Finite Mordell-Tornheim Multiple Zeta Values

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Abstract.
We investigate a finite analogue of the Mordell-Tornheim multiple zeta values (the *finite Mordell-Tornheim multiple zeta values*). These values can be expressed by a linear combination of finite multiple zeta values, and its rules are described by the shuffle product. Using this expression, we give a certain relation among finite multiple zeta values.

1. Multiple zeta values and Mordell-Tornheim multiple zeta values

Definition (Multiple Zeta Values)

$$\zeta(k_1,\ldots,k_r) := \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

for $k_1,\ldots,k_r \in \mathbb{Z}_{>0}$ *with* $k_1 \geq 2$.

The case $r=2$ was studied by Euler, and general cases have been studied by many authors from the 1990s.
The following types of sums were first studied by Tornheim(1950) and Mordell(1958).

Definition (Mordell-Tornheim multiple zeta values)

$$\zeta^{MT}(k_1,\ldots,k_r;k_{r+1}) := \sum_{m_1,\ldots,m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}(m_1+\cdots+m_r)^{k_{r+1}}}.$$

for $k_1,\ldots,k_{r+1} \in \mathbb{Z}_{>0}$.

2. Finite Mordell-Tornheim multiple zeta values

Definition (Finite multiple zeta values, introduced by Kaneko-Zagier)

Let $\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z} \,/\, \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ *where* p *runs over all primes. Then*

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) := \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \right)_p \in \mathcal{A}$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$.

Proposition 1

For $k_1, \ldots, k_r, k \in \mathbb{Z}_{>0}$, *the following identities hold:*

1. $\zeta_{\mathcal{A}}(k, \ldots, k) = 0$.
2. $\zeta_{\mathcal{A}}(k_1, \ldots, k_r) = (-1)^{k_1 + \cdots + k_r} \zeta_{\mathcal{A}}(k_r, \ldots, k_1)$.
3. $\zeta_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} \frac{B_{p - k_1 - k_2}}{k_1 + k_2}$ *where* B_n *is the* n*-th Bernoulli number.*
4. $\displaystyle\sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{A}}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = 0$ *where* $\mathfrak{S}_r$ *is the symmetric group of degree* r.

The following theorem was proved by Saito-Wakabayashi (2015).

Theorem 2 (Sum formula)

For $1 \leq i \leq r \leq k-1$, we have

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_i \geq 2,\, k_l \geq 1 (l \neq i)}} \zeta_{\mathcal{A}}(k_1, \ldots, k_r) = \left((-1)^{i-1} \left(\binom{k-1}{i-1} + (-1)^r \binom{k-1}{r-i} \right) \frac{B_{p-k}}{k} \right)_p .$$

Definition (Finite Mordell-Tornheim multiple zeta values)

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; k_{r+1}) = \left(\sum_{\substack{m_1, \ldots, m_r > 0 \\ m_1+\cdots+m_r < p}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}} \right)_p \in \mathcal{A}$$

for $k_1, \ldots, k_{r+1} \in \mathbb{Z}_{\geq 0}$.

It is clear that $\zeta_{\mathcal{A}}^{MT}(k_1, 0; k_3) = \zeta_{\mathcal{A}}(k_3, k_1)$.

3. Main results

Let $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial ring over $\mathbb{Q}$ in two indeterminates x and y. The shuffle product ш on $\mathfrak{H}$ is a $\mathbb{Q}$-bilinear map $\mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ satisfying

$$w ш 1 = 1 ш w = w,$$
$$(u_1 w_1) ш (u_2 w_2) = u_1(w_1 ш (u_2 w_2)) + u_2((u_1 w_1) ш w_2)$$

for $w, w_i \in \mathfrak{H}$ and $u_i = x$ or y $(i = 1, 2)$. We denote $x^{k-1}y$ by z_k for $k \geq 1$, and define the $\mathbb{Q}$-linear map $Z_{\mathcal{A}} : \mathfrak{H} \to \mathcal{A}$ satisfying $Z_{\mathcal{A}}(z_{k_1} z_{k_2} \cdots z_{k_r}) = \zeta_{\mathcal{A}}(k_1, k_2, \ldots, k_r)$. For example, $Z_{\mathcal{A}}(x^2yxy) = Z_{\mathcal{A}}(z_3 z_2)$ $= \zeta_{\mathcal{A}}(3, 2)$.

