible wormhole

⇒ WH is Unstable

(A) with positive energy pulse ⇒ Black Hole

(B) with negative energy pulse ⇒ Inflationary expansion

⇒ (A) confirms duality conjecture between BH and WH.

⇒ (B) provides a mechanism for enlarging a quantum wormhole to macroscopic size.

- **We answered to the question of : what happens if our hero (or heroine) attempts to traverse the wormhole.**
- **New discoveries of the critical behaviour.**

"Science can be stranger than science fiction."

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Quantum foot in the door

ALL around us are tiny doors that lead to the rest of the Universe. Predicted by Einstein's equations, these quantum wormholes offer a faster-than-light short cut to the rest of the cosmos—at least in principle. Now physicists believe they could open these doors wide enough to allow someone to travel through.

Quantum wormholes are thought to be much smaller than even protons and electrons, and until now no one has modelled what happens when something passes through one. So Sean Hayward at Ewha Womans University in Korea and Hisa-aki Shinkai at the Riken Institute of Physical and Chemical Research in Japan decided to do the sums.

They have found that any matter travelling through adds positive energy to the wormhole. That unexpectedly collapses it into a black hole, a supermassive region with a gravitational pull so strong not even light can escape.

But there's a way to stop any would-be traveller being crushed into oblivion. And it lies with a strange energy field nicknamed "ghost radiation". Predicted by quantum theory, ghost radiation is a negative energy field that dampens normal positive energy. Similar effects have been shown experimentally to exist.

Ghost radiation could therefore be used to offset the positive energy of the travelling matter, the researchers have found. Add just the right amount and it should be possible to prevent the wormhole collapsing—a lot more and the wormhole could be widened just enough for someone to pass through.

It would be a delicate operation, however. Add too much negative energy, the scientists discovered, and the wormhole will briefly explode into a new universe that expands at the speed of light, much as astrophysicists say ours did immediately after the big bang.

For now, such space travel remains in the realm of thought experiments. The CERN Large Hadron Collider in Switzerland is expected to generate one mini-black hole per second, a potential source of wormholes through which physicists could try to send quantum-sized particles. But sending a person would be another thing. To keep the wormhole open wide enough would take a negative field equivalent to the energy that would be liberated by converting the mass of Jupiter.

Charles Choi

More at: www.arxiv.org/abs/gr-qc/0205041

Unsolved Problems in GR

.... many many

Grav. Wave Physics: How can we achieve precise numerical simulations of coalescence of binary neutron stars and/or black holes? Can we determine equation of state of neutron star? Physically reasonable initial data? How identify black-hole horizons? Validity of new approximations?

BH Uniqueness Theorems, No-hair Conjecture: Are colored BHs realistic? In higher dim.? Stable configuration of Black String?

Cosmic Censorship Conjecture: Counter-examples? Strong version?

Hoop Conjecture: definition of quasi-local mass? Validity? In higher dim.?

BH Thermodynamics: Why area, not volume? Under dynamical situation?

Dynamical Wormholes: topology change in dynamical transition? New critical behavior for forming black-hole mass? Time-machine? (closed timelike curve? Chronological protection conjecture?) Wormhole thermodynamics? .... etc etc

All realistic discussion requires numerical simulations.

Our understanding for numerical procedures are accumulating.

Ready to go conjecture hunting !