Theorem 3 (Main Theorem)

For $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$, we have

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; l) = Z_{\mathcal{A}}(x^l(z_{k_1} ш \cdots ш z_{k_r})). \tag{1}$$

The right-hand side of (1) can be expressed by a linear combination of finite multiple zeta values, hence the finite Mordell-Tornheim multiple zeta values can be expressed by a linear combination of finite multiple zeta values.

Remark

Main Theorem is proved by using the partial fraction decomposition, and this method can be applied for the classical Mordell-Tornheim multiple zeta values. In fact, we can prove the identity

$$\zeta^{MT}(k_1,\dots,k_r;l) = Z(x^l(z_{k_1} ш \cdots ш z_{k_r})),$$

where $Z:\mathfrak{H}\to\mathbb{R}$ is the $\mathbb{Q}$-linear map satisfying $Z(z_{k_1}\cdots z_{k_r}) = \zeta(k_1,\dots,k_r)$.

We note

$$\zeta_{\mathcal{A}}^{MT}(k_1,\dots,k_r;l) = (-1)^{k_1+l}\zeta_{\mathcal{A}}^{MT}(l,k_2,\dots,k_r;k_1).$$

This is obtained by changing the variables as $m_1 \mapsto p - m_1 - \cdots - m_r$ in the summation. Hence we obtain the following identity.

Corollary 4

For $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$, we have

$$Z_{\mathcal{A}}(x^l(z_{k_1} ш \cdots ш z_{k_r})) = (-1)^{k_1+l} Z_{\mathcal{A}}(x^{k_1}(z_l ш z_{k_2} ш \cdots ш z_{k_r})) \quad (l \geq 1),$$

$$Z_{\mathcal{A}}(z_{k_1} ш \cdots ш z_{k_r}) = (-1)^{k_1} Z_{\mathcal{A}}(z_{k_1}(z_{k_2} ш \cdots ш z_{k_r})).$$

These identities give linear relations among finite multiple zeta values.

4. Examples

For $(k_1, \ldots, k_r) = (k_1, \ldots, k_{r-t}, \overbrace{1, \ldots, 1}^{t})$ $(k_{r-t} \geq 2)$, we define $u_{k_1,\ldots,k_r} = t + 1$. For example, $u_{2,3} = 1$, $u_{3,1} = 2$ and $u_{1,1,1} = 4$. Then the following identity holds.

Proposition 5

For $m, l, r \in \mathbb{Z}_{>0}$, we have

$$(-1)^l \sum_{\substack{k_1+\cdots+k_r=m+r \\ k_i \geq 1}} u_{k_1,\ldots,k_r} \zeta_{\mathcal{A}}(l+1, k_1, \ldots, k_r)$$

$$= (-1)^m \sum_{\substack{k_1+\cdots+k_r=l+r \\ k_i \geq 1}} u_{k_1,\ldots,k_r} \zeta_{\mathcal{A}}(m+1, k_1, \ldots, k_r).$$

We give some relations derived from the above proposition.

Example

When $l = 3$, $m = 1$ and $r = 2$, we obtain

$$\begin{aligned}&\zeta_{\mathcal{A}}(4,1,2) + 2\zeta_{\mathcal{A}}(4,2,1)\\&= \zeta_{\mathcal{A}}(2,1,4) + \zeta_{\mathcal{A}}(2,2,3) + \zeta_{\mathcal{A}}(2,3,2) + 2\zeta_{\mathcal{A}}(2,4,1).\end{aligned}$$

With the help of Prop. 1, we have

$$4\zeta_{\mathcal{A}}(4,1,2) = \zeta_{\mathcal{A}}(2,2,3).$$

Similarly, when $l = 1$, $m = 2$ and $r = 3$, we have

$$\begin{aligned}2\zeta_{\mathcal{A}}(3,1,2,1) + 3\zeta_{\mathcal{A}}(3,2,1,1) + 2\zeta_{\mathcal{A}}(2,1,3,1)&\\+ 3\zeta_{\mathcal{A}}(2,3,1,1) + 2\zeta_{\mathcal{A}}(2,2,2,1) = 0.&\end{aligned}$